



Fractal nonlocal thermoelastic dynamics of imperfect curved functionally graded nanobeam with memory effect

Rajendran Selvamani, Thangamuni Prabhakaran and Marin Marin

Abstract

This study presents an exact analytical investigation into the vibrational behavior of imperfect curved nanobeams under uniform thermal conditions, incorporating fractal geometry to model material heterogeneity at multiple scales. The analysis employs a combined nonlocal-strain gradient model to account for scale-dependent behavior and introduces a memory-dependent constitutive model to reflect time-history influences. Material properties vary through the thickness according to a temperature-dependent modified power-law, accounting for porosity induced during manufacturing. The governing equations, derived via Timoshenko beam theory with memory and fractal effects, are solved using the Navier method. Results are validated against existing literature and emphasize the influence of nonlocality, porosity, thermal loading, gradient index, geometry, fractal scaling, and memory effects on natural frequencies. The study confirms that memory-dependent and fractal behaviors significantly affect dynamic responses, offering benchmark solutions for future research on functionally graded nanostructures with porosity and time-dependent effects.

Key Words: FG curved beam, Porosity, Strain gradient theory, Thermal-mechanical vibration, Memory-dependent derivative, fractal dimensions.

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1 Introduction

Functionally graded materials (FGMs) represent a novel category of composite materials that have garnered significant attention within the domain of smart structures and materials. Unlike conventional composites, FGMs minimize issues such as interface delamination and stress concentrations [1–3]. These materials are characterized by the continuous variation of their properties along one or more spatial directions. Owing to their superior thermal resistance and mechanical strength, FGMs are widely applied in various high-performance systems, including thermal protection for spacecraft, nuclear components, advanced jet structures, and engine parts [4–6]. Marin [7] investigates thermoelasticity in micropolar bodies, where material points possess both displacement and independent micro-rotation. Using mathematical analysis, the paper establishes the existence and uniqueness of solutions to the governing coupled equations. The results confirm the well-posed nature of micropolar thermoelastic models. Recent advancements have extended the application of FGMs to nanoscale structures, enabling the tailoring of nanomaterials with enhanced and optimized performance. As the utility of FGM-based nanostructures continues to grow in modern technology, understanding their mechanical characteristics has become a focal point of research. To address this, enhanced frameworks such as Eringen’s theory of nonlocal [8, 9] and the theory of strain gradient [10, 11] have been introduced. These models consider the impact of long-range interatomic forces and material microstructure, which are crucial at small scales [12–16]. Researcher [17, 18] examined the vibration characteristics of functionally graded nanobeams based on a nonlocal elasticity framework. Nonlinear vibrations were also examined by Nazemnezhad and Hosseini-Hashemi [19, 20] using Eringen’s theory. Despite its effectiveness, Eringen’s model overlooks certain stiffening behaviors observed experimentally, particularly at micro and nanoscales [21]. To bridge this gap, strain gradient theories have been developed, accounting for both nonlocal interactions and strain gradients via the introduction of two distinct length-scale parameters [22]. This dual-parameter model enables a more accurate representation of nanoscale mechanical responses. For example, Li and Hu [23] analyzed buckling in nonlinear beams, while Li et al. [24] explored the vibrational behavior of FG nanobeams using Navier-type solutions. They concluded that increasing the nonlocal parameter reduces natural frequencies, whereas a larger length scale leads to the opposite effect. [25] studied thermoelastic bodies with microstructure, considering intrinsic rotations and micro-expansions/contractions. The authors formulate and study the initial boundary value problem incorporating these microstructural effects. They prove well-posedness and show how micro-rotational dynamics influence ther-

mal-mechanical wave behavior.

