



Sums of bases-exponents positive integer powers

A. Zaroni and M. Zaroni

Abstract

We consider natural numbers that can be represented as sums of positive integer powers greater than 1, such that the set of bases coincides with the set of exponents. We address this issue by indicating some equalities and relations, defining some new integer sequences recently introduced in the OEIS, reporting some experimental results and showing some graphs regarding the obtained numbers.

1 Introduction

Let $\mathbb{N}_2 = \mathbb{N} \setminus \{0, 1\}$, $0 < t \in \mathbb{N}$ and $B = \{b_1, \dots, b_t\}, E = \{e_1, \dots, e_t\} \subset \mathbb{N}_2$. Consider the power sums

$$n = \sum_{i=1}^t b_i^{e_i} \in \mathbb{N}_2$$

The case of equal exponents $e_1 = \dots = e_t$ is nowadays quite classical (four squares theorem, Waring problem, etc.), but many more interesting issues arise when exponents are different. The case $e_{i+1} = e_i + 1$ is very common [1], [4], [5], [6], [7], [8] and [10], but also sparse exponents have recently been analyzed [2], [3].

In this work the case $B = E$ is considered for the first time, focusing on the possible values of n : they will be noted as BE -numbers (Bases-Exponents

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numbers) in the rest of the paper. The base/exponent 1 is kept excluded, as otherwise for e.g. $t = 2$ everything would be pretty trivial. In fact, if $B = E = \{1, b\}$ with $b > 1$ then $n = 1^b + b^1 = 1 + b$ and every $n > 2$ could be then represented (something analogous happens for $t > 2$). Really interesting questions for this new number theory problem arise therefore only when $1 \notin B$.

We consider in section 2 some sequences of integers that can be represented with t addends under this constraint and their union. For $t = 1, 2, 3$ we will indicate for simplicity $B = \{a\}$, $B = \{a, b\}$ and $B = \{a, b, c\}$, respectively: these are the easiest cases. Section 3 is dedicated to other sequences and some issues concerning BE -numbers, while in section 4 some graphs are reported.

In the following, abusing notation, B and E can be understood both as sets and as t -uples, with B sorted in ascending order and $E = \pi(B)$, where $\pi \in S(t)$ is the permutation connecting the two t -uples. As it will be shown, the (B, E) -representation of a number n is in general not unique.

2 Sum of powers with equal sets of bases and exponents

We first consider the very basic cases $t = 1, 2, 3$.

$t = 1$ This is trivial: the only possibility is $n = a^a \in S_1$, the A000312 sequence (apart the first term, because of the above position) in OEIS [9].

$$S_1 = \{4, 27, 256, 3125, 46656, 823543, 16777216, 387420489, 10000000000, \dots\}$$

$t = 2$ In this case $n = a^b + b^a$ or $n = a^a + b^b$. The set of such integers is the recently added A385232 sequence.

$$S_2 = \{17, 31, 32, 57, 100, 145, 177, 260, 283, 320, 368, 593, 945, 1124, 1649, \dots\}$$

$t = 3$ This case contains the six* possibilities $n = a^a + b^b + c^c$, $a^a + b^c + c^b$, $a^b + b^a + c^c$, $a^b + b^c + c^a$, $a^c + b^a + c^b$ and $a^c + b^b + c^a$. The resulting set is

$$S_3 = \{59, 84, 89, 105, 127, 149, 166, 204, 273, 276, 287, 289, 313, 347, 356, \dots\}$$

(sequence A385233) and so on for $t > 2$.

The union of all such sequences $S = \bigcup_{t>0} S_t$ gives the sequence A385969.

$$S = \{4, 17, 27, 31, 32, 57, 59, 84, 89, 100, 105, 127, 145, 149, 166, 177, 202, 204, 245, 254, 256, 260, 273, 276, 283, 287, 289, 313, 320, 322, 340, 347, 348, 356, 368, 372, 377, 383, 400, 422, 433, 460, 465, 468, 480, 532, 545, 548, 568, 576, 593, 603, 620, 624, 628, 673, 688, 700, 724, 730, 736, 738, 739, 773, 777, 801, 804, 845, 849, 865, 876, 892, 922, 932, 936, 944, 945, \dots\}$$

* In the general case, there are $t!$ possibilities, corresponding to the permutations of B .

The minimum number of powers whose sums are the BE -numbers in S gives the sequence $\tilde{S}$ (A385970).

$$\begin{aligned}\tilde{S} = \{ & 1, 2, 1, 2, 2, 2, 3, 3, 3, 2, 3, 3, 2, 4, 3, 4, 4, 1, 2, 3, 3, 2, 3, 3, 2, 4, 4, 3, \\ & 4, 3, 2, 3, 4, 4, 4, 3, 4, 4, 4, 3, 4, 4, 4, 3, 2, 4, 3, 3, 4, 3, 4, 4, 5, 4, 4, 4, 3, \\ & 3, 5, 5, 4, 3, 4, 4, 4, 3, 4, 5, 2, 3, 4, 4, 5, 4, 4, 5, 4, 4, 3, 4, 5, 4, 2, 4, 3, \dots \}\end{aligned}$$

The minimum value of S_t can be easily obtained by considering that the function

$$h : \mathbb{N}_2^2 \ni (b, e) \mapsto h(b, e) = b^e \in \mathbb{N}_2 \quad (1)$$

is strictly monotonically increasing with respect to both variables and that the smallest possible values for the bases and exponents for S_t are, obviously, $\{2, 3, \dots, t+1\}$. Larger bases must be associated with smaller exponents in order to obtain the minimum, so the sum must include the term $(t+1)^2$ and, by finite induction on the remaining bases and exponents, the following formula is obtained.

$$\min\{S_t\} = \sum_{a=2}^{t+1} a^{t+1-(a-2)} = \sum_{a=2}^{t+1} a^{t+3-a} \quad (2)$$

Let S_0 be the sequence of these minima: its first terms are reported below.

$$S_0 = \{4, 17, 59, 202, 724, 2771, 11367, 49852, 232756, 1151901, 6018771, \dots\}$$

It is possible to obtain new BE -numbers with longer representations (more addends) by combining shorter ones: the following proposition is the core of the idea.

