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Binomial sequences

Abstract. We present a description of all binomial sequences of polynomials in one variable over a field of characteristic zero.

1. Introduction

Throughout this article K is a field of characteristic zero, $K[x]$ is the ring of polynomials in one variable x over K, and $K[x,y]$ is the ring of polynomials in two variables x, y over K. Moreover, $K[x][[t]]$ is the ring of formal power series in one variable t over $K[x]$.

Let $\mathcal{F} = (F_n(x))_{n\geqslant 0}$ be a nonzero sequence of polynomials in $K[x]$. We say that $\mathcal{F}$ is a *binomial sequence* if

$$F_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} F_k(x)\, F_{n-k}(y)$$

for all $n \geqslant 0$. The equalities are in the ring $K[x,y]$. The assumption that $\mathcal{F}$ is nonzero means that there exists a nonnegative integer n such that $F_n(x) \neq 0$. We will say that a binomial sequence $\mathcal{F} = (F_n(x))_{n\geqslant 0}$ is *strict* if every polynomial $F_n(x)$ is nonzero.

The well known binomial theorem can be stated by saying that $(x^n)_{n\geqslant 0}$ is a strict binomial sequence. Several other such strict sequences exist. The sequence of *lower factorials* $(x_{(n)})_{n\geqslant 0}$, defined by $x_{(0)} = 1$ and

$$x_{(n)} = x(x-1)(x-2)\cdots(x-n+1) \qquad \text{for } n \geqslant 1$$

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is a strict binomial sequence. The same property has the sequence of *upper factorials* $(x^{(n)})_{n\geqslant 0}$, defined by $x^{(0)}=1$ and

$$x^{(n)} = x(x+1)(x+2)\cdots(x+n-1) \qquad \text{for } n \geqslant 1.$$

The sequence of Abel's polynomials $(A_n(x))_{n\geqslant 0}$, defined by $A_0(x)=1$ and $A_n(x) = x(x-n)^{n-1}$ for $n \geqslant 1$, is a strict binomial sequence (see Subsection 7.3). Many interesting results concerning binomial sequences can be find for example in [3], [5], [13], [15], [17], [20], [21], [23] and others.

There exists a full description of all strict binomial sequences. The important role of such description play results of I. M. Sheffer [24], on linear operators of type zero, published in 1939. Later, in 1957, H. L. Krall [13], applying these results, proved that $\mathcal{F} = (F_n(x))_{n\geqslant 0}$ is a strict binomial sequence if and only if there exists a formal power series $H(t) = \sum_{n=1}^{\infty} a_n t^n$, belonging to $K[[t]]$ with $a_1 \neq 0$ and without the constant term, such that

$$\sum_{n=0}^{\infty} \frac{F_n(x)}{n!} t^n = e^{xH(t)}.$$

In Section 6 we present his proof and some basic properties of linear operators of type zero. Several other proofs and applications of this result can be find; see for example: [21], [20] and [12]. We have here the assumption that $\mathcal{F}$ is strict. In the known proofs this assumption is very important. In this case every polynomial $F_n(x)$ is nonzero and moreover, $\deg F_n(x) = n$ for all $n \geqslant 0$.

However, there exist non-strict binomial sequences. We have the obvious example $\mathcal{F} = (1, 0, 0, \dots)$. The sequence $(F_n(x))_{n\geqslant 0}$ defined by

$$F_{2m}(x) = \frac{(2m)!}{m!} x^m \quad \text{and} \quad F_{2m+1}(x) = 0 \qquad \text{for all } m \geqslant 0,$$

is also a non-strict binomial sequence. Some other such examples are in Section 7.

In this article we present a description of all binomial sequences. We prove (see Theorem 5.5) that if in the above mentioned result of Krall [13] we omit the assumption $a_1 \neq 0$, then this result is also valid for non-strict binomial sequences.

2. Notations and preliminary facts

We denote by $\mathbb{N}$ the set $\{1, 2, 3, \dots\}$, of all natural numbers, and by $\mathbb{N}_0$ the set $\{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$, of all nonnegative integers. If $i_1, \dots, i_s \in \mathbb{N}_0$, where $s \geqslant 1$, then we denote by $\langle i_1, \dots, i_s\rangle$ the generalized Newton integer

$$\frac{(i_1 + \cdots + i_s)!}{i_1! \cdots i_s!}.$$

In particular, $\langle i_1\rangle = 1$, $\langle i_1, i_2\rangle = \binom{i_1+i_2}{i_1}$, $\langle i_1, i_2, i_3\rangle = \langle i_1 + i_2, i_3\rangle\langle i_1, i_2\rangle$.

Let $\mathcal{F} = (F_n)_{n\geqslant 0}$ be a nonzero sequence of polynomials belonging to $K[x]$. Let us repeat that $\mathcal{F}$ is a *binomial sequence* if

$$F_n(x+y) = \sum_{i+j=n} \langle i, j\rangle F_i(x) F_j(y) \qquad \text{for all } n \geqslant 0.$$

We shall say that $\mathcal{F}$ is a *principal sequence*, if

$$F_n(x+y) = \sum_{i+j=n} F_i(x)F_j(y) \qquad \text{for all } n \geqslant 0.$$

Here the sums range over all pairs of nonnegative integers (i,j) such that $i+j=n$. Recall that a binomial sequence $\mathcal{F}$ is *strict* if all the polynomials F_n are nonzero. Moreover, we say that a principal sequence $\mathcal{F}$ is *strict* if all the polynomials F_n are nonzero.

PROPOSITION 2.1
Let $\mathcal{F} = (F_n)_{n\geqslant 0}$ and $\mathcal{P} = (P_n)_{n\geqslant 0}$ be nonzero sequences of polynomials from $K[x]$ such that

$$P_n = \frac{1}{n!}F_n \qquad \textit{for } n \geqslant 0.$$

The sequence $\mathcal{F}$ is binomial if and only if the sequence $\mathcal{P}$ is principal. Moreover, $\mathcal{F}$ is a strict binomial sequence if and only if $\mathcal{P}$ is a strict principal sequence.

Proof. Assume that $\mathcal{F}$ is binomial. Then we have

$$\begin{aligned} P_n(x+y) &= \frac{1}{n!}F_n(x+y) = \frac{1}{n!}\sum_{i+j=n}\langle i,j\rangle F_i(x)F_j(y) \\ &= \sum_{i+j=n}\Big(\frac{1}{i!}F_i(x)\Big)\Big(\frac{1}{j!}F_j(y)\Big) = \sum_{i+j=n} P_i(x)P_j(y). \end{aligned}$$

Hence, it is clear that $\mathcal{P}$ is principal. The opposite implication is also clear.

Thus, if we have a result for principal sequences, then by the above proposition we obtain a similar result for binomial sequences.

Let R be a commutative ring with identity. We shall denote by $R\langle\!\langle t\rangle\!\rangle$ the ring of *formal power series with divided powers* ([2], [18]). Every element of this ring is an ordinary formal power series of the form $\sum_{n=0}^{\infty} r_n t^n$ with $r_n \in R$. It is the ring with the usual addition and with the multiplication $*$ defined by the formulas $a * t^n = t^n * a = at^n$ for $a \in R$, and

$$t^n * t^m = \langle n,m\rangle\, t^{n+m} = \binom{n+m}{n} t^{n+m}.$$

This multiplication $*$ is called the *binomial convolution* ([10], [18]).

If $f = \sum_{n=0}^{\infty} a_n t^n$ and $g = \sum_{n=0}^{\infty} b_n t^n$ are elements of $R\langle\!\langle t\rangle\!\rangle$, then the binomial convolution of f and g is

$$f * g = \sum_{n=0}^{\infty}\Big(\sum_{i+j=n}\langle i,j\rangle a_i b_j\Big)t^n.$$

The ring $R\langle\!\langle t\rangle\!\rangle$ is commutative with identity. Note that if $f = \sum_{n=0}^{\infty} a_n t^n$, $g = \sum_{n=0}^{\infty} b_n t^n$ and $h = \sum_{n=0}^{\infty} c_n t^n$, then

$$(f*g)*h = f*(g*h) = \sum_{n=0}^{\infty}\Big(\sum_{i+j+k=n}\langle i,j,k\rangle a_i b_j c_k\Big)t^n.$$

If R is a domain containing $\mathbb{Q}$, then $R\langle\langle t\rangle\rangle$ is also a domain.

Proposition 2.2
If $\mathbb{Q} \subset R$, then the rings $R\langle\langle t\rangle\rangle$ and $R[[t]]$ are isomorphic. More exactly, the mapping $\sigma\colon R\langle\langle t\rangle\rangle \to R[[t]]$ defined by

$$\sigma\Big(\sum_{n=0}^{\infty} f_n t^n\Big) = \sum_{n=0}^{\infty} \frac{f_n}{n!} t^n$$

is an isomorphism of rings.

Proof. It is clear that σ is a bijection, $\sigma(1) = 1$ and $\sigma(f+g) = \sigma(f) + \sigma(g)$ for $f, g \in R\langle\langle t\rangle\rangle$. Put $f = \sum_{n=0}^{\infty} f_n t^n$ and $g = \sum_{n=0}^{\infty} g_n t^n$. Then $f*g = \sum_{n=0}^{\infty} (f*g)_n t^n$ and we have

$$\begin{aligned}
\sigma(f*g) &= \sum_{n=0}^{\infty} \frac{1}{n!}(f*g)_n t^n = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\sum_{i+j=n} \langle i,j\rangle f_i g_j\Big) t^n \\
&= \sum_{n=0}^{\infty} \Big(\sum_{i+j=n} \Big(\frac{f_i}{i!}\Big)\Big(\frac{g_j}{j!}\Big)\Big) t^n = \Big(\sum_{n=0}^{\infty} \frac{f_n}{n!} t^n\Big)\Big(\sum_{n=0}^{\infty} \frac{g_n}{n!} t^n\Big) \\
&= \sigma(f)\sigma(g).
\end{aligned}$$

This completes the proof.