Studies have also explored wave propagation and hygrothermal responses [26] within the same theoretical context. Curved nanobeams and nanorings have received growing interest due to their unique mechanical properties and potential in nanoscale devices such as switches, valves, and filters [27]. Yan and Jiang [28] focused on electromechanical responses considering surface effects, while Kananipour et al. [29] introduced a polar coordinate-based numerical approach. Porosity is typically incorporated through variations in local material density. Many FGMs, especially ceramic-metal composites, are fabricated using sintering techniques, during which differential shrinkage can introduce pores [30, 31]. In processes like sequential infiltration, incomplete penetration in the central regions of specimens often leads to higher porosity. Hence, accurately modeling the porosity effect is essential for reliable structural analysis and design. Despite numerous studies on the overall dynamic performance of FG structures, those specifically focused on porous FG beams are relatively sparse. Yahia et al. [32] conducted nonlinear vibration analyses of porous FGM annular plates, finding that both porosity type and volume fraction significantly influence dynamic responses. Mechab et al. [33] extended these findings to FG porous nanoplates on elastic foundations using nonlocal models. Researchers [34, 36] applied analytical methods such as the Chebyshev collocation method to analyze linear and nonlinear vibrations of porous FG beams. Selvamani et al. [37–43] have extensively contributed to this field by employing Bernstein polynomials, wave propagation and dynamic behavior of functionally graded and porous nanobeams under various loading conditions. Their work spans the investigation of boundary effects, fluid–structure interaction, thermoviscoelastic damping, nonlinear and parametric vibrations, as well as magneto-piezo-thermoelastic and magneto-aerothermal responses. These studies provide valuable frameworks for enhancing the accuracy of nano-scale device modeling under realistic boundary constraints and coupled field environments. Qi et al. [44] and Abouelregal and Dargail [45] Examine the influence of memory effects on the thermoelastic nanobeams, accounting for small-scale effects and functionally graded (FG) materials. Abouelregal [46] examines a thermoelastic material governed by a non-local Moore–Gibson–Thompson heat transfer law with memory-dependent derivatives and coupled model and analyze how advanced heat conduction affects thermal and elastic wave responses. Their results highlight the significant impact of non-local and memory effects on thermoelastic behavior. Li et al. [47] examine the response of bilayered structures subjected to laser heating, applying fractional and memory-based models to capture transient thermal effects. Yu et al. [48] introduce a generalized thermo-elastic model that integrates memory-dependent derivatives, improving the characterization of nonlocal heat transfer and wave propa-

gation. El-Nabulsi and Anukool [49] investigate the fractal thermoelastic properties in nanobeams with unconventional thermal conductivity. Tarasov [50] provides foundational insights into fractional dynamics across various physical domains. Li and Ostoja-Starzewski [51] examine wave propagation fractal-structured solids, expanding understanding of wave behavior in complex geometries. Studies by El-Nabulsi and Anukool [52,53] extend quantum mechanics into fractal dimensions with position-dependent masses, focusing on implications for quantum dots and nanocrystals. Hobiny et al. [54] employed the hyperbolic two-temperatures theory to investigate wave propagation in semiconductor materials with spherical cavities, emphasizing the importance of thermal effects on dynamic responses. Extending this framework, Abouelregal et al. [55] developed a thermoelastic model incorporating memory-dependent dynamic responses for thermo-piezoelectric functionally graded rotating rods, thereby capturing both functional gradation and time-dependent behaviors. Complementing these approaches, Zeeshan et al. [56] utilized computational intelligence techniques to optimize the magnetohydrodynamic (MHD) flow of Casson ternary hybrid nanofluids over shrinking sheets with radiative effects, demonstrating the role of smart computational methods in tackling nonlinear multi-physics problems. Collectively, these works advance the understanding of thermoelasticity, wave propagation, and nanofluid dynamics by integrating thermal memory, functional gradation, and computational optimization.