Proposition 1. *Let $B_j \subset \mathbb{N}_2$ be disjoint finite sets ($j \in \mathcal{J} = [1, J]$), $B = \bigcup_{j \in \mathcal{J}} B_j$ and $n_j = \sum_{b \in B_j} b^{e_b}$ be BE -numbers. Then $n = \sum_{b \in B} b^{e_b}$ is a BE -number.*

Proof. Note that the exponents e_b in the representation of n are uniquely determined because of the disjointness hypothesis. As n_j is a BE -number, $E_j = \{e_b \mid b \in B_j\} = B_j \Rightarrow E = \bigcup_{j \in \mathcal{J}} E_j = \bigcup_{j \in \mathcal{J}} B_j = B$: the thesis is proved. $\square$

As a consequence of the proposition, let $n = n(B, E)$ be a BE -number: by choosing $B' = E'$ with $B \cap B' = E \cap E' = \emptyset$, the integer $n' = n'(B \cup B', E \cup E')$ is a new BE -number (the easiest case is of course $B' = \{b\}$, so that $n' = n + b^b$). This leads to the distinction of two types of BE -numbers:

Definition 1. *Let $n = n(B, E)$ be a BE -number. If $B = E$ can be partitioned into two sets $B_1 = E_1$, $B_2 = E_2$ such that $n_1 = n_1(B_1, E_1)$ and*

$n_2 = n_2(B_2, E_2)$ are *BE-numbers*, the (B, E) representation is said to be reducible. A not reducible representation (i.e. the permutation π is cyclic) is called irreducible.

Definition 2. A *BE-number* is said to be irreducible if all of its representations are irreducible, otherwise it is reducible.

The easiest cases are the ones with $t = 1, 2$: all *BE-numbers* whose only representation is a^a or $a^b + b^a$ are obviously irreducible. The set S can therefore be considered as partitioned into irreducibles and reducibles, the former set being, in a certain sense, a kind of additive base for the latter. Note that it is not minimal, as irreducibles themselves can in general be expressed as sums of irreducibles, e.g. $32 + 57 = 89$ or $57 + 320 = 377$ (see next section).

3 Issues concerning *BE-numbers*

There are many issues that can be considered regarding *BE-numbers*: some of them are listed below.

[1] Representations of *BE-numbers* are generally not unique. The subsequence of numbers $n \in S$ with more than one representation is

$$\begin{aligned} S_D = \{ & 545, 1407, 1492, 2409, 3370, 3605, 3718, 4516, 4684, 5441, 7346, 7737, \\ & 7865, 7922, 8122, 8538, 10037, 10458, 10651, 10953, 10298, 10458, \\ & 10554, 10651, 10953, 11047, 11986, 12025, 13621, 14078, 14098, \\ & 14535, 14970, 16138, 16449, 16705, 16905, 17401, 18149, 18161, \\ & 18509, 18538, 18638, 19253, 19283, 19574, 19787, 19830, 19893, \dots \} \end{aligned}$$

Definition 3. A *BE-number* $n \in S_D$ is called duplicate. We call k -duplicate a duplicate with exactly k representations.

As “curiosity” examples of duplicates, we note (in particular the first two ones) that

$$\begin{aligned} 18446744073743106673 &= 2^{25} + 16^{16} + 25^2 = 4^{32} + 5^4 + 32^5 \\ 23283064365386996445313 &= 2^{25} + 16^2 + 25^{16} = 4^4 + 5^{32} + 32^5 \\ 324518553658426726785977517904201 &= 6^{16} + 16^{27} + 27^6 = 8^{36} + 9^9 + 36^8 \end{aligned}$$

The OEIS sequences of $[2, 3, 4, 5, 6]$ -duplicates are, respectively, A386966, A386967, A386968, A387099, A388009. Values and representations of many *BE-numbers*, as well as duplicates and irreducibles, can be found in the website <https://be-numbers.jimdosite.com>. The representations of the first duplicates (smaller than 10^4), that are in particular all 2-duplicates, follow:

$$\begin{aligned}
545 &= 2^6 + 3^2 + 4^4 + 6^3 &= 2^7 + 3^5 + 5^3 + 7^2 \\
1407 &= 2^{10} + 3^3 + 4^4 + 10^2 &= 2^7 + 3^4 + 4^5 + 5^3 + 7^2 \\
1492 &= 2^{10} + 3^5 + 5^3 + 10^2 &= 2^7 + 3^6 + 4^4 + 6^2 + 7^3 \\
2409 &= 2^6 + 4^5 + 5^2 + 6^4 &= 2^8 + 3^6 + 4^3 + 6^4 + 8^2 \\
3370 &= 2^6 + 3^4 + 4^3 + 5^5 + 6^2 &= 2^{11} + 3^6 + 4^4 + 6^3 + 11^2 \\
3605 &= 2^7 + 3^2 + 5^5 + 7^3 &= 2^7 + 3^3 + 4^5 + 5^2 + 7^4 \\
3718 &= 2^9 + 5^5 + 9^2 &= 2^{10} + 3^6 + 4^5 + 5^4 + 6^3 + 10^2 \\
4516 &= 2^4 + 3^5 + 4^6 + 5^3 + 6^2 &= 2^{10} + 3^7 + 4^4 + 7^2 + 10^3 \\
4523 &= 2^{12} + 3^3 + 4^4 + 12^2 &= 2^7 + 3^6 + 4^5 + 5^2 + 6^3 + 7^4 \\
4684 &= 2^8 + 3^5 + 4^3 + 5^2 + 8^4 &= 2^7 + 3^4 + 4^6 + 6^2 + 7^3 \\
5441 &= 2^4 + 3^7 + 4^3 + 5^5 + 7^2 &= 2^{12} + 3^6 + 4^4 + 6^3 + 12^2 \\
6348 &= 2^{12} + 3^5 + 4^4 + 5^2 + 12^3 &= 2^{11} + 3^7 + 4^5 + 5^4 + 7^3 + 11^2 \\
7346 &= 2^3 + 3^4 + 4^6 + 5^5 + 2^2 &= 2^5 + 3^8 + 4^3 + 5^4 + 8^2 \\
7737 &= 2^3 + 3^7 + 4^2 + 5^5 + 7^4 &= 2^8 + 3^7 + 4^5 + 5^3 + 7^2 + 8^4 \\
7865 &= 2^6 + 5^2 + 6^5 &= 2^{12} + 3^6 + 4^2 + 6^4 + 12^3 \\
7922 &= 3^7 + 4^6 + 6^4 + 7^3 &= 2^9 + 3^8 + 4^4 + 8^3 + 9^2 \\
8122 &= 2^9 + 4^5 + 5^2 + 9^4 &= 2^9 + 3^8 + 4^4 + 8^2 + 9^3 \\
8538 &= 2^3 + 3^6 + 5^2 + 6^5 &= 2^6 + 3^2 + 4^3 + 5^4 + 6^5 \\
9046 &= 2^{12} + 3^4 + 4^2 + 5^5 + 12^3 &= 2^{10} + 3^7 + 4^6 + 6^4 + 7^3 + 10^2
\end{aligned}$$

The first 3-duplicates (smaller than 10^5) are reported below.

$$\begin{aligned}
23506 &= 2^9 + 4^7 + 7^2 + 9^4 = 2^3 + 3^4 + 4^2 + 5^6 + 6^5 = 2^4 + 3^8 + 4^5 + 5^6 + 6^3 + 8^2 \\
23778 &= 2^{11} + 3^8 + 4^2 + 8^3 + 11^4 = 2^{12} + 3^2 + 4^5 + 5^6 + 6^4 + 12^3 \\
&= 2^{10} + 3^8 + 4^6 + 5^3 + 6^5 + 8^4 + 10^2 \\
41682 &= 2^3 + 3^9 + 4^7 + 5^5 + 7^4 + 9^2 = 2^9 + 3^8 + 4^7 + 5^4 + 7^5 + 8^2 + 9^3 \\
&= 2^{14} + 3^8 + 4^6 + 5^2 + 6^5 + 8^4 + 14^3 \\
50261 &= 2^7 + 3^2 + 5^5 + 6^6 + 7^3 = 2^7 + 3^3 + 4^5 + 5^2 + 6^6 + 7^4 \\
&= 2^{14} + 3^9 + 4^6 + 5^5 + 6^3 + 9^4 + 14^2 \\
53554 &= 2^8 + 3^7 + 4^2 + 6^6 + 7^3 + 8^4 = 2^{13} + 3^5 + 4^7 + 5^3 + 7^2 + 13^4 \\
&= 2^{13} + 3^9 + 5^6 + 6^5 + 9^2 + 13^3 \\
76754 &= 2^{11} + 3^{10} + 4^2 + 10^3 + 11^4 = 2^{10} + 3^6 + 4^8 + 5^4 + 6^5 + 8^2 + 10^3 \\
&= 2^{12} + 3^8 + 4^7 + 6^6 + 7^4 + 8^3 + 12^2 \\
78289 &= 2^{16} + 4^6 + 5^4 + 6^5 + 16^2 = 2^{12} + 3^9 + 4^5 + 5^3 + 6^6 + 9^4 + 12^2 \\
&= 2^{10} + 3^8 + 4^7 + 5^5 + 6^6 + 7^3 + 8^4 + 10^2 \\
92030 &= 2^9 + 3^4 + 4^7 + 5^6 + 6^2 + 7^3 + 9^5 = 2^{11} + 3^7 + 4^8 + 5^2 + 7^5 + 8^4 + 11^3 \\
&= 2^{13} + 3^9 + 4^6 + 5^4 + 6^3 + 9^5 + 13^2 \\
96981 &= 2^3 + 3^6 + 4^2 + 5^7 + 6^4 + 7^5 = 2^4 + 3^8 + 4^6 + 5^7 + 6^5 + 7^3 + 8^2 \\
&= 2^6 + 3^8 + 4^2 + 5^7 + 6^5 + 7^3 + 8^4 \\
99913 &= 2^{14} + 3^5 + 4^2 + 5^7 + 7^4 + 14^3 = 2^7 + 3^2 + 4^8 + 5^6 + 6^4 + 7^5 + 8^3 \\
&= 2^8 + 3^4 + 4^6 + 5^7 + 6^2 + 7^5 + 8^3
\end{aligned}$$