3. Initial properties of principal sequences

Proposition 3.1
If $\mathcal{P} = (P_n)_{n\geqslant 0}$ is a principal sequence, then $P_0 = 1$.

Proof. Suppose $P_0 = 0$. Let $n \geqslant 1$ and assume that $P_0 = P_1 = \cdots = P_{n-1} = 0$. Then

$$\begin{aligned}
P_n(x) = P_n(x+0) &= P_0(x)P_n(0) + P_n(x)P_0(0) + \sum_{k=1}^{n-1} P_k(x)P_{n-k}(0) \\
&= 0 + 0 + \sum_{k=1}^{n-1} 0 = 0.
\end{aligned}$$

Hence, by induction, $P_n = 0$. Thus, if $P_0 = 0$ then $\mathcal{P}$ is the zero sequence; but it is a contradiction because by definition every principal sequence is nonzero. Therefore, $P_0 \neq 0$. Let $P_0 = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_0$, where $n \geqslant 0$, $p_0, \ldots, p_n \in K$, and $p_n \neq 0$. Since $P_0(x+x) = P_0(x)P_0(x)$, we have the equality

$$2^n p_n x^n + 2^{n-1} p_{n-1} x^{n-1} + \cdots + 2p_1 x + p_0 = p_n^2 x^{2n} + \cdots + p_0^2.$$

If $n \geqslant 1$, then $p_n^2 = 0$ but this contradicts the assumption $p_n \neq 0$. Thus, $n = 0$ and $P_0 = p_0 \in K \setminus \{0\}$. Moreover, $p_0 = p_0^2$, because $P_0(0) = P_0(0+0) = P_0(0)^2$. Hence, $P_0 = p_0 = 1$.

PROPOSITION 3.2
If $\mathcal{P} = (P_n)_{n\geqslant 0}$ is a principal sequence, then $P_n(0) = 0$ for all $n \geqslant 1$.

Proof. We already know from Proposition 3.1 that $P_0 = 1$. Let $n \geqslant 1$ and assume that $P_1(0) = P_2(0) = \cdots = P_n(0) = 0$. Then

$$P_{n+1}(0) = P_{n+1}(0+0) = \sum_{i+j=n+1} P_i(0)P_j(0) = P_{n+1}(0) + P_{n+1}(0)$$

and so, $P_{n+1}(0) = 0$.

Assume that $\mathcal{P} = (P_n)_{n\geqslant 0}$ is an arbitrary principal sequence. We do not assume that $\mathcal{P}$ is strict. There exist many non-strict such sequences. For example $\mathcal{P} = (1, 0, 0, \dots)$ is a non-strict principal sequence. Next such examples we may obtain by the following proposition.

PROPOSITION 3.3
Let $(P_n)_{n\geqslant 0}$ be a principal sequence and let s be a positive integer. Let $(R_n)_{n\geqslant 0}$ be a sequence of polynomials defined by

$$R_{ms} = P_m \qquad \text{for } m \geqslant 0,$$

and $R_n = 0$ when $s \nmid n$. Then $(R_n)_{n\geqslant 0}$ is a non-strict principal sequence.

Proof. It is obvious that

$$R_n(x+y) = 0 = \sum_{i+j=n} R_i(x)R_j(y)$$

in the case when $s \nmid n$. If $n = sm$ with $m \in \mathbb{N}_0$, then

$$\begin{aligned}\sum_{i+j=sm} R_i(x)R_j(y) &= \sum_{si+sj=sm} R_{si}(x)R_{sj}(y) = \sum_{i+j=m} P_i(x)P_j(y)\\ &= P_m(x+y) = R_{sm}(x+y).\end{aligned}$$

Note also the following general property of principal sequences.

PROPOSITION 3.4
Let $(P_n(x))_{n\geqslant 0}$ be a principal sequence of polynomials from $K[x]$ and let $0 \neq a \in K$. Let

$$R_n(x) = a^n P_n(x) \qquad \text{for } n \geqslant 0.$$

Then $(R_n(x))_{n\geqslant 0}$ is a principal sequence.

Proof. We have

$$\begin{aligned}R_n(x+y) &= a^n P_n(x+y) = a^n \sum_{i+j=n} P_i(x)P_j(y)\\ &= \sum_{i+j=n} (a^i P_i(x))(a^j P_j(y)) = \sum_{i+j=n} R_i(x)R_j(y).\end{aligned}$$

This completes the proof.

In the next proposition we characterize strict principal sequences.

PROPOSITION 3.5
Let $(P_n(x))_{n\geqslant 0}$ be a strict principal sequence. Then

(1) *$P_1(x) = ax$ for some $0 \neq a \in K$;*

(2) *$\deg P_n(x) = n$ for all $n \geqslant 0$;*

(3) *the initial monomial of each $P_n(x)$ is equal to $\frac{1}{n!}a^n x^n$.*

Proof. Let $P_1(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$, where $m \geqslant 0$ and $a_0, \ldots, a_m \in K$ with $a_m \neq 0$. Since $P_1(x+y) = P_0(x)P_1(y) + P_1(x)P_0(y)$ and $P_0(x) = P_0(y) = 1$, we have (putting $y = x$) $P_1(2x) = 2P_1(x)$ and so, $(2^m - 2)a_m = 0$. Hence, $m = 1$ because $a_m \neq 0$. We know also that $P_1(0) = 0$ (see Proposition 3.2). Therefore,

$$P_1(x) = ax \qquad \text{for some } 0 \neq a \in K.$$

Now let $s \geqslant 2$ and assume that the initial monomial of every $P_k(x)$, for $k = 1, \ldots, s-1$, is equal to $\frac{1}{k!}a^k x^k$. Look at the equality

$$P_s(2x) - 2P_s(x) = \sum_{k=1}^{s-1} P_k(x)P_{s-k}(x).$$

On the right side we have a polynomial and its initial monomial is equal to

$$\sum_{k=1}^{s-1} \Big(\frac{1}{k!}a^k x^k\Big)\Big(\frac{1}{(s-k)!}a^{s-k}x^{s-k}\Big) = \frac{1}{s!}a^s x^s \sum_{k=1}^{s-1}\binom{s}{k} = \frac{2^s - 2}{s!}a^s x^s \neq 0.$$

This implies that $P_s(x) \neq 0$. Let $P_s(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$, where $m \geqslant 0$ and $a_0, \ldots, a_m \in K$ with $a_m \neq 0$. Then the initial monomial of $P_s(2x) - 2P_s(x)$ is equal to $(2^m - 2)a_m x^m$. Hence,

$$(2^m - 2)a_m x^m = \frac{2^s - 2}{s!}a^s x^s$$

and hence, $m = s$ and $a_m = \frac{1}{s!}a^s$. Therefore, $\deg P_s(x) = s$ and the initial monomial of $P_s(x)$ equals $\frac{1}{s!}a^s x^s$. This completes the proof.

COLORARY 3.6
A principal sequence $(P_n)_{n\geqslant 0}$ is strict if and only if $P_1 \neq 0$.

4. Principal power series

In this section $K[x][[t]]$ is the ring of formal power series over $K[x]$ in one variable t. Every element of this ring is of the form

$$P(x) = \sum_{n=0}^{\infty} P_n(x)t^n,$$

where $(P_n(x))_{n\geqslant 0}$ is a sequence of polynomials belonging to $K[x]$. We shall say that the series $P(x)$ is *principal* if $(P_n(x))_{n\geqslant 0}$ is a principal sequence.

PROPOSITION 4.1
Let $P(x) = \sum_{n=0}^{\infty} P_n(x)t^n \in K[x][[t]]$. The series $P(x)$ is principal if and only if in the ring $K[x, y][[t]]$ it satisfies the equality

$$P(x + y) = P(x)P(y).$$

Proof. Assume that the series $P(x)$ is principal. Then $(P_n(x))_{n\geqslant 0}$ is a principal sequence, and then

$$\begin{aligned} P(x+y) &= \sum_{n=0}^{\infty} P_n(x+y)t^n = \sum_{n=0}^{\infty} \Big(\sum_{i+j=n} P_i(x)P_j(y)\Big)t^n \\ &= \Big(\sum_{n=0}^{\infty} P_n(x)t^n\Big)\Big(\sum_{n=0}^{\infty} P_n(y)t^n\Big) = P(x)P(y). \end{aligned}$$

Thus if $P(x)$ is principal, then it is clear that $P(x+y) = P(x)P(y)$. The opposite implication is also clear.

Let $F = \sum_{n=0}^{\infty} F_n t^n$ be a formal power series belonging to $K[x][[t]]$, and let $G = \sum_{n=1}^{\infty} G_n t^n \in K[x][[t]]$ be a formal power series without the constant term. Consider the substitution

$$F(G) = \sum_{n=0}^{\infty} F_n\Big(\sum_{j=1}^{\infty} G_j t^j\Big)^n.$$

Since G has no the constant term, $F(G)$ is a formal power series belonging to $K[x][[t]]$. Let us use this substitution for the power series $F = e^t = \sum_{n=0}^{\infty} \frac{1}{n!}t^n$ and $G = xH(t)$, where $H(t) = \sum_{n=1}^{\infty} a_n t^n \in K[[t]]$. Denote this substitution by $P(x)$. Thus, we have

$$P(x) = e^{xH(t)} = P_0(x) + P_1(x)t^1 + P_2(x)t^2 + P_3(x)t^3 + \cdots,$$

where each $P_j(x)$ is a polynomial belonging to $K[x]$. Initial examples

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= a_1 x, \\ P_2(x) &= \frac{1}{2}a_1^2 x^2 + a_2 x, \\ P_3(x) &= \frac{1}{6}a_1^3 x^3 + a_1 a_2 x^2 + a_3 x, \\ P_4(x) &= \frac{1}{24}a_1^4 x^4 + \frac{1}{2}a_1^2 a_2 x^3 + a_1 a_3 x^2 + \frac{1}{2}a_2^2 x^2 + a_4 x, \\ P_5(x) &= \frac{1}{120}a_1^5 x^5 + \frac{1}{6}a_1^3 a_2 x^4 + \frac{1}{2}(a_1^2 a_3 + a_1 a_2^2)x^3 + (a_1 a_4 + a_2 a_3)x^2 + a_5 x. \end{aligned}$$

Proposition 4.2
Let $H(t) \in K[[t]]$ be a formal power series without the constant term, and let

$$P(x) = e^{xH(t)}.$$

Then $P(x)$ is a formal power series belonging to $K[x][[t]]$ and this series is principal. Moreover, if $P(x) = \sum_{n=0}^{\infty} P_n(x)t^n$ and $H(t) = \sum_{n=1}^{\infty} a_n t^n$, then $a_n = P_n'(0)$ for all $n \geqslant 1$, where each $P_n'(x)$ is the derivative of $P_n(x)$.