A thorough review of existing studies on the vibration behavior of functionally graded (FG) curved nanobeams reveals a noticeable gap in the literature, particularly regarding the simultaneous consideration of small-scale effects, strain gradient theory, porosity, thermal influences, fractal geometry, and memory-dependent behavior. To fill this gap, the present study focuses on analyzing the dynamic response of functionally graded porous nanobeams under thermal environments, incorporating scale-dependent effects through advanced elasticity models that consider long-range interactions, internal length parameters, and fractal-based formulations to capture geometrical and dynamic complexity. Additionally, a memory-dependent constitutive model is adopted to account for the thermo-mechanical conditions influencing the current dynamic response. Since curvature is inherently present in most practical beam structures, unlike previous studies that predominantly utilized simplified straight beam assumptions, this work emphasizes a more realistic curved beam configuration. The gradual variation in thermo-mechanical properties across the beam thickness is described using a modified power-law distribution. Using Hamilton's principle, the governing equations are formulated and solved analytically. The resulting dimensionless natural frequencies are analyzed with respect to various influencing factors, including curvature angle, length scale effects, temperature variations, vibration modes, material grada-

tion index, fractal scaling effects, memory parameters, and nonlocal effects. A comparison with previously published results demonstrates the reliability and precision of the proposed model.

2 Memory-dependent derivative

In this study, the heat conduction equations proposed by Li et al. [47] are utilized and expressed as follows:

$$q_i + \tau D_\tau q_i = -\aleph \theta_i \quad (2.1)$$

$$D_\tau q_i(t) = \frac{1}{\tau} \int_{t-\tau}^t G(t-\xi) \frac{\partial q_i(\xi)}{\partial \xi} d\xi \quad (2.2)$$

$$G(t-\xi) = \left(1 - \frac{t-\xi}{\tau}\right)^j \quad (2.3)$$

In this context The flux vector of heat is represented by q_i , the thermal conductivity is represented by $\aleph$, the present time and the time-delay factor are indicated by t and τ , respectively; the memory-dependent derivative of q_i is $D_\tau q_i$; the MDD kernel function is $G(t-\xi)$; and j is a given positive real number. The energy equation in the absence of a heat source is as follows [48]:

$$\rho C_E \partial_t \theta + \gamma T_0 \partial_t \varepsilon_{kk} = -q_{i,i} \quad (2.4)$$

where $\partial_t = \frac{\partial}{\partial t}$ represents the partial derivative taken with respect to the time variable t . It is possible to obtain the energy equation based on MDD [44] by replacing Eq. (2.1) with Eq. (2.4).

$$(1 + \tau D_\tau) [\rho C_E \partial_t \theta + \gamma T_0 \partial_t \varepsilon_{kk}] = \aleph \nabla^2 \theta \quad (2.5)$$

where C_E is the specific heat at constant volume and ρ is the mass density.

3 Formulation of the nonlocal fractal thermoelasticity model

This section establishes the fundamental principles of linearly organized scale analysis (LOSA - Li and Ostoj a-Starzewski [50]) and develops a nonlocal fractal thermoelasticity model. Consider an anisotropic medium with spatial dimensions x_1, x_2, x_3 , mass m , and density ρ distributed over a fractal domain is represented as [53]:

$$\wp = \alpha_1 + \alpha_2 + \alpha_3, \quad \text{where } 0 < \alpha_k \leq 1 \quad (3.1)$$

The mass distribution in such a medium is represented as:

$$m = \int_V \rho(x_1, x_2, x_3) \prod_{k=1}^3 d\mu_k(x_k) \quad (3.2)$$

where each

$$d\mu_k(x_k) = c_1^k(\alpha_k, x_k) dx_k$$

denotes the corresponding measure along x_k . The line transformation coefficient in its direction is given by $c_1^k(\alpha_k, x_k)$. With this approach, the resulting fractal volume element is

$$dV_\varphi = c_3 dx_1 dx_2 dx_3 = c_3 dV_3 \quad (3.3)$$

where $c_3 = c_1^{(1)} c_2^{(2)} c_3^{(3)}$, represents the volume transformation coefficient. Here, α_k plays the role of a fractal dimension along x_k . For the surface transformation coefficient c_2^k , the relationship is expressed as:

$$c_2^k = c_1^i c_1^j = \frac{c_3}{c_1^k}, \quad i \neq j, \quad i, j \neq k \quad (3.4)$$