For 4-duplicates one has to look at larger numbers: the first ones are

$$\begin{aligned} 331979 &= 2^2 + 3^8 + 4^9 + 5^3 + 8^4 + 9^5 = 2^{15} + 4^5 + 5^6 + 6^7 + 7^4 + 15^2 \\ &= 2^5 + 3^{10} + 4^9 + 5^2 + 9^3 + 10^4 = 2^6 + 3^{10} + 4^9 + 5^5 + 6^2 + 9^4 + 10^3 \end{aligned}$$

$$\begin{aligned} 536134 &= 2^6 + 3^{12} + 4^3 + 5^5 + 6^4 + 12^2 = 2^{16} + 4^8 + 5^5 + 6^7 + 7^6 + 8^4 + 16^2 \\ &= 2^{18} + 3^{11} + 4^4 + 5^7 + 7^5 + 11^3 + 18^2 \\ &= 2^{16} + 3^{10} + 4^9 + 5^7 + 6^6 + 7^5 + 9^4 + 10^3 + 16^2 \end{aligned}$$

$$\begin{aligned} 602342 &= 2^{16} + 3^{12} + 4^4 + 5^5 + 12^3 + 16^2 = 2^8 + 3^{12} + 4^3 + 5^5 + 6^6 + 8^2 + 12^4 \\ &= 2^{13} + 3^8 + 4^2 + 5^3 + 6^7 + 7^5 + 8^6 + 13^4 \\ &= 2^{15} + 3^{10} + 4^8 + 5^7 + 6^4 + 7^2 + 8^6 + 10^5 + 15^3 \end{aligned}$$

$$\begin{aligned} 848707 &= 2^3 + 3^9 + 4^6 + 6^4 + 7^7 + 9^2 = 2^{12} + 3^5 + 4^3 + 5^2 + 7^7 + 12^4 \\ &= 2^7 + 3^{12} + 4^9 + 6^6 + 7^2 + 9^4 + 12^3 \\ &= 2^{10} + 3^{11} + 4^9 + 5^5 + 6^7 + 7^6 + 9^4 + 10^3 + 11^2 \end{aligned}$$

$$\begin{aligned} 1007017 &= 2^{13} + 3^{10} + 4^9 + 5^6 + 6^7 + 7^2 + 9^3 + 10^4 + 13^5 \\ &= 2^{13} + 3^6 + 4^9 + 5^8 + 6^7 + 7^2 + 8^4 + 9^5 + 13^3 \\ &= 2^{15} + 3^9 + 4^8 + 5^7 + 6^6 + 7^2 + 8^4 + 9^3 + 15^5 \\ &= 2^{17} + 3^{11} + 4^7 + 5^8 + 6^6 + 7^2 + 8^3 + 11^5 + 17^4 \end{aligned}$$

$$\begin{aligned} 1360430 &= 2^9 + 3^{12} + 5^5 + 7^7 + 9^2 + 12^3 = 2^{10} + 3^9 + 4^3 + 5^4 + 6^7 + 7^2 + 9^5 + 10^6 \\ &= 2^{11} + 4^9 + 5^8 + 6^7 + 7^4 + 8^6 + 9^2 + 11^5 \\ &= 2^{17} + 3^6 + 4^{10} + 5^7 + 6^4 + 7^3 + 10^5 + 17^2 \end{aligned}$$

The hunt for 5-duplicates involves even larger numbers, beginning with

$$\begin{aligned} 5260225 &= 2^{22} + 3^8 + 4^5 + 5^2 + 7^7 + 8^3 + 22^4 \\ &= 2^{21} + 3^8 + 4^{10} + 7^3 + 8^7 + 10^4 + 21^2 \\ &= 2^7 + 3^{14} + 4^5 + 5^8 + 6^6 + 7^3 + 8^2 + 14^4 \\ &= 2^{15} + 3^{10} + 4^9 + 5^5 + 6^4 + 7^6 + 9^7 + 10^3 + 15^2 \\ &= 2^{11} + 3^7 + 4^{10} + 5^9 + 6^8 + 7^3 + 8^5 + 9^6 + 10^4 + 11^2 \end{aligned}$$

$$\begin{aligned} 7923882 &= 2^8 + 3^5 + 4^{11} + 5^9 + 6^3 + 8^4 + 9^2 + 11^6 \\ &= 2^{12} + 3^9 + 4^5 + 5^6 + 6^2 + 7^8 + 8^7 + 9^3 + 12^4 \\ &= 2^{14} + 3^{13} + 4^4 + 5^5 + 6^7 + 7^8 + 8^6 + 13^2 + 14^3 \\ &= 2^{18} + 3^{14} + 4^6 + 5^9 + 6^3 + 7^7 + 9^5 + 14^4 + 18^2 \\ &= 2^{19} + 3^{14} + 4^8 + 5^9 + 6^6 + 8^2 + 9^4 + 14^5 + 19^3 \end{aligned}$$

$$\begin{aligned} 11054875 &= 2^3 + 3^6 + 4^{10} + 5^5 + 6^2 + 7^4 + 10^7 \\ &= 2^{15} + 3^8 + 4^6 + 5^2 + 6^9 + 7^7 + 8^3 + 9^5 + 15^4 \\ &= 2^{22} + 3^{12} + 4^2 + 5^6 + 6^7 + 7^8 + 8^5 + 12^3 + 22^4 \\ &= 2^{11} + 3^{12} + 4^8 + 5^7 + 6^9 + 7^6 + 8^3 + 9^2 + 11^5 + 12^4 \\ &= 2^9 + 3^{13} + 4^{10} + 5^3 + 6^5 + 7^8 + 8^7 + 9^6 + 10^4 + 13^2 \end{aligned}$$

$$\begin{aligned} 11224211 &= 2^{18} + 3^2 + 4^9 + 6^6 + 7^8 + 8^3 + 9^7 + 18^4 \\ &= 2^{17} + 3^{14} + 4^4 + 6^7 + 7^8 + 8^6 + 14^3 + 17^2 \\ &= 2^{20} + 3^4 + 4^9 + 5^{10} + 6^6 + 9^3 + 10^5 + 20^2 \\ &= 2^{11} + 3^8 + 4^9 + 5^{10} + 6^4 + 7^7 + 8^6 + 9^3 + 10^5 + 11^2 \\ &= 2^{12} + 3^8 + 4^{11} + 5^9 + 6^3 + 7^6 + 8^4 + 9^7 + 11^5 + 12^2 \end{aligned}$$

$$\begin{aligned} 11870046 &= 2^{22} + 3^{12} + 4^7 + 5^9 + 7^3 + 9^2 + 12^4 + 22^5 \\ &= 2^{17} + 3^5 + 4^7 + 5^{10} + 6^8 + 7^2 + 8^6 + 10^4 + 17^3 \\ &= 2^{18} + 3^{13} + 4^7 + 5^{10} + 6^5 + 7^6 + 10^3 + 13^2 + 18^4 \\ &= 2^{21} + 3^{14} + 4^{11} + 5^8 + 6^6 + 8^2 + 11^5 + 14^3 + 21^4 \\ &= 2^{21} + 3^{13} + 4^{11} + 5^9 + 6^8 + 8^6 + 9^5 + 11^3 + 13^4 + 21^2 \end{aligned}$$