Proof. Since $H(t)$ is without the constant term, the substitution $e^{xH(t)}$ is well defined and it is really an element of $K[x][[t]]$. Moreover,

$$P(x+y) = e^{(x+y)H(t)} = e^{xH(t)+yH(t)} = e^{xH(t)}e^{yH(t)} = P(x)P(y).$$

Hence, by Proposition 4.1, the series $P(x)$ is principal.

Now we use the derivation $\frac{d}{dx}$ of the ring $K[x][[t]]$, and we have

$$\sum_{n=1}^{\infty} P_n'(x)t^n = P'(x) = (e^{xH(t)})' = H(t)e^{xH(t)}.$$

Hence,

$$\sum_{n=1}^{\infty} P_n'(0)t^n = H(t)e^0 = H(t) = \sum_{n=1}^{\infty} a_n t^n$$

and hence, $a_n = P_n'(0)$ for all $n \geqslant 1$.

Now we shall prove that every principal power series is of the above form $e^{xH(t)}$, where $H(t) \in K[[t]]$ is a power series without the constant term. Before our proof, let us recall some well known properties of formal power series.

Assume that R is a commutative ring with identity containing the field $\mathbb{Q}$, of rational numbers, and let $R[[t]]$ be the ring of formal power series over R. Denote by $\mathcal{M}$ the ideal $tR[[t]]$, and let $1 + \mathcal{M} = \{1 + f;\ f \in \mathcal{M}\}$. Note that $\mathcal{M}$ is the set of all power series from $R[[t]]$ without the constant terms, and $1 + \mathcal{M}$ is the set of all power series from $R[[t]]$ with constant terms equal to 1. We have two classical functions $\mathrm{Log} : 1 + \mathcal{M} \to \mathcal{M}$ and $\mathrm{Exp} : \mathcal{M} \to 1 + \mathcal{M}$, defined by

$$\mathrm{Log}\,(1+\xi) = \xi - \frac{1}{2}\xi^2 + \frac{1}{3}\xi^3 - \frac{1}{4}\xi^4 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\xi^n,$$

$$\mathrm{Exp}\,(\xi) = 1 + \xi + \frac{1}{2!}\xi^2 + \frac{1}{3!}\xi^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\xi^n = e^{\xi}$$

for all $\xi \in \mathcal{M}$. It is well known that $\mathrm{Log}\,(\mathrm{Exp}\,(\xi)) = \xi$ and $\mathrm{Exp}\,(\mathrm{Log}\,(1+\xi)) = 1+\xi$, for all $\xi \in \mathcal{M}$. As a consequence of these facts we obtain

LEMMA 4.3
With the above notations:

(1) *if* $\xi, \nu \in \mathcal{M}$ *and* $e^{\xi} = e^{\nu}$, *then* $\xi = \nu$;

(2) *for every* $\xi \in \mathcal{M}$ *there exists a unique* $\nu \in \mathcal{M}$ *such that* $e^{\nu} = 1 + \xi$.

Now let R be the polynomial ring $K[x]$, where K is a field of characteristic zero.

LEMMA 4.4
Let $F(x)$ *be a polynomial from* $K[x]$ *such that*

$$(x + y)F(x + y) = xF(x) + yF(y).$$

Then $F(x) \in K$.

Proof. Suppose that $F(x) \notin K$. Let $\deg F(x) = n \geqslant 1$, and let $F(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_0, \ldots, a_n \in K$ with $a_n \neq 0$. Putting $y = x$, we have $2xF(2x) = 2xF(x)$ and so, $F(2x) = F(x)$. This implies that $2^n a_n = a_n$, so $2^n = 1$ and we have the contradiction: $0 = n \geqslant 1$.

Now we are ready to prove the following main result of this section.

THEOREM 4.5
Let $\mathcal{P} = (P_n(x))_{n \geqslant 0}$ *be a nonzero sequence of polynomials from* $K[x]$. *Then* $\mathcal{P}$ *is a principal sequence if and only if there exists a formal power series* $H(t)$, *belonging to* $K[[t]]$ *and without the constant term, such that*

$$\sum_{n=0}^{\infty} P_n(x)t^n = e^{xH(t)}.$$

Proof. Put $P(x) = \sum_{n=0}^{\infty} P_n(x)t^n$. We already know (see Proposition 4.2) that if $P(x) = e^{xH(t)}$ where $H(t) \in K[[t]]$ is without the constant term, then $\mathcal{P}$ is principal.

Now assume that $\mathcal{P}$ is principal. Since $\mathcal{P}$ is nonzero, we know by Proposition 3.1 that $P_0(x) = 1$. Thus, by Lemma 4.3(2), there exists a formal power series $B(x) \in K[x][[t]]$, without the constant term, such that $P(x) = e^{B(x)}$. Put $B(x) = \sum_{n=1}^{\infty} B_n(x)t^n$, where each $B_n(x)$ is a polynomial from $K[x]$. Observe that, by Proposition 3.2, we have $P(0) = 1$. Hence, $1 = P(0) = e^{B(0)}$ and hence, by Lemma 4.3(1), we have the equality $B(0) = 0$. Therefore, $B_n(0) = 0$ for all $n \geqslant 1$. This implies that for every $n \geqslant 1$ there exists a polynomial $A_n(x) \in K[x]$ such that $B_n(x) = xA_n(x)$. Put $A(x) = \sum_{n=1}^{\infty} A_n(x)t^n$. Then $B(x) = xA(x)$, and we have

$$P(x) = e^{xA(x)},$$

where $A(x)$ is a power series from $K[x][[t]]$ without the constant term. Since $\mathcal{P}$ is principal, we know, by Proposition 4.1, that $P(x + y) = P(x)P(y)$. Hence

$$e^{(x+y)A(x+y)} = P(x + y) = P(x)P(y) = e^{xA(x)}e^{yA(y)} = e^{xA(x)+yA(y)}$$

and hence, $(x+y)A(x+y) = xA(x) + yA(y)$ (see Lemma 4.3(1)), that is,

$$\sum_{n=1}^{\infty}((x+y)A_n(x+y))t^n = \sum_{n=1}^{\infty}(xA_n(x) + yA_n(y))t^n.$$

So $(x+y)A_n(x+y) = xA_n(x)+yA_n(y)$ for all $n \geqslant 1$ and so, by Lemma 4.4, every $A_n(x)$ is a constant a_n belonging to K. Consequently $A(x) = \sum_{n=1}^{\infty} a_n t^n$. Thus, $P(x) = e^{xH(t)}$, where $H(t) = \sum_{n=1}^{\infty} a_n t^n$. This completes the proof.

The next propositions are consequences of the above theorem.

PROPOSITION 4.6
If $A(x), B(x) \in K[x][[t]]$ are principal power series, then the product $A(x)B(x)$ is a principal power series.

Proof. It follows from Theorem 4.5 that $A(x) = e^{xH_1(t)}$ and $B(x) = e^{xH_2(t)}$, where $H_1(t), H_2(t)$ are some formal power series from $K[[t]]$ without the constant terms. Then $A(x)B(x) = e^{xH(t)}$, where $H(t) = H_1(t) + H_2(t)$ is a formal power series from $K[[t]]$ without the constant term. Hence, again by Theorem 4.5, the power series $A(x)B(x)$ is principal.

PROPOSITION 4.7
Let $P(x) = \sum_{n=0}^{\infty} P_n(x)t^n \in K[x][[t]]$ be a principal power series. Then $P(x)$ is invertible in $K[x][[t]]$, and the inverse $P(x)^{-1}$ is a principal power series. Moreover,

$$P(x)^{-1} = \sum_{n=0}^{\infty} P_n(-x)t^n.$$

Proof. It follows from Theorem 4.5 that $P(x) = e^{xH(t)}$, where $H(t)$ is a formal power series from $K[[t]]$ without the constant term. Then $P(x)P(-x) = e^{xH(t)}e^{-xH(t)} = e^0 = 1$, and hence $P(x)^{-1} = P(-x) = e^{x(-H(t))}$, and, again by Theorem 4.5, the series $P(x)^{-1}$ is principal.

Thus, the set of all principal power series from $K[x][[t]]$ is a subgroup of the multiplicative group of the ring $K[x][[t]]$.

5. Properties of binomial sequences

In the previous sections we proved several essential properties of principal sequences. Let us recall (see Proposition 2.1) that a sequence of polynomials $(P_n(x))_{n\geqslant 0}$ is principal if and only if $(n!P_n(x))_{n\geqslant 0}$ is a binomial sequence. The following propositions are immediate consequences of Proposition 2.1 and the suitable propositions from Section 3.

PROPOSITION 5.1
Let $\mathcal{F} = (F_n(x))_{n\geqslant 0}$ be a binomial sequence. Then

(1) $F_0(x) = 1$;

(2) $F_n(0) = 0$ *for all* $n \geqslant 1$.

(3) *Let s be a positive integer, and let $\mathcal{G} = (G_n(x))_{n\geqslant 0}$ be a sequence defined by*

$$G_{ms}(x) = \frac{(ms)!}{m!}F_m(x) \qquad \text{for } m \geqslant 0,$$

and $G_n(x) = 0$ when $s \not| n$. Then $\mathcal{G}$ is a binomial sequence.