The sum $d^{(k)} = \alpha_i + \alpha_j$ characterizes $S_d^{(k)}$ along the diagonals where $|x_i| = |x_j|$. A number of variants were used in the literature to express $c_1^k(\alpha_k, x_k)$: the first is based on the Riesz fractional integral, and the second is based on the modified Riemann-Liouville fractional integral or the Jumarie fractional integral [49]. The second formulation, utilized in the LOSA framework, is expressed as [50]:

$$c_1^k(\alpha_k, x_k) = \alpha_k \left(\frac{l_k - x_k}{l_{k0}} \right)^{\alpha_k - 1} \quad (3.5)$$

where $0 < x_k < l_k$ varies with length l_k , and l_k is a characteristic length. In Li and Ostoja-Starzewski [50], l_{k0} is the mean pore size, while l_k is the total length in the x_k direction. Assuming that dl_a , dS_d and dV_φ are the line, surface, and volume elements in the fractal body, respectively, and that dx , dS_2 and dV_3 represent those in the homogenized model where the relations between the two images are given by the coefficients c_1 , c_2 and c_3 then the following requirements hold: The density of states is used to express the measure in a given direction:

$$dl_{l\alpha k}(X_k) = c_1^k(\alpha_k, X_k) dX_k \quad (3.6)$$

dV_φ related to the cubic volume element

$$dV_3 = dx dy dz, \text{ by } dV_\varphi = dl_{\alpha_1}(X_1) dl_{\alpha_2}(X_2) dl_{\alpha_3}(X_3) = c_3 dV_3, \text{ where} \\ c_3 = c_1^{(1)} c_1^{(2)} c_1^{(3)} \quad (3.7)$$

dS_d is related to the plane surface element

$$dS_d^{(k)} = dx_i dx_j = c_2^k dS_2, \text{ with } c_2^k = c_1^{(i)} c_1^{(j)} = \frac{c_3}{c_1^{(k)}}, \quad i \neq j, \quad i, j \neq k \quad (3.8)$$

The following differential operators appear on a fractal domain $\wp$:

The gradient of fractal

$$\nabla^\wp = e_k \nabla_k^\wp \quad (3.9)$$

where $\nabla_k^\wp = (c_1^{(k)})^{-1} \nabla_k$, The base vectors in the Euclidean space E_3 are denoted by e_k .

The fractal Laplacian operator

$$\Delta^\alpha = \nabla^\alpha \cdot \nabla^\alpha \quad (3.10)$$

which arise from applying Gauss's theorem to the specific case

$$c_2^k = c_1^{(i)} c_1^{(j)} = \frac{c_3}{c_1^{(k)}}, \quad i \neq j, \quad i, j \neq k \quad (3.11)$$

For any vector field $\bar{f} = f_k e_k$, with the following Gauss and Stokes theorems for fractals hold true when $\hat{n} = n_k e_k$ serves as the unit normal vector to a surface, with the Einstein summation convention applied [49]:

$$\int_{\partial\wp} \bar{f} \cdot \hat{n} dS_d = \int_{\partial\wp} f_k n_k dS_d = \int_{\wp} f_{k,k} \frac{c_2^k}{c_3} dV_d = \int_{\wp} \frac{f_{k,k}}{c_1^k} dx_1 dx_2 dx_3 \quad (3.12)$$

$$\int_A \hat{n} \cdot (\nabla_\wp \times f) dS_d = \int_l f dl_{\alpha_1} \quad (3.13)$$

We will apply the LOSA framework to develop formulations for thermoelasticity and thermo-poromechanics in fractal media. Specifically, we aim to develop a new model for fractal nonlocal thermoelastic vibrations of porous Euler–Bernoulli sensor nanobeams with Klein-Gordon nonlocality and variable thermal conductivity. This problem is significant in nonlocal continuum mechanics, where the impact of long-range interactions in elastic materials is considered, and it has notable applications in the field of nanomechanics.

4 Problem formulation

4.1 Curved FG porous nanobeams material characteristics

In the θ direction, A curved functionally graded porous nanobeam with uniform porosity distribution is considered, having a length L and a rectangular

cross-section defined by its width b and thickness h , as illustrated in figure 1. The nanobeam is composed of two constituent materials, Si_3N_4 and $SUS304$, whose thermo-mechanical properties are listed in Table 1. The structure is subjected to a thermal load.