We also found numbers with six representations: the first four ones are

$$\begin{aligned}
10630307 &= 2^{15} + 3^{12} + 4^2 + 5^{10} + 10^3 + 12^5 + 15^4 \\
&= 2^8 + 3^{12} + 4^9 + 5^{10} + 8^2 + 9^5 + 10^4 + 12^3 \\
&= 2^{11} + 3^{14} + 4^7 + 6^6 + 7^8 + 8^2 + 11^4 + 14^3 \\
&= 2^{13} + 3^{12} + 4^8 + 5^{10} + 8^3 + 10^4 + 12^5 + 13^2 \\
&= 2^{11} + 3^2 + 4^{10} + 5^9 + 6^8 + 7^4 + 8^3 + 9^7 + 10^6 + 11^5 \\
&= 2^{19} + 3^{11} + 4^{10} + 5^5 + 6^3 + 7^8 + 8^7 + 10^6 + 11^4 + 19^2 \\
28797910 &= 2^6 + 3^{10} + 4^4 + 5^9 + 6^5 + 7^3 + 8^8 + 9^2 + 10^7 \\
&= 2^9 + 3^{13} + 4^2 + 5^{10} + 6^3 + 8^8 + 9^6 + 10^5 + 13^4 \\
&= 2^{14} + 3^{13} + 4^9 + 5^{10} + 8^8 + 9^3 + 10^4 + 13^5 + 14^2 \\
&= 2^{11} + 3^{10} + 4^{12} + 5^4 + 6^8 + 7^6 + 8^3 + 10^7 + 11^5 + 12^2 \\
&= 2^{12} + 3^{13} + 4^3 + 5^{10} + 6^7 + 7^6 + 8^8 + 10^4 + 12^5 + 13^2 \\
&= 2^{21} + 3^{11} + 4^4 + 5^{10} + 6^9 + 7^7 + 9^3 + 10^2 + 11^6 + 21^5 \\
110307490 &= 2^7 + 3^8 + 4^5 + 5^{11} + 6^{10} + 7^2 + 8^4 + 10^6 + 11^3 \\
&= 2^{10} + 3^{16} + 4^6 + 5^2 + 6^3 + 7^9 + 8^8 + 9^5 + 10^7 + 16^4 \\
&= 2^{10} + 3^{16} + 4^8 + 5^2 + 6^3 + 7^9 + 8^4 + 9^5 + 10^7 + 16^6 \\
&= 2^{16} + 3^{10} + 4^5 + 5^2 + 6^3 + 7^9 + 8^4 + 9^8 + 10^7 + 16^6 \\
&= 2^{19} + 3^{10} + 4^{11} + 5^9 + 6^3 + 7^7 + 8^6 + 9^2 + 10^8 + 11^4 + 19^5 \\
&= 2^{25} + 3^9 + 4^{10} + 5^{11} + 6^6 + 7^5 + 8^8 + 9^3 + 10^7 + 11^4 + 25^2 \\
135712207 &= 2^{11} + 3^{10} + 4^{13} + 5^6 + 6^5 + 7^8 + 8^4 + 10^2 + 11^3 + 13^7 \\
&= 2^{11} + 3^{16} + 4^{13} + 5^2 + 6^5 + 7^8 + 8^6 + 11^7 + 13^4 + 16^3 \\
&= 2^{11} + 3^{17} + 4^{10} + 5^9 + 6^8 + 8^5 + 9^3 + 10^2 + 11^6 + 17^4 \\
&= 2^{13} + 3^{10} + 4^8 + 5^5 + 6^4 + 7^7 + 8^9 + 9^6 + 10^2 + 13^3 \\
&= 2^{13} + 4^{10} + 5^{11} + 6^7 + 7^9 + 8^4 + 9^8 + 10^2 + 11^6 + 13^5 \\
&= 2^{16} + 3^{10} + 4^{12} + 5^9 + 6^3 + 8^2 + 9^5 + 10^8 + 12^4 + 16^6
\end{aligned}$$

As it can be seen, generally speaking the more representations are requested, the rarer the numbers are. We were not able to find examples with seven or more representations, but we are confident that considering larger values of t and/or larger bases it could be possible to discover them.