(4) *Let $0 \neq a \in K$. Let $G_n(x) = a^n F_n(x)$ for $n \geqslant 0$. Then $(G_n(x))_{n\geqslant 0}$ is a binomial sequence.*

PROPOSITION 5.2
If $\mathcal{F} = (F_n(x))_{n\geqslant 0}$ is a strict binomial sequence, then

(1) $F_1(x) = ax$ *for some* $0 \neq a \in K$;

(2) $\deg F_n(x) = n$ *for all* $n \geqslant 0$;

(3) *the initial monomial of each $F_n(x)$ equals $a^n x^n$.*

Proof. Use Propositions 3.5 and 2.1.

COLORARY 5.3
A binomial sequence $(F_n)_{n\geqslant 0}$ is strict if and only if $F_1 \neq 0$.

PROPOSITION 5.4
Let $H(t) \in K[[t]]$ be a formal power series without the constant term, and let

$$e^{xH(t)} = \sum_{n=0}^{\infty} \frac{F_n(x)}{n!}t^n.$$

Then $(F_n(x))_{n\geqslant 0}$ is a binomial sequence. Moreover, if $H(t) = \sum_{n=1}^{\infty} a_n t^n$, then $n!a_n = F_n'(0)$ for all $n \geqslant 1$, where each $F_n'(x)$ is the derivative of $F_n(x)$.

The following theorem is the main result of this article. It is an extension of Krall's result [13] mentioned in Introduction.

THEOREM 5.5
Let $\mathcal{F} = (F_n(x))_{n\geqslant 0}$ be a nonzero sequence of polynomials from $K[x]$. Then $\mathcal{F}$ is a binomial sequence if and only if there exists a formal power series $H(t)$, belonging to $K[x][[t]]$ and without the constant term, such that

$$\sum_{n=0}^{\infty} \frac{F_n(x)}{n!}t^n = e^{xH(t)}.$$

Proof. Use Theorem 4.5 and 2.1.

Let us recall (see Section 2) that we denote by $K[x]\langle\langle t\rangle\rangle$ the ring of formal power series with divided powers over $K[x]$. If $\sum_{n=0}^{\infty} F_n(x)t^n$ is a formal power series belonging to $K[x]\langle\langle t\rangle\rangle$, then we shall say that this series is *binomial* if $(F_n(x))_{n\geqslant 0}$ is a binomial sequence.

The following propositions are immediate consequences of Proposition 2.1 and the suitable facts from the previous section.

PROPOSITION 5.6
If $F, G \in K[x]\langle\langle t\rangle\rangle$ are binomial power series, then the binomial convolution $F * G$ is a binomial power series.

PROPOSITION 5.7
Let $F = \sum_{n=0}^{\infty} F_n(x)t^n$ be a formal power series belonging to $K[x]\langle\langle t\rangle\rangle$. If F is binomial, then F is invertible in $K[x]\langle\langle t\rangle\rangle$, and the inverse F^{-1} is a binomial power series, and moreover

$$F^{-1} = \sum_{n=0}^{\infty} F_n(-x)t^n.$$

PROPOSITION 5.8
The set of all binomial series from $K[x]\langle\langle t\rangle\rangle$ is a subgroup of the multiplicative group of the ring $K[x]\langle\langle t\rangle\rangle$.

It follows from Theorem 5.5 that every binomial sequence $(F_n(x))_{n\geqslant 0}$ is uniquely determined by the formula $\sum_{n=0}^{\infty} \frac{F_n(x)}{n!}t^n = e^{xH(t)}$, where $H(t) \in K[[t]]$ is a formal power series without the constant term. Thus for every sequence $\mathcal{H} = (h_1, h_2, \dots)$ of elements of K we obtain the unique nonzero binomial sequence $\mathcal{F} = (F_n(x))_{n\geqslant 0}$ defined by the formula $\sum_{n=0}^{\infty} \frac{F_n(x)}{n!}t^n = e^{xH(t)}$, where $H(t) = \sum_{n=1}^{\infty} h_n t^n$. In this case we shall say that $\mathcal{F}$ is the binomial sequence *determined by* $H(t)$.

PROPOSITION 5.9
Let $H(t) = \sum_{n=1}^{\infty} h_n t^n \in K[[t]]$, and let $(F_n)_{n\geqslant 0}$ be the binomial sequence determined by $H(t)$. Let $0 \neq a \in K$. Then $(a^n F_n)_{n\geqslant 0}$ is the binomial sequence determined by $H(at) = \sum_{n=1}^{\infty} h_n a^n t^n$.

Proof. This proposition follows from Theorem 5.5 and Proposition 5.1(4).

6. Linear operators of type zero

In this section we consider strict binomial sequences. We recall some important results of I. M. Sheffer [24] and H. L. Krall [13], mentioned in Introduction. Throughout this section we denote by d the ordinary derivative $\frac{d}{dx}$.

Assume that F is a polynomial belonging to $K[x]$. We know that $d^n(F) = 0$ for all $n > \deg F$. Moreover, $d^n(x^n) = n!$ and

$$d^n(x^m) = m(m-1)\cdots(m-n+1)x^{m-n} \qquad \text{for } n \leqslant m.$$

PROPOSITION 6.1
Let $J\colon K[x] \to K[x]$ be a K-linear map. Then there exists a unique sequence $(L_n(x))_{n\geqslant 0}$ of polynomials from $K[x]$, such that

$$J(F) = \sum_{n=0}^{\infty} L_n(x) d^n(F)$$

for every $F \in K[x]$.

Proof. Put $F_n = J(x^n)$ for all $n \in \mathbb{N}_0$. We define the $L_n(x)$ recurrently by the relation

$$F_n = J(x^n) = \sum_{k=0}^{n} L_k(x) \cdot n(n-1)\cdots(n-k+1)x^{n-k},$$

for $n \geqslant 0$. That is,

$$\begin{aligned}
L_0 &= F_0,\\
L_1 &= F_1 - xL_0,\\
L_2 &= \frac{1}{2}(F_2 - x^2L_0 - 2xL_1),\\
L_3 &= \frac{1}{6}(F_3 - x^3L_0 - 3x^2L_1 - 6xL_2),\\
L_4 &= \frac{1}{24}(F_4 - x^4L_0 - 4x^3L_1 - 12x^2L_2 - 24xL_3),
\end{aligned}$$

and so on. Then, for every $m \in \mathbb{N}_0$, we have the equality

$$J(x^m) = \sum_{n=0}^{\infty} L_n(x) d^n(x^m).$$

But the mappings J and d are K-linear, hence $J(F) = \sum_{n=0}^{\infty} L_n(x)d^n(F)$, for all $F \in K[x]$. It is obvious that such sequence $(L_n(x))_{n\geqslant 0}$ is unique.

Thus, for every K-linear mapping $J\colon K[x] \to K[x]$ we have the unique sequence $(L_n(x))_{n\geqslant 0}$ *associated with* J. In this case the mapping J is said to be an *operator of type zero* ([24], [13]) if its associated sequence is of the following form: $L_n(x) = c_n \in K$ for all $n \geqslant 0$ with $c_0 = 0$ and $c_1 \neq 0$, that is, if

$$J(F) = c_1 d(F) + c_2 d^2(F) + c_3 d^3(F) + \cdots$$

for all $F \in K[x]$, where $c_n \in K$ for $n \geqslant 1$ and $c_1 \neq 0$. There are many interesting papers on operators of type zero, their generalizations and applications ([1], [25]).

Now we present some properties of operators of type zero.

PROPOSITION 6.2 ([24])
Let J be an operator of type zero. If $F \in K[x]$ is a nonzero polynomial of degree $n \geqslant 1$, then $J(F)$ is a nonzero polynomial of degree $n-1$.

Proof. Put $J = c_1 d + c_2 d^2 + \cdots$, with $c_1 \neq 0$, and let $F = a_n x^n + \cdots + a_1 x + a_0$, where $a_0, \ldots, a_n \in K$, $a_n \neq 0$. Then $d(F) = n a_n x^{n-1} + \cdots$ is a nonzero polynomial of degree $n-1$, and the degrees of all the polynomials $d^2(F), d^3(F), \ldots$ are smaller than $n-1$. Since $c_1 \neq 0$, the polynomial $J(F) = c_1 d(F) + c_2 d^(F) + \cdots$ is nonzero, and its degree is equal to $n-1$.

PROPOSITION 6.3
Let J be an operator of type zero, and let $G \in K[x]$ be a nonzero polynomial of degree $n-1 \geqslant 0$. Then there exists a unique polynomial $F \in K[x]$ of degree n such that $J(F) = G$ and $F(0) = 0$.

Proof. **(I)**. Put $J = c_1 d + c_2 d^2 + \cdots$, with $c_1 \neq 0$, and $G = g_0 + g_1 x + \cdots + g_{n-1} x^{n-1}$, where $g_0, \ldots, g_{n-1} \in K$, $g_{n-1} \neq 0$. We shall construct a polynomial

$$F = f_1 x + f_2 x^2 + \cdots + f_n x^n$$

with $f_1, \ldots, f_n \in K$ and $f_n \neq 0$, such that $J(F) = G$.