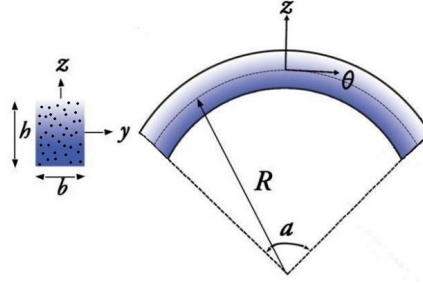


Figure 1: Geometry of representation

According to Setoodeh et al. [35], (θ) denotes the relationship between the length of a circularly curved beam and (α) denotes the curvature of angle can be expressed as follows.

$$\theta = R\alpha \quad (4.1.1)$$

The effective properties of the curved functionally graded porous beam vary smoothly along the thickness using a modified power-law distribution. Based on this approach, the properties (P_f) of the porous FG nanobeam are determined, as described by [34]:

$$P_f = P_u \left(V_u - \frac{\vartheta}{2} \right) + P_l \left(V_l - \frac{\vartheta}{2} \right) \quad (4.1.2)$$

where ϑ represents the porosity volume fraction, and (P_l, P_u) denotes the material properties at the bottom and top surfaces, respectively. Additionally, (V_l, V_u) are the associated volume fractions related through the following expression:

$$V_u = \left(\frac{z}{h} + \frac{1}{2} \right)^p \quad (4.1.3)$$

$$V_u + V_l = 1 \quad (4.1.4)$$

Therefore, the unique material characteristics of nanobeam can be described as follows using Eqs.(4.1.2) and (4.1.3):

$$P(z) = (P_u - P_l) \left(\frac{z}{h} + \frac{1}{2} \right)^p + P_l - (P_u + P_l) \frac{\vartheta}{2} \quad (4.1.5)$$

The power-law index is denoted by p , which governs how the material composition is distributed along the beam's thickness in the z -direction. This

gradation model indicates that at $(z = -h/2)$, the material is fully composed of Si_3N_4 , while at $(z = h/2)$, it consists entirely of $SUS304$, which is made up of only one constituent material. The temperature dependence of material properties can be expressed through a nonlinear relationship between thermoelastic properties and temperature T (K), as described by [36]:

$$P = P_0 \left(P_{-1} \frac{1}{T} + 1 + P_1 T + P_2 \frac{1}{T^2} + P_3 \frac{1}{T^3} \right) \quad (4.1.6)$$

where, the temperature dependent factors P_0, P_{-1}, P_1, P_2 and P_3 that accessible in the table 1.

4.2 Governing Equation

Using theory of Timoshenko beam, the field of displacement at each point in the curved beam model may be explained as follows [28]:

$$u_\theta(\theta, r, t) = \left(1 + \frac{r}{R} \right) u(\theta, t) + r \phi(\theta, t) \quad (4.2.1a)$$

$$u_z(\theta, r, t) = w(\theta, t) \quad (4.2.1b)$$

Here, u and w denote the radial and tangential displacements of the curved FG porous beam, respectively, and ϕ represents the total bending rotation. The nonzero stress components based on the Timoshenko beam model are given by

$$\varepsilon_{\theta\theta}^0 = \partial_\theta w - \frac{u}{R} \quad (4.2.2a)$$

$$\kappa = \partial_\theta \phi \quad (4.2.2b)$$

$$\gamma_{\theta z} = \partial_\theta u - \phi + \frac{w}{R} \quad (4.2.2c)$$

In the curved beam model, γ stands for shear strain.

$$\varepsilon_{\theta\theta} = \varepsilon_{\theta\theta}^0 + z \kappa \quad (4.2.3)$$

The equation of motion is obtained using the extended form of Hamilton's principle, which yields:

$$\int_0^t \delta (U_s - T + W_{\text{ext}}) dt = 0 \quad (4.2.4)$$

The strain energy, kinetic energy, and work performed by externally applied loads are denoted by U_s , T and W_{ext} , respectively. One way to express the initial strain energy (U_s) fluctuation is as follows:

$$\delta U_s = \int_V (\sigma_{\theta\theta} \delta \varepsilon_{\theta\theta} + \sigma_{\theta z} \delta \gamma_{\theta z}) dV \quad (4.2.5)$$