[2] The sequence of irreducibles is A387302:

$$\begin{aligned}
S_I = \{ &4, 17, 27, 32, 57, 89, 100, 105, 145, 166, 177, 254, 256, 276, 289, 320, 348, \\
&368, 377, 480, 548, 568, 593, 673, 730, 739, 773, 777, 845, 865, 892, 922, \\
&932, 945, 1036, 1065, 1124, 1138, 1164, 1174, 1230, 1236, 1250, \dots \}
\end{aligned}$$

There are BE -numbers that are both irreducible and 2-duplicates (sequence A387898), such as

$$\begin{aligned}
\{ &4684, 7865, 8538, 10953, 11986, 12449, 18638, 19893, 40765, 53318, 53444, \\
&65379, 66681, 72292, 82741, 85485, 90135, 95290, 108197, 118261, 119482, \dots \}
\end{aligned}$$

while the rarer irreducible 3-duplicates are (sequence A387899)

$$\begin{aligned}
\{ &313323, 550979, 2035918, 4094727, 4853691, 5122577, 5129782, 6057059, \\
&6248742, 6785690, 6891027, 7586030, 8374709, 10176486, 10815379, \dots \}
\end{aligned}$$

We also found one irreducible 4-duplicate:

$$\begin{aligned}
 71286675 &= 2^6 + 3^2 + 4^9 + 6^{10} + 7^8 + 8^3 + 9^7 + 10^4 \\
 &= 2^9 + 3^2 + 4^3 + 6^{10} + 7^8 + 8^6 + 9^7 + 10^4 \\
 &= 2^8 + 3^{14} + 4^{11} + 5^3 + 6^{10} + 8^5 + 10^2 + 11^6 + 14^4 \\
 &= 2^{20} + 3^{10} + 4^{12} + 5^6 + 6^9 + 8^2 + 9^8 + 10^5 + 12^3 + 20^4
 \end{aligned}$$

Two daring conjectures are the following:

Conjecture 1. *The set of k -duplicates is not empty $\forall k \in \mathbb{N}_2$.*

Conjecture 2. *The set of irreducible k -duplicates is not empty $\forall k \in \mathbb{N}_2$.*

[3] The smallest and the largest BE -numbers whose representations with t powers have b as smallest base and β as largest base (with $1 \leq t \leq \beta - b + 1$) are, respectively,

$$n(b, t) = \sum_{m=b}^{b+t-1} m^{2b+t-1-m} \quad ; \quad N(\beta, t) = \sum_{m=\beta-t+1}^{\beta} m^m \quad (3)$$

The number of possible t -uples (bases) to be chosen is $\binom{\beta-b+1}{t}$ and for each of them there are $t!$ possible choices for the exponents. Duplicates may occur, but for a rough estimate the probability $P_{BE}(b, \beta, t)$ of observing a BE -number $n(B, E)$ with t powers with $\{b, B\} \subseteq B \subseteq [b, \beta]$ can be somehow approximated with

$$P_{BE}(b, \beta, t) \simeq \frac{\binom{\beta-b+1}{t} t!}{N(\beta, t) - n(b, t) + 1} \quad (4)$$

By taking all possible values of t and considering the minimum and maximum obtainable BE -numbers

$$n(b) = n(b, 1) = b^b \quad ; \quad \mathcal{N}(b, \beta) = N(\beta, \beta - b + 1) = \sum_{m=b}^{\beta} m^m$$

the probability to observe a BE -number $n \in [n(b), \mathcal{N}(b, \beta)]$ having representation containing only bases in $[b, \beta]$ results to be approximatively

$$P_{BE}(b, \beta) \simeq \frac{\sum_{t=1}^{\beta-b+1} \binom{\beta-b+1}{t} t!}{\mathcal{N}(b, \beta) - b^b + 1} \quad (5)$$

Now, considering relation (5) with $b = 2$ we can derive an approximated formula for the probability to find BE -numbers whose representations do not contain bases larger than β :

$$P_{BE}(\beta) \simeq \frac{\sum_{t=1}^{\beta-1} \binom{\beta-1}{t} t!}{\mathcal{N}(2, \beta) - 3} = \frac{\sum_{t=1}^{\beta-1} \prod_{i=1}^t (\beta - i)}{\mathcal{N}(2, \beta) - 3} \quad (6)$$

[4] An interesting problem is to check whether a particular $n \in \mathbb{N}_2$ is a BE -number. Actually it is not particularly complicated to check if $n \in S_1$: as $2 \leq a \Rightarrow 1 \leq \log_2 a$, one has

$$n = a^a \Rightarrow 2 \leq a \leq a \cdot 1 \leq \lfloor a \log_2 a \rfloor = \lfloor \log_2 n \rfloor$$

a binary search in the interval $[2, \lfloor \log_2 n \rfloor]$ is sufficient to detect if such a base a exists or not. As binary search has logarithmic complexity, the problem can therefore be solved in $O(\log \log n)$ steps.

Testing if $n \in S_2$ is a bit more complicated, as essentially pairs must be considered. Anyway, things are not so bad. Consider the case $n = a^a + b^b$, with $b > a \geq 2$ (so that $b \geq a + 1 \geq 3$): one has

$$n = a^a + b^b > a^a + a^{a+1} = a^a(a+1) \geq 3a^a \Rightarrow a < \lfloor a \log_2 a \rfloor \leq \lfloor \log_2(n/3) \rfloor$$

For each such value a_i , a S_1 -problem $b^b = n - a_i^{a_i}$ has to be solved. Considering the worst case complexity $O(\log \log n)$ of the problem $b^b = n - 4$, one obtains (after checking that $a \neq b$, of course) that the global complexity is therefore $O(\log n \log \log n)$. The case $n = a^b + b^a$ can be treated similarly:

$$\begin{aligned} n = a^b + b^a &> a^{a+1} + a^a \geq 3a^a \Rightarrow a < \lfloor \log_2(n/3) \rfloor \\ n = a^b + b^a &\geq a^b + 3^a \Rightarrow b \leq \left\lfloor \frac{\log_2(n - 3^a)}{\log_2 a} \right\rfloor \end{aligned}$$

For each admissible value a_i the corresponding value of b can be determined, if it exists, by means of a binary search in an interval of width at most $O(\log n)$, so that the complexity is again $O(\log n \log \log n)$.