If $1 \leqslant j \leqslant m$, the we use the notation:

$$w(m, j) = m(m-1)\cdots(m-j+1).$$

Observe that, for all $j-1, \ldots, n$, we have $d^j(F) = \sum_{k=j}^{n} w(k, j) f_k x^{k-j}$. If $G = J(F)$, then we have the following equalities:

$$\begin{aligned}
G &= \sum_{j=1}^{n} c_j d^j(F) = \sum_{j=1}^{n} c_j \sum_{k=j}^{n} w(k, j) f_k x^{k-j} \\
&= c_1\big(w(1,1) f_1 x^0 + w(2,1) f_2 x^1 + w(3,1) f_3 x^2 + \cdots + w(n,1) f_n x^{n-1}\big) \\
&\quad + c_2\big(w(2,2) f_2 x^0 + w(3,2) f_3 x^1 + w(4,2) f_4 x^2 + \cdots + w(n,2) f_n x^{n-2}\big) \\
&\quad + c_3\big(w(3,3) f_3 x^0 + w(4,3) f_4 x^1 + w(5,3) f_5 x^2 + \cdots + w(n,3) f_n x^{n-3}\big) \\
&\qquad\qquad \vdots \\
&\quad + c_{n-1}\big(w(n-1,n-1) f_{n-1} x^0 + w(n,n-1) f_n x^1\big) \\
&\quad + c_n\big(w(n,n) f_n x^0\big).
\end{aligned}$$

Comparing the coefficients of x^{n-1}, we have $g_{n-1} = c_1 w(n,1) f_n = n c_1 f_n$. But $n c_1 \neq 0$, so $f_n = \frac{1}{n c_1} g_{n-1}$. Thus, if $J(F) = G$, then the coefficient f_n uniquely determined. Now compare the coefficients of x^{n-2}. We have $g_{n-2} = (n-1) c_1 f_{n-1} + c_2 w(n,2) f_n$. But f_n is already constructed and $(n-1)c_1 \neq 0$, so the coefficient f_{n-1} is also uniquely determined. Repeating this procedure we obtain the coefficients $f_n, f_{n-1}, \ldots, f_2$. In the final step, we compare the coefficients of x^0 and we obtain the equality

$$g_0 = c_1 f_1 + (2!) c_2 f_2 + \cdots + (n!) c_n f_n.$$

But the coefficients $f_2, f_3, \ldots, f_n$ are already uniquely determined and $c_1 \neq 0$, so the coefficient f_1 is also uniquely determined. This completes the proof.

One of the reviewers suggests to prove Proposition 6.3 by induction on n. He presents the following very short proof.

Proof. **II.** Let G be the polynomial of degree n. One can easily check that $J(x^{n+1})$ has also degree n. Hence there exists a constant c such that the polynomial

$$G' = G - cJ(x^{n+1})$$

has a smaller degree. By the inductive assumption there is a polynomial F' such that $J(F') = G'$. Then $G = J(F' + cx^{n+1})$.

As a consequence of Proposition 6.3 we obtain

PROPOSITION 6.4 ([24])
If J is an operator of type zero, then there exists a unique sequence $(B_n(x))_{n\geqslant 0}$, of nonzero polynomials from $K[x]$, such that

(1) $B_0(x) = 1$;

(2) $B_n(0) = 0$ *for* $n \geqslant 1$;

(3) $J(B_n(x)) = B_{n-1}(x)$ *for* $n \geqslant 0$, *where* $B_{-1}(x) = 0$.

Proof. Put $B_0(x) = 1$. Then of course $J(B_0(x)) = 0 = B_{-1}(x)$. Let $n \geqslant 0$ and assume that the polynomials $B_0(x), B_1(x), \ldots, B_n(x)$ are already defined. Then, by Proposition 6.3, there exists a unique nonzero polynomial $B_{n+1}(x) \in K[x]$ such that $B_{n+1}(0) = 0$ and $J(B_{n+1}(x)) = B_n(x)$. Thus, by induction, we obtain a uniquely determined sequence $(B_n(x))_{n\geqslant 0}$ satisfying the given conditions.

The polynomial sequence $(B_n(x))_{n\geqslant 0}$ from the above proposition is said to be the *basic sequence of* J (see [24], [13]). We will prove that this sequence is principal.

Let $J = c_1 d + c_2 d^2 + \ldots$ be a fixed operator of type zero. Let us recall that $c_1 \neq 0$ and $c_n \in K$ for $n \geqslant 1$. Denote by $M(t)$ the formal power series from $K[[t]]$, defined by

$$M(t) = c_1 t^1 + c_2 t^2 + c_3 t^3 + \cdots .$$

Since $M(t)$ is without the constant term and $c_1 \neq 0$, there exists a unique formal power series

$$H(t) = s_1 t^1 + s_2 t^2 s_3 t^3 + \cdots \in K[[t]]$$

such that $s_1 = c_1^{-1} \neq 0$ and $H(M(t)) = M(H(t)) = t$. Consider the formal power series $A(x) = e^{xH(t)}$. This series belongs to $K[x][[t]]$. Put

$$A(x) = e^{xH(t)} = A_0(x) + A_1(x)t + A_2(x)t^2 + \cdots ,$$

where $A_n \in K[x]$ for all $n \geqslant 0$. It is clear that $A_0(x) = 1$, $A_n(0) = 0$ for $n \geqslant 1$. Moreover, each $A_n(x)$ is nonzero and $\deg A_n(x) = n$.

LEMMA 6.5 ([24])
If J and A are as above, then

$$J(A_n(x)) = A_{n-1}(x) \qquad \text{for } n \geqslant 1.$$

Proof. Let us extend the derivative $d = \frac{d}{dx}\colon K[x] \to K[x]$ to the derivative $d\colon K[x][[t]] \to K[x][[t]]$ putting $d(t) = 0$. Then

$$d\Big(\sum_{n=0}^{\infty} f_n(x)t^n\Big) = \sum_{n=0}^{\infty} d(f_n(x))t^n$$

and, for every $k \geqslant 0$, we have

$$d^k\Big(\sum_{n=0}^{\infty} f_n(x)t^n\Big) = \sum_{n=0}^{\infty} d^k(f_n(x))t^n.$$

Let us extend also the operator $J\colon K[x] \to K[x]$ to the $K[[t]]$-linear mapping $J\colon K[x][[t]] \to K[x][[t]]$ defined by

$$J(\varphi) = \sum_{n=1}^{\infty} c_n d^n(\varphi),$$

for $\varphi \in K[x][[t]$. Since for every $F \in K[x]$ there exists an m such that $d^m(F) = 0$, the extended operator J is well defined. Observe that $J(\sum_{p=0}^{\infty} A_p(x)t^p) = \sum_{p=0}^{\infty} J(A_p(x))t^p$. In fact,

$$\begin{aligned} J\Big(\sum_{p=0}^{\infty} A_p(x)t^p\Big) &= \sum_{n=1}^{\infty} c_n d^n\Big(\sum_{p=0}^{\infty} A_p(x)t^p\Big) = \sum_{n=1}^{\infty} c_n\Big(\sum_{p=0}^{\infty} d^n(A_p(x))t^p\Big) \\ &= \sum_{n=1}^{\infty}\sum_{p=0}^{\infty} c_n d^n(A_p(x))t^p = \sum_{p=0}^{\infty}\Big(\sum_{n=1}^{\infty} c_n d^n(A_p(x))\Big)t^p \\ &= \sum_{p=0}^{\infty} J(A_p(x))t^p. \end{aligned}$$

Observe also that $d(e^{xH(t)}) = H(t)e^{xH(t)}$ and $d^k(e^{xH(t)}) = H(t)^k e^{xH(t)}$ for all $k \geqslant 0$. Hence,

$$\begin{aligned} J\Big(\sum_{p=0}^{\infty} A_p(x)t^p\Big) &= J\big(e^{xH(t)}\big) = \sum_{n=1}^{\infty} c_n d^n\big(e^{xH(t)}\big) = \sum_{n=1}^{\infty} c_n H(t)^n e^{xH(t)} \\ &= \Big(\sum_{n=1}^{\infty} c_n H(t)^n\Big)e^{xH(t)} = M(H)e^{xH(t)} = te^{xH(t)} \\ &= t\Big(\sum_{p=0}^{\infty} A_p(x)t^p\Big) = \sum_{p=1}^{\infty} A_{p-1}(x)t^p. \end{aligned}$$

Hence, we proved that

$$\sum_{p=1}^{\infty} J(A_p(x))t^p = \sum_{p=1}^{\infty} A_{p-1}(x)t^p$$

and this implies that $J(A_n(x)) = A_{n-1}(x)$ for all $n \geqslant 1$. This completes the proof.

THEOREM 6.6 ([24])
If $(B_n(x))_{n\geqslant 0}$ is the basic sequence of an operator $J = \sum_{n=1} c_n d^n$ of type zero, then

$$\sum_{n=0}^{\infty} B_n(x)t^n = e^{xH(t)},$$

where $H(t) \in K[[t]]$ is the formal power series (without the constant term) such that $M(H) = H(M) = t$, where $M(t) = \sum_{n=1}^{\infty} c_n t^n$.

Proof. Put $e^{xH(t)} = \sum_{n=0}^{\infty} A_n(x)t^n$. It is clear that $A_0(x) = 1$ and $A_n(0) = 0$ for $n \geqslant 1$. Moreover we know, by Lemma 6.5, that $J(A_n(x)) = A_{n-1}(x)$ for all $n \geqslant 0$. Hence, by Proposition 6.4, the sequence $(A_n(x))_{n\geqslant 0}$ is the basic sequence of J. Thus, $B_n(x) = A_n(x)$ for $n \geqslant 0$, and we have the equality $\sum_{n=0}^{\infty} B_n(x)t^n = e^{xH(t)}$.

THEOREM 6.7 ([24], [13])
The basic sequence of every operator of type zero is a strict principal sequence.

Proof. This is an immediate consequence of Theorem 6.6 and Proposition 4.2.

Now we shall prove that every strict principal sequence is the basic sequence of an operator of type zero. For this aim, first we prove two lemmas. Let us recall that K is a field of characteristic zero.

LEMMA 6.8
Let $F(x), G(x)$ be two polynomials from $K[x]$ such that

$$F(x+y) - F(x) - F(y) = G(x+y) - G(x) - G(y).$$

Then $F(x) = G(x) + px$ for some $p \in K$.

Proof. Let $F(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $G(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$, where $a_0, \ldots, a_n, b_0, \ldots, b_n \in K$. We do not assume that $a_n \neq 0$ and $b_n \neq 0$. Putting $y = x$, we have the equality $F(2x) - 2F(x) = G(2x) - 2G(x)$, that is,

$$\begin{aligned}(2^n - 2)a_n x^n + (2^{n-1} - 2)a_{n-1}x^{n-1} + \cdots + 4a_2 x^2 + a_0 \\ = (2^n - 2)b_n x^n + (2^{n-1} - 2)b_{n-1}x^{n-1} + \cdots + 4b_2 x^2 + b_0.\end{aligned}$$

Observe that we do not have the monomials $a_1 x$ and $b_1 x$. This equality implies that $a_j = b_j$ for $j = 2, 3, \ldots, n$ and $a_0 = b_0$. Thus, $F(x) = G(x) + px$ where $p = a_1 - b_1 \in K$.