When Eqs. (4.1.6) and Eqs.(4.2.1) substituted into Eq. (4.2.5), the strain energy's initial variation is as follows:

$$\delta U_s = \int_0^L \left[N \left(-\frac{\delta u}{R} + \partial_\theta \delta w \right) + M (\partial_\theta \delta \phi) + Q \left(\partial_\theta \delta u - \delta \phi + \frac{\delta w}{R} \right) \right] d\theta \quad (4.2.6)$$

where, the final expression involves the following variable definitions: N , M and Q stand for shear force, axial force and bending moment of cross section, respectively. The following is an expression of these stressors:

$$\begin{aligned} N &= \int_A \sigma_{\theta\theta} dA, \\ M &= \int_A \sigma_{\theta\theta} z dA, \\ Q &= \int_A K_{\text{shear}} \sigma_{\theta z} dA. \end{aligned} \quad (4.2.7)$$

where the shear correction factor is expressed by K_{shear} . The current curve beam model's initial variational of the virtual kinetic energy, as follows

$$T = \frac{1}{2} \int_0^L \int_A \rho(z, T) \left[\left(\frac{\partial u_\theta}{\partial t} \right)^2 + \left(\frac{\partial u_z}{\partial t} \right)^2 \right] dA d\theta \quad (4.2.8)$$

$$\begin{aligned} \delta T &= I_0 (\partial_t u \partial_t \delta u) + \left(I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2} \right) (\partial_t w \partial_t \delta w) + I_2 (\partial_t \phi \partial_t \delta \phi) \\ &+ \left(I_1 + \frac{I_2}{R} \right) (\partial_t \phi \partial_t \delta w + \partial_t w \partial_t \delta \phi) \end{aligned} \quad (4.2.9)$$

In the context, (I_0, I_1, I_2) denotes mass moments of inertias are yields:

$$(I_0, I_1, I_2) = \int_A (1, z, z^2) \rho(z, T) dA \quad (4.2.10)$$

Work done can be applied as follows:

$$\delta W_{\text{ext}} = \frac{1}{2} \int_0^L (\aleph \nabla^2 \theta + N_T) (\partial_\theta w \delta \partial_\theta w) d\theta \quad (4.2.11)$$

Here, N_T denotes the in-plane thermal load, which can be represented as:

$$N_T = \int_{-h/2}^{h/2} E(z, T) \lambda_1(z, T) \Delta T dz \quad (4.2.12)$$

The corresponding Euler-Lagrange equations for FG curved porous nanobeams under fractal thermal loading are found by putting coefficients δu , δw , $\delta \varphi$ and $\delta \psi$ to zero [49]:

$$\frac{N}{R} + \partial_\theta Q = I_0 \partial_t^2 u \quad (4.2.13a)$$

$$\partial_\theta N - \frac{Q}{R} - (\aleph \nabla^2 \theta + N_T) \Delta^\alpha w = \left(I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2} \right) \partial_t^2 w + \left(I_1 + \frac{I_2}{R} \right) \partial_t^2 \phi \quad (4.2.13b)$$

$$\partial_\theta M + Q = \left(I_1 + \frac{I_2}{R} \right) \partial_t^2 w + I_2 \partial_t^2 \phi \quad (4.2.13c)$$

The following boundary conditions are associated with equations of motion:

$$N = 0 \quad \text{at } \theta = 0, \theta = L \quad (4.2.14a)$$

$$Q = 0 \quad \text{at } \theta = 0, \theta = L \quad (4.2.14b)$$

$$M = 0 \quad \text{at } \theta = 0, \theta = L \quad (4.2.14c)$$

4.3 Strain gradient theory

This study proposed the strain gradient theory, which incorporates the effects of stress and strain fields. Accordingly, they formulated the stress expression as [30]:

$$\sigma_{ij} = \sigma_{ij}^{(0)} - \frac{d \sigma_{ij}^{(1)}}{dx} \quad (4.3.1)$$