Generally speaking, the largest base β that can appear in the representation of a BE -number n must have the smallest possible exponent, that is 2. If $n > 4$ the representation of n should contain a “twin” power $b^\beta \geq 2^\beta$. To better frame the situation, consider the monotonically increasing function

$$g_2(x) = \begin{cases} 4 & \text{if } x = 2 \\ 2^x + x^2 & \text{if } x > 2 \end{cases} \quad \text{with } g_2^{-1}(x) \in O(\log x)$$

Then $\beta = \lfloor g_2^{-1}(n) \rfloor$. This establishes an upper bound for the bases that can appear in the searched representation.

An upper bound T for t can be easily obtained by finding the index of the smallest value in S_0 greater than n (in case of equality a BE -representation of n is immediately found). For $2 < t \leq T$ essentially a recursive approach should be set up, taking care of the largest possible base/exponent. It is given by a generalization of the function $g_2(x)$, as t powers are needed: now the minimum possible value is controlled by

$$g_t(x) = 2^x + \sum_{i=3}^t i^{t+3-i} + x^2 \quad (7)$$

so that $b_t = \lfloor g_t^{-1}(n) \rfloor$ for $S_0[t-1] < n \leq S_0[t]$.

The easiest case is to consider $n' = n - b^b$ for all the $O(\log n)$ meaningful values of b : it is possible to recurse to the $t-1$ case hoping to find a representation for n' not containing b to recover a valid representation for n . If no representation is found, the $t! \left(\sum_{k=0}^t (-1)^k / k! \right)$ derangements remain to be considered, representing one third of the possible cases for $t=3$, the 37.5% for $t=4$ and $< 36.81\%$ for $t \geq 5$ (the percentage actually rapidly converges to $e^{-1} \simeq 36.7879\%$).

In the general case $n' = n - b^e$ what in practice has to be done is a depth-first search on a tree: for every value $2 \leq b \leq b_t$ consider all its possible compatible exponents $e = 2, \dots, \bar{e}(b) \leq \lfloor \log_b n \rfloor$ – and recurse on n' , by applying the same reasoning with $t-1$ but coping with the restriction that e as base and b as exponent must necessarily appear in all (sub-)representations that will be checked, discarding the ones not having them.

[5] The correlation (actually, an equality) between B and E is the main reason for the generating function of whatever of the considered sequences not to have a simple (or not at all) closed form; in fact, the problem begins even with the very basic S_1 .

4 Experimental data

Several BE -number representations were generated automatically and checked for duplicateness, irreducibility and both. Some considerations helped to reduce the needed time and optimize the computations: the goal is to compute all BE -numbers up to a certain upper bound U and to do this the first thing is to compute T and $b_t = \max\{B\}$ for each $1 \leq t \leq T$.

Consider the partial ordering on sequences of length t defined as $B \leq B' \Leftrightarrow b_i \leq b'_i$ for $i = 1, \dots, t$. If $B < B'$ then $n(B, \pi(B)) > U \Rightarrow n(B', \pi(B')) > U$ because of monotonicity of (1).

Example 1. Let $U = 600$: then $T = 4$ (the fifth element of S_0 is 723) and $b_t = \max\{B\} = 9$, as $9^2 + 2^9 = 593$ while $10^2 + 2^{10} = 1124 > U$. Considering the case $t = 3$ the bound b_t is smaller, as $g_3(9) = 2^9 + 3^3 + 9^2 = 620 > U$: for all $\binom{b_t-2+1}{t} = \binom{7}{3} = 35$ sequences $B \in \{(2, 3, 4), (2, 3, 5), \dots, (6, 7, 8)\}$ and the $6 = t!$ permutations for each B , in total 210 numbers should be considered, but many of them are greater than U .

The idea is to understand which permutations can be avoided. Consider e.g. $B = (2, 3, 5)$: as $2^2 + 3^3 + 5^5 = 3156 > U$ and things get even worse with all other $B' > B$, the identity permutation can always be avoided for all $B \neq (2, 3, 4)$. The same consideration applies to $\pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. This means that for all B but one, only 4 permutations - $136 = 34 \cdot 4$ numbers in total - have to be analyzed, with a global saving of circa 32.4%. Actually much more can be done by considering sequences $B' > B$ and applying the same reasoning to exclude other permutations (cases) on a subset of sequences $B'' > B'$.

The approach introduced in the above example can be efficiently generalized, in order to obtain a drastic reduction of the numbers of cases. For each $t = 1, \dots, T$ let $B_0 = (2, \dots, t+1)$ and $P \subseteq S(t)$ be the set of “surviving” permutations for which $n(B_0, \pi(B_0)) \leq U$ for all $\pi \in P$. Let $i = t$ and consider all strictly increasing sequences $B = (b_1, \dots, b_t) > B_0$ one after the other, in lexicographic order. If b_i is increased for the first time with respect to the corresponding entry in the previous sequence, update P by removing all permutations π such that $n(B, \pi(B)) > U$ and if $i = 1$ set $i = t$, otherwise decrease i by 1 and continue, terminating when $P = \emptyset$ or all the sequences have been considered. This shortcut procedure can be further refined, but it is already enough to achieve the results presented here. The results obtained from the computations are synthetically shown in three graphs.

Let $f(x) = \mathbb{N} \ni x \mapsto \#\{m \in S \mid m \leq x\} \in \mathbb{N}$ be the function counting the number of BE -numbers up to x . Figure 1 shows its graph for $x \leq 10^8$. Similarly, the graph of $i(x) : \mathbb{N} \ni x \mapsto \#\{m \in S_I \mid m \leq x\} \in \mathbb{N}$ is reported in figure 2, while the graph of $d(x) : \mathbb{N} \ni x \mapsto \#\{m \in S_D \mid m \leq x\} \in \mathbb{N}$, the number of duplicates smaller or equal than x , is in figure 3. A somewhat linear behavior can be observed in all cases.