LEMMA 6.9
Let $(P_n)_{n\geqslant 0}$ be a strict principal sequence. Then there exists a sequence $(c_n)_{n\geqslant 1}$, of elements of K, such that $c_1 \neq 0$, and for every $n \geqslant 1$,

$$V_n(P_j) = P_{j-1} \qquad \text{for } j = 1, 2, \ldots, n,$$

where $V_n = c_1 d + c_2 d^2 + \cdots + c_n d^n$.

Proof. ([13]). We define the sequence $(c_n)_{n\geqslant 1}$ recurrently by the following way. We know (see Proposition 3.5) that $P_1 = ax$ for some $0 \neq a \in K$, and the initial coefficient of each polynomial P_n, for $n \geqslant 1$, is equal to $\frac{1}{n!}a^n$. Let $c_1 = \frac{1}{a}$ and $V_1 = c_1 d$. Then

$$V_1(P_1) = \frac{1}{a}d(ax) = \frac{a}{a} = 1 = P_0.$$

Thus, c_1 is determined. Let $n \geqslant 2$ and assume that the elements $c_1, \ldots, c_{n-1}$ are already determined. Consider the operator $V_{n-1} = c_1 d + c_2 d^2 + \cdots + c_{n-1}d^{n-1}$. We already know that $V_{n-1}(P_j) = P_{j-1}$ for $j = 1, 2, \ldots, n-1$. Since V_{n-1} is an operator of type zero, there exists the basic sequence $(B_m)_{m\geqslant 0}$ of V_{n-1} (see Proposition 6.4). It follows from Proposition 6.3 that then $B_j = P_j$ for all $j = 0, 1, \ldots, n-1$. Moreover, we know from Theorem 6.7 that $(B_m)_{m\geqslant 0}$ is a principal sequence. Hence,

$$\begin{aligned} P_n(x+y) - P_n(x) - P_n(y) &= \sum_{k=1}^{n-1} P_k(x)P_{n-k}(y) = \sum_{k=1}^{n-1} B_k(x)B_{n-k}(y) \\ &= B_n(x+y) - B_n(x) - B_n(y) \end{aligned}$$

and hence, by Lemma 6.9, $P_n = B_n + px$ for some $p \in K$. Moreover, since $B_1 = P_1 = ax$, the initial coefficient of B_n is equal to $\frac{1}{n!}a^n$ (see Proposition 3.5). We define

$$c_n = -\frac{p}{a^{n+1}}.$$

Let $V_n = c_1 d + \cdots + c_n d^n = V_{n-1} + c_n d^n$. Then it is clear that $V_n(P_j) = P_{j-1}$ for all $j = 1, 2, \ldots, n-1$. We shall show that it is also true for $j = n$, that is, that $V_n(P_n) = P_{n-1}$. In fact,

$$\begin{aligned} V_n(P_n) &= V_{n-1}(P_n) + c_n d^n(P_n) = V_{n-1}(B_n + px) + c_n d^n(B_n + px) \\ &= V_{n-1}(B_n) + pV_{n-1}(x) + c_n d^n(B_n) = B_{n-1} + pc_1 - \frac{p}{a^{n+1}}a^n \\ &= B_{n-1} + \frac{p}{a} - \frac{p}{a} = B_{n-1} = P_{n-1}. \end{aligned}$$

This completes the proof.

THEOREM 6.10 ([13])
Every strict principal sequence is the basic sequence of an operator of type zero.

Proof. Let $\mathcal{P} = (P_n)_{n\geqslant 0}$ be a strict principal sequence. Let $(c_n)_{n\geqslant 1}$ be the sequence of elements from K, defined in Lemma 6.9. It follows from this lemma that $\mathcal{P}$ is the basic sequence of the operator $\sum_{n=1}^{\infty} c_n d^n$.

Now, by Proposition 2.1 and the above facts, we obtain

THEOREM 6.11 ([13])
A sequence $(F_n)_{n\geqslant 0}$ of polynomials from $K[x]$, is a strict binomial sequence if and only if $\left(\frac{F_n}{n!}\right)_{n\geqslant 0}$ is the basic sequence of an operator of type zero.

We will say that $(c_n)_{n\geqslant 1}$ is a *strict sequence*, if $c_n \in K$ for all $n \geqslant 1$ and $c_1 \neq 0$. Given an arbitrary strict sequence $C = (c_n)_{n\geqslant 1}$, we obtain a unique strict binomial sequence $(F_n)_{n\geqslant 0}$ such that $\left(\frac{F_n}{n!}\right)_{n\geqslant 0}$ is the basic sequence of the operator

$$J = c_1 d + c_2 d^2 + c_3 d^3 + \cdots .$$

We call it the *C-sequence*. Recall that d is the ordinary derivative $\frac{d}{dx}$. Every polynomial $F_n(x)$ is here nonzero, and its degree equals n. Moreover, every strict binomial sequence is a C-sequence for some strict sequence C.

7. Examples of binomial sequences

7.1. Successive powers of x

It is well known that $(x^n)_{n\geqslant 0}$ is a strict binomial sequence of polynomials. It is the first classical example of binomial sequences. It is not difficult to verify that it is the C-sequence for $C = (1, 0, 0, \dots)$, and it is the binomial sequence determined by $H(t) = t$. The binomial sequence $(a^n x^n)_{n\geqslant 0}$, where $0 \neq a \in K$, is determined by $H(t) = at$.

Example 7.1
Let $F_{2n}(x) = \frac{(2n)!}{n!}x^n$ and $F_{2n+1}(x) = 0$ for all $n \geqslant 0$. Then $(F_n(x))_{n\geqslant 0}$ is the binomial sequence determined by $H(t) = t^2$. This sequence is non-strict.

Let $0 \neq a \in K$ and let s be a positive integer. Let $\mathcal{F} = (F_n(x))_{n\geqslant 0}$, where

$$F_{ms}(x) = \frac{(ms)!}{m!}a^n x^n \qquad \text{for } m \geqslant 0,$$

and $F_n(x) = 0$ when $s \not| \, n$. Then $\mathcal{F}$ is the binomial sequence determined by $H(t) = (at)^s$. If $s \geqslant 2$, then this sequence is non-strict.

7.2. Lower and upper factorials

Let $a \in K$. Consider the polynomial sequence $(W_n(x))_{n\geqslant 0}$ defined by

$$W_n(x) = \begin{cases} 1, & \text{for } n = 0, \\ x(x+a)(x+2a)\cdots(x+(n-1)a), & \text{for } n \geqslant 1. \end{cases}$$

In particular, $W_1(x) = x$, $W_2(x) = x^2 + ax$, $W_3(x) = x^3 + 3ax^2 + 2a^2x$, and

$$W_{n+1}(x) = (x + na)W_n(x) \qquad \text{for all } n \geqslant 0.$$

Proposition 7.2
The sequence $(W_n(x))_{n\geqslant 0}$ is binomial.

Proof. We shall show, by induction, that for all $n \geqslant 0$,

$$W_n(x+y) = \sum_{i+j=n} \langle i, j\rangle W_i(x) W_j(y).$$

It is obvious for $n \leqslant 1$. Assume that it is true for some $n \geqslant 1$. Then

$$\begin{aligned}
W_{n+1}(x+y) &= (x+y+na)W_n(x+y) \\
&= (x+y+na)\sum_{k=0}^{n}\binom{n}{k}W_k(x)W_{n-k}(y) \\
&= \sum_{k=0}^{n}\binom{n}{k}(x+ka)W_k(x)W_{n-k}(y) \\
&\quad + \sum_{k=0}^{n}\binom{n}{k}(y+(n-k)a)W_k(x)W_{n-k}(y) \\
&= \sum_{k=0}^{n}\binom{n}{k}W_{k+1}(x)W_{n-k}(y) + \sum_{k=0}^{n}\binom{n}{k}W_k(x)W_{n+1-k}(y) \\
&= W_{n+1}(x) + W_{n+1}(y) + \sum_{k=0}^{n-1}\binom{n}{k}W_{k+1}(x)W_{n-k}(y) \\
&\quad + \sum_{k=1}^{n}\binom{n}{k}W_k(x)W_{n+1-k}(y) \\
&= W_{n+1}(x) + W_{n+1}(y) + \sum_{k=1}^{n}\binom{n}{k-1}W_k(x)W_{n+1-k}(y) \\
&\quad + \sum_{k=1}^{n}\binom{n}{k}W_k(x)W_{n+1-k}(y) \\
&= W_{n+1}(x) + W_{n+1}(y) + \sum_{k=1}^{n}\left(\binom{n}{k-1} + \binom{n}{k}\right)W_k(x)W_{n+1-k}(y) \\
&= W_{n+1}(x) + W_{n+1}(y) + \sum_{k=1}^{n}\binom{n+1}{k}W_k(x)W_{n+1-k}(y) \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}W_k(x)W_{n+1-k}(y).
\end{aligned}$$

This completes the proof.

Note that $(W_n(x))_{n\geqslant 0}$ is the binomial sequence determined by

$$H(t) = \sum_{n=1}^{\infty}\frac{a^{n-1}}{n}t^n.$$

Two special cases of such sequences $(W_n(x))_{n\geqslant 0}$ are well known. For $a=-1$ we have the sequence $(x_{(n)})_{n\geqslant 0}$ of *lower factorials*, defined by $x_{(0)}=1$ and

$$x_{(n)} = x(x-1)(x-2)\cdots(x-n+1) \qquad \text{for } n \geqslant 1.$$

In particular, $x_{(1)} = x$, $x_{(2)} = x^2 - x$, $x_{(3)} = x^3 - 3x^2 + 2x$, and $x_{(n+1)} = (x-n)x_{(n)}$. For $a = 1$ we have the sequence $(x^{(n)})_{n\geqslant 0}$ of *upper factorials*, defined by $x^{(0)} = 1$ and

$$x^{(n)} = x(x+1)(x+2)\cdots(x+n-1) \qquad \text{for } n \geqslant 1.$$

In particular, $x^{(1)} = x$, $x^{(2)} = x^2 + x$, $x^{(3)} = x^3 + 3x^2 + 2x$, and $x^{(n+1)} = (x+n)x^{(n)}$. It follows from Proposition 7.2 that $(x_{(n)})_{n\geqslant 0}$ and $(x^{(n)})_{n\geqslant 0}$ are strict binomial sequences. Moreover,

PROPOSITION 7.3
The sequence $(x_{(n)})_{n\geqslant 0}$ is the C-sequence for $C = (\frac{1}{n!})_{n\geqslant 1}$, and it is the binomial sequence determined by $H(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t^n$.