If strain ε_{xx} and strain gradient $\varepsilon_{xx,x}$ are associated with stresses $\sigma_{xx}^{(0)}$ and $\sigma_{xx}^{(1)}$, respectively, and are represented as follows:

$$\sigma_{ij}^{(0)} = \int_0^L C_{ijkl} \alpha_0 \varepsilon'_{kl}(x') (x, x', e_0 a) dx' \quad (4.3.2a)$$

$$\sigma_{ij}^{(1)} = l^2 \int_0^L C_{ijkl} \alpha_1 \varepsilon'_{kl,x}(x') (x, x', e_1 a) dx' \quad (4.3.2b)$$

In the context C_{ijkl} denotes constants of elasticity and the parameters $e_1 a$ and $e_0 a$ represent the contributions of the nonlocal stress, while length scale parameter is denoted by l that reflects the impact of higher-order strain gradient effects. When the nonlocal kernel functions $\alpha_1(x, x', e_1 a)$ and $\alpha_0(x, x', e_0 a)$

meet the criteria established by Eringen [8], constitutive equation for a functionally graded nanobeam with fractal scaling can be written as [49]:

$$\begin{aligned} [1 - (e_1 a)^2 \Delta^\alpha] [1 - (e_0 a)^2 \Delta^\alpha] \sigma_{ij} &= C_{ijkl} \varepsilon_{kl} [1 - (e_1 a)^2 \Delta^\alpha] \\ &\quad - C_{ijkl} l^2 \Delta^\alpha \varepsilon_{kl} [1 - (e_0 a)^2 \Delta^\alpha] \end{aligned} \quad (4.3.3)$$

In the context, fractal Laplacian operator is represented by Δ^α . Assuming a constant $e_1 = e = e_0$, the constitutive relation in Eq. (4.3.1) with fractal scaling can be written as [49]:

$$\sigma_{ij} [1 - (ea)^2 \Delta^\alpha] = C_{ijkl} \varepsilon_{kl} [1 - l^2 \Delta^\alpha] \quad (4.3.4)$$

Thus, this represents the formulation describing stress behavior in a deformable nanostructure:

$$\sigma_{\theta\theta} - \mu^2 \Delta^\alpha \sigma_{\theta\theta} = E(z, T) (\varepsilon_{\theta\theta} - \lambda \varepsilon_{\theta\theta}) \quad (4.3.5a)$$

$$\sigma_{\theta z} - \mu^2 \Delta^\alpha \sigma_{\theta z} = G(z, T) (\gamma_{\theta z} - \lambda \gamma_{\theta z}) \quad (4.3.5b)$$

The force-strain and moment-strain relations for the fractal nonlocal curved FG porous Timoshenko beam are obtained by integrating Eqs. (4.3.5a) and (4.3.5b) over the cross-sectional area.

$$N - \mu^2 \Delta^\alpha N = A \left(\partial_\theta w - \frac{u}{R} - \lambda (\partial_\theta^3 w - \frac{1}{R} \partial_\theta^2 u) \right) + B (\partial_\theta \phi - \lambda \partial_\theta^3 \phi) \quad (4.3.6a)$$

$$M - \mu^2 \Delta^\alpha M = B \left(\partial_\theta w - \frac{u}{R} - \lambda (\partial_\theta^3 w - \frac{1}{R} \partial_\theta^2 u) \right) + D (\partial_\theta \phi - \lambda \partial_\theta^3 \phi) \quad (4.3.6b)$$

$$Q - \mu^2 \Delta^\alpha Q = K_{\text{Shear}} C \left[\left(\frac{w}{R} + \partial_\theta u - \phi \right) - \lambda \left(\frac{1}{R} \partial_\theta^2 w + \partial_\theta^3 u - \partial_\theta^2 \phi \right) \right] \quad (4.3.6c)$$

where the coefficients are obtained as follows: The correction factor K_{Shear} is taken as $\frac{5}{6}$.

$$\{J, K, L\} = \