5 Conclusions

In this paper we defined a subset of the natural numbers $\mathbb{N}$, generated by sums of powers under a base-exponent set symmetry. The numbers contained in this subset are referred to as BE -numbers, and they are related to new sequences recently introduced in the OEIS. We pointed out BE -numbers with multiple representations, an additive base and provided some graphs related to the density of all these numbers.

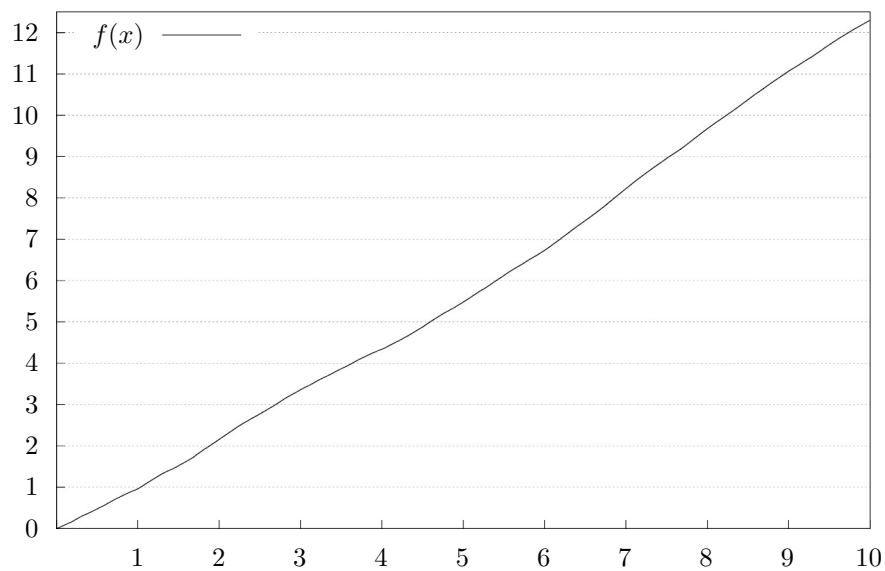


Figure 1: BE -numbers up to $x \leq 10^8$ (abscissas: $\times 10^7$, ordinates: $\times 10^6$).

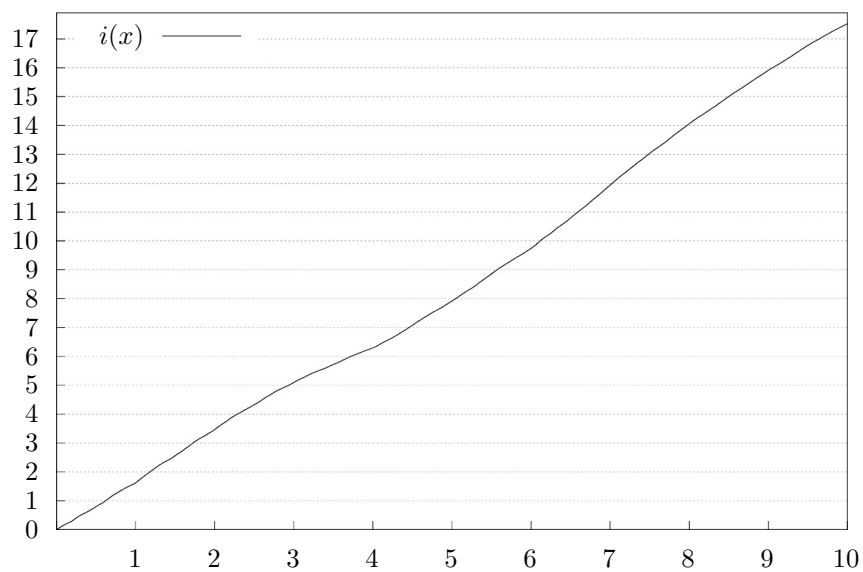


Figure 2: Irreducibles up to $x \leq 10^8$ (abscissas: $\times 10^7$, ordinates: $\times 10^5$).

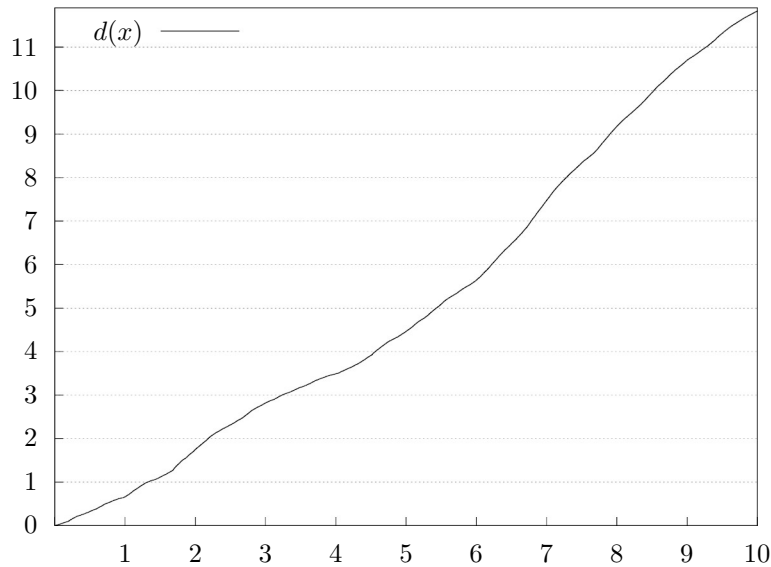


Figure 3: Duplicates up to $x \leq 10^8$ (abscissas: $\times 10^7$, ordinates: $\times 10^5$).

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Alberto Zanoni,
Epigenesys s.r.l.
Via Edoardo D'Onofrio 304
00155 Roma, Italy
Email: zanoni@epigenesys.com

Marina Zanoni,
BDO DIGITAL GmbH
Fuhlentwiete 12
20355 Hamburg – Germany
Email: marina.zanoni@bdo.de