The sequence $(x^{(n)})_{n\geqslant 0}$ is the C-sequence for $C = \left(\frac{(-1)^{n+1}}{n!}\right)_{n\geqslant 1}$, and it is the binomial sequence determined by $H(t) = \sum_{n=1}^{\infty} \frac{1}{n} t^n$.

7.3. Abel polynomials

Now we examine the sequence $(A_n(x))_{n\geqslant 0}$ of *Abel polynomials*, defined by

$$A_n(x) = x(x - an)^{n-1},$$

where a is an element of K. The first few polynomials are

$$A_0(x) = 1, \quad A_1(x) = x, \quad A_2(x) = x(x-2a), \quad A_3(x) = x(x^2 - 6ax + 9a^2).$$

We will show that this sequence is binomial. We will prove this fact trough a series of lemmas below. Let

$$B_n(x) = \frac{1}{n!} A_n(x) \qquad \text{for } n \geqslant 0.$$

LEMMA 7.4
For every $n \geqslant 1$ and all $0 \leqslant k \leqslant n-1$,

$$B_n^{(k)}(x) = \frac{1}{(n-k)!}(x - ka)(x - na)^{n-1-k}.$$

Here $B_n^{(k)}(x)$ is the k-th derivative of $B_n(x)$.

Proof. By induction on k. It is obvious for $k = 0$. Assume that it is true for some $k \geqslant 0$. Then

$$\begin{aligned} B_n^{(k+1)}(x) &= \big(B_n^{(k)}(x)\big)' = \Big(\frac{1}{(n-k)!}(x - ka)(x - na)^{n-1-k}\Big)' \\ &= \frac{1}{(n-k)!}\big((x - na)^{n-1-k} + (n - 1 - k)(x - ka)(x - na)^{n-2-k}\big) \\ &= \frac{1}{(n-k)!}(x - na)^{n-2-k}\big((x - na) + (n - 1 - k)(x - ka)\big) \end{aligned}$$

$$= \frac{1}{(n-k)!}(x-na)^{n-2-k}(n-k)(x-a(1+k))$$
$$= \frac{1}{(n-(k+1))!}(x-na)^{n-1-(k+1)}(x-a(1+k)).$$

This completes the proof of this lemma.

Lemma 7.5

$$\sum_{p=1}^{n}\binom{n}{p}pa^{n-p}z^{p-1} = n(z+a)^{n-1}.$$

Proof. Use the derivative $\frac{d}{dz}$ for the equality $\sum_{p=1}^{n}\binom{n}{p}a^{n-p}z^p = (z+a)^n - 1$.

Lemma 7.6

$$\sum_{p=1}^{n}\binom{n}{p}(z+pa)a^{n-p}z^{p-1} = (z+a+an)(z+a)^{n-1} - a^n.$$

Proof. By Lemma 7.5, we have

$$\begin{aligned}\sum_{p=1}^{n}\binom{n}{p}a^{n-p}(z+pa)z^{p-1} &= \sum_{p=1}^{n}\binom{n}{p}a^{n-p}z^p + a\sum_{p=1}^{n}\binom{n}{p}pa^{n-p}z^{p-1}\\ &= (z+a)^n - a^n + an(z+a)^{n-1}\\ &= (z+a+an)(z+a)^{n-1} - a^n.\end{aligned}$$

This completes the proof.

Lemma 7.7

$$\sum_{k=0}^{n-1}\binom{n}{k}a^k(x-(k+1)a)(x-(n+1)a)^{n-1-k} = x(x-an)^{n-1} - a^n.$$

Proof. Using Lemma 7.6 for $z = x-(n+1)a$, we obtain that the left side of the above equality is equal to

$$\begin{aligned}\sum_{k=0}^{n-1}\binom{n}{k}a^k(z+(n-k)a)z^{n-k-1} &= \sum_{p=1}^{n}\binom{n}{p}a^{n-p}(z+pa)z^{p-1}\\ &= (z+a+an)(z+a)^{n-1} - a^n\\ &= x(x-an)^{n-1} - a^n.\end{aligned}$$

Proposition 7.8
$(A_n(x))_{n\geqslant 0}$ is a strict binomial sequence. It is the C-sequence for

$$C = \Big(1, a, \frac{1}{2!}a^2, \frac{1}{3!}a^3, \frac{1}{4!}a^4, \dots\Big).$$

Proof. Put $c_1 = 1$ and $c_n = \frac{1}{(n-1)!}a^{n-1}$ for all $n \geqslant 2$, and let $J = c_1 d + c_2 d^2 + \cdots$. We need to show that $J(B_{n+1}(x)) = B_n(x)$, that is, that

$$B_n(x) = c_1 B_{n+1}^{(1)}(x) + c_2 B_{n+1}^{(2)}(x) + \cdots + c_{n+1} B_{n+1}^{(n+1)}(x)$$

for all $n \geqslant 0$. For $n = 0$ and $n = 1$ it is obvious. Assume that $n \geqslant 2$. Then, by the previous lemmas, we have

$$\begin{aligned}
J(B_{n+1}(x)) &= \sum_{k=1} B_{n+1}^{(k)}(x) \\
&= \sum_{k=1}^{n} \frac{a^{k-1}}{(k-1)!(n-(k-1))!}(x-ka)(x-(n+1)a)^{n-k} + \frac{1}{n!}a^n \\
&= \sum_{k=0}^{n-1} \frac{a^k}{k!(n-k)!}(x-(k+1)a)(x-(n+1)a)^{n-(k+1)} + \frac{1}{n!}a^n \\
&= \frac{1}{n!}\sum_{k=0}^{n-1} \binom{n}{k} a^k (x-(k+1)a)(x-(n+1)a)^{n-(k+1)} + \frac{1}{n!}a^n \\
&= \frac{1}{n!}(x(x-an)^{n-1} - a^n + a^n) = \frac{1}{n!}x(x-an)^{n-1} = B_n(x).
\end{aligned}$$

This completes the proof.

Thus, we already know that $(A_n(x))_{n\geqslant 0}$ is a binomial sequence. It is not difficult to check that this sequence is determined by

$$H(t) = \sum_{n=1}^{\infty} \frac{(-na)^{n-1}}{n!} t^n.$$

The fact that $(A_n(x))_{n\geqslant 0}$ is a binomial sequence means that in the polynomial ring $K[x, y]$ we have the equalities $A_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} A_k(x) A_{n-k}(y)$ for all $n \geqslant 0$. Hence, for $a \in K$ and $n \geqslant 0$, the following identity holds

$$(x+y)(x+y-na)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} x(x-ka)^{k-1} y(y-(n-k)a)^{n-k-1}. \tag{1}$$

Now we present a second proof of the above identity (1). In 1826, Abel deduced an identity which is

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x-ka)^{k-1}(y+ka)^{n-k}, \tag{2}$$

for $a \in K$. Many authors offered different proofs of this identity ([9], [7], [22], [8]). In 2004, M. Lipnowski [14] and G. Zheng [27] presented elegant and short proofs in Solutions of Problem 310 of Mathematical Olympiads' Correspondence Program. There are many applications of the Abel identity ([7], [22], [11], [19], [26]).

Proposition 7.9
The identity (1) *follows from the Abel identity.*

Proof. Substitute in (2) the element $-a$ to the places of a, and next substitute $y+na$ to the places of y. Then we get

$$\sum_{k=0}^{n}\binom{n}{k}x(x+ka)^{k-1}(y+(n-k)a)^{n-k}=(x+y+na)^n. \tag{3}$$

Call $U_n(x,y,a)$ the left hand side of (3). Then $U_n(x,y,a)=(x+y+na)^n$, and looking at $U_{n-1}(x,y+a,a)$, we obtain the identity

$$\sum_{k=0}^{n-1}\binom{n-1}{k}x(x+ka)^{k-1}(y+(n-k)a)^{n-1-k}=(x+y+na)^{n-1}. \tag{4}$$

Put $P=\sum_{k=0}^{n}\binom{n}{k}x(x+ka)^{k-1}y(y+(n-k)a)^{n-k-1}$. Then we have

$$\begin{aligned}(x+y+na)^n&=\sum_{k=0}^{n}\binom{n}{k}x(x+ka)^{k-1}(y+(n-k)a)^{n-k}\\&=\sum_{k=0}^{n}\binom{n}{k}x(x+ka)^{k-1}(y+(n-k)a)^{n-1-k}(y+(n-k)a)\\&=P+Q,\end{aligned}$$

where $Q=\sum_{k=0}^{n}\binom{n}{k}x(x+ka)^{k-1}(n-k)a(y+(n-k)a)^{n-1-k}$. Using (4) and the identity $(n-k)\binom{n}{k}=n\binom{n-1}{k}$ we get $Q=na(x+y+na)^{n-1}$. Hence, $P=(x+y+na)^n-Q=(x+y+na)^n-na(x+y+na)^{n-1}=(x+y+na)^{n-1}(x+y)$, and hence,

$$(x+y)(x+y+na)^{n-1}=\sum_{k=0}^{n}\binom{n}{k}x(x+ka)^{k-1}y(y+(n-k)a)^{n-k-1}.$$

Now, putting $-a$ instead of a, we obtain (1). This completes the proof.

Note also the following proposition.

Proposition 7.10
The Abel identity follows from the identity (1).

Proof. Substitute in (1) the element $-a$ to the places of a, and next substitute $y+na$ to the places of y. Then we get

$$(x+y+na)(x+y)^{n-1}=\sum_{k=0}^{n}\binom{n}{k}x(x-ka)^{k-1}(y+na)(y+ka)^{n-k-1}. \tag{5}$$

We prove the Abel identity (2) by induction. When $n=0$, then it is obvious. Assume that for $n\geqslant 1$,

$$(x+y)^{n-1}=\sum_{k=0}^{n-1}\binom{n-1}{k}x(x-ka)^{k-1}(y+ka)^{n-1-k}. \tag{6}$$

Then, by (5), (6) and the identity $n\binom{n-1}{k} = (n-k)\binom{n}{k}$, we have

$$\begin{aligned}
(x+y)^n &= (x+y+na)(x+y)^{n-1} - na(x+y)^{n-1} \\
&= \sum_{k=0}^{n} \binom{n}{k} x(x-ka)^{k-1}((y+ka)+(n-k)a)(y+ka)^{n-k-1} \\
&\quad - na\sum_{k=0}^{n-1} \binom{n-1}{k} x(x-ka)^{k-1}(y+ka)^{n-1-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x(x-ka)^{k-1}(y+ka)^{n-k} \\
&\quad + na\sum_{k=0}^{n-1} \binom{n-1}{k} x(x-ka)^{k-1}(y+ka)^{n-1-k} \\
&\quad - na\sum_{k=0}^{n-1} \binom{n-1}{k} x(x-ka)^{k-1}(y+ka)^{n-1-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x(x-ka)^{k-1}(y+ka)^{n-k}.
\end{aligned}$$

This completes the proof.

7.4. Laguerre polynomials

Let $(L_n(x))_{n\geqslant 0}$ be the sequence of polynomials from $K[x]$ defined by $L_0(x) = 1$ and

$$L_n(x) = \sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}x^k \qquad \text{for } n \geqslant 1.$$

They are called the *Laguerre*[1] *polynomials* ([4], [6], [20], [12]). The first few polynomials are

$$\begin{aligned}
L_1(x) &= x, \\
L_2(x) &= (x+2)x, \\
L_3(x) &= (x^2+6x+6)x, \\
L_4(x) &= (x^3+12x^2+36x+24)x, \\
L_5(x) &= (x^4+20x^3+120x^2+240x+120)x, \\
L_6(x) &= (x^5+30x^4+300x^3+1200x^2+1800x+720)x, \\
L_7(x) &= (x^6+42x^5+630x^4+4200x^3+12600x^2+15120x+5040)x.
\end{aligned}$$

PROPOSITION 7.11
$(L_n(x))_{n\geqslant 0}$ is the a strict binomial sequence. This sequence is determined by

$$H(t) = \sum_{n=1}^{\infty} t^n.$$

[1]Edmond Nicolas Laguerre (1834-1886), a French mathematician.

7.5. Other examples

Example 7.12 ([13])
Consider the strict sequence $C = (1, 0, -\frac{1}{3!}, 0, \frac{1}{5!}, 0, -\frac{1}{7!}, 0, \dots)$. The initial terms of the C-sequence $(F_n(x))$ are

$$F_0(x) = 1, \qquad F_3(x) = x^3 + x,$$
$$F_1(x) = x, \qquad F_4(x) = x^4 + 4x^2,$$
$$F_2(x) = x^2, \qquad F_5(x) = x^5 + 10x^3 + 9x.$$

Example 7.13
Initial terms of the C-sequence for $C = (1, 1, 0, 0, 0, \dots)$ are

$$F_0(x) = 1,$$
$$F_1(x) = x,$$
$$F_2(x) = x(x-2),$$
$$F_3(x) = x(x^2 - 6x + 12),$$
$$F_4(x) = x(x^3 - 12x^2 + 60x - 120),$$
$$F_5(x) = x(x^4 - 20x^3 + 180x^2 - 840x + 1680),$$
$$F_6(x) = x(x^5 - 30x^4 + 420x^3 - 3360x^2 + 15120x - 30240),$$
$$F_7(x) = x(x^6 - 42x^5 + 840x^4 - 10080x^3 + 75600x^2 - 332640x + 665280).$$

Example 7.14
Initial terms of the C-sequence for $C = (1, 0, 1, 0, 0, 0, \dots)$ are

$$F_0(x) = 1,$$
$$F_1(x) = x,$$
$$F_2(x) = x^2,$$
$$F_3(x) = (x^2 - 6)x,$$
$$F_4(x) = (x^2 - 24)x^2,$$
$$F_5(x) = (x^4 - 60x^2 + 360)x,$$
$$F_6(x) = (x^2 - 120x^2 + 2520)x^2,$$
$$F_7(x) = (x^6 - 210x^4 + 10080x^2 - 60480)x.$$

Example 7.15
Initial terms of the C-sequence for $C = (1, 0, 0, 1, 0, 0, 0, \dots)$ are

$$F_0(x) = 1, \qquad F_4(x) = (x^3 - 24)x,$$
$$F_1(x) = x, \qquad F_5(x) = (x^3 - 120)x^2,$$
$$F_2(x) = x^2, \qquad F_6(x) = (x^3 - 360)x^3,$$
$$F_3(x) = x^3, \qquad F_7(x) = (x^6 - 840x^3 + 20160)x.$$

In [16], we find a description of C-sequences for $J = ad - bd^{p+1}$ where $a, b \in \mathbb{R}$, $a \neq 0$ and $p \geqslant 1$.

EXAMPLE 7.16
Initial terms of the C-sequence for $C = (1, 1, 1, 0, 0, 0, 0, \dots)$ are

$$\begin{aligned}
F_0(x) &= 1,\\
F_1(x) &= x,\\
F_2(x) &= (x-2)x,\\
F_3(x) &= (x^2-6x+6)x,\\
F_4(x) &= (x^2-6)^2x,\\
F_5(x) &= (x^4-20x^3+120x^2-120x-480)x,\\
F_6(x) &= (x^5-30x^4+300x^3-840x^2-2520x+10080)x,\\
F_7(x) &= (x^6-42x^5+630x^4-3360x^3-5040x^2+90720x-151200)x.
\end{aligned}$$

EXAMPLE 7.17
Let $(F_n(x))_{n\geqslant 0}$ be the binomial sequence determined by $H(t) = t - \frac{1}{120}t^5$. Then $F_n(x) = x^n$ for $0 \leqslant n \leqslant 4$ and

$$\begin{aligned}
F_5(x) &= (x^4-1)x, & F_8(x) &= (x^4-56)x^4,\\
F_6(x) &= (x^4-6)x^2, & F_9(x) &= (x^4-126)x^5,\\
F_7(x) &= (x^4-21)x^3, & F_{10}(x) &= (x^8-252x^4+126)x^2.
\end{aligned}$$

The next example is a generalization of the previous example.

EXAMPLE 7.18
Let $(F_n(x))_{n\geqslant 0}$ be the binomial sequence determined by $H(t) = t - \frac{1}{(s+1)!}t^{s+1}$ with $s \geqslant 1$. Then $F_n(x) = x^n$ for $0 \leqslant n \leqslant s$ and

$$F_{s+k}(x) = \Big(x^s - \binom{s+k}{s+1}\Big)x^k,$$

for $k = 1, 2, \dots, s+1$.

In the next examples we present initial terms of two non-strict binomial sequences.

EXAMPLE 7.19
Let $(F_n(x))_{n\geqslant 0}$ be the binomial sequence determined by $H(t) = \frac{1}{2}t^2 + \frac{1}{6}t^3$. Then

$$\begin{aligned}
F_0(x) &= 1, & F_7(x) &= 105x^3,\\
F_1(x) &= 0, & F_8(x) &= 35(3x+8)x^3,\\
F_2(x) &= x, & F_9(x) &= 140(9x+2)x^3,\\
F_3(x) &= x, & F_{10}(x) &= 315(3x+20)x^4,
\end{aligned}$$

$$F_4(x) = 3x^2, \qquad F_{11}(x) = 1925(9x+8)x^4,$$
$$F_5(x) = 10x^2, \qquad F_{12}(x) = 385(27x^2+360x+40)x^4,$$
$$F_6(x) = 5(3x+2)x^2, \qquad F_{13}(x) = 30030(9x+20)x^5.$$

EXAMPLE 7.20
Let $(F_n(x))_{n\geqslant 0}$ be the binomial sequence determined by $H(t) = \frac{1}{6}t^3 + \frac{1}{24}t^4$. Then

$$F_0(x) = 1, \qquad F_{13}(x) = a_{13}x^4,$$
$$F_1(x) = 0, \qquad F_{14}(x) = a_{14}x^4,$$
$$F_2(x) = 0, \qquad F_{15}(x) = a_{15}(8x+15)x^4,$$
$$F_3(x) = x, \qquad F_{16}(x) = a_{16}(32x+3)x^4,$$
$$F_4(x) = 3x, \qquad F_{17}(x) = a_{17}x^5,$$
$$F_5(x) = 0, \qquad F_{18}(x) = a_{18}(8x+45)x^5,$$
$$F_6(x) = 10x^2, \qquad F_{19}(x) = a_{19}(32x+15)x^5,$$
$$F_7(x) = 35x^2, \qquad F_{20}(x) = a_{20}(80x+3)x^5,$$
$$F_8(x) = 35x^2, \qquad F_{21}(x) = a_{21}(8x+105)x^6,$$
$$F_9(x) = 280x^3, \qquad F_{22}(x) = a_{22}(32x+45)x^6,$$
$$F_{10}(x) = 2100x^3, \qquad F_{23}(x) = a_{23}(16x+3)x^6,$$
$$F_{11}(x) = 5775x^3, \qquad F_{24}(x) = a_{24}(128x^2+3360x+63)x^6,$$
$$F_{12}(x) = 1925(8x+3)x^3, \qquad F_{25}(x) = a_{25}(32x+105)x^7,$$

where $a_{13}, a_{14}, \ldots, a_{25}$ are some positive integers.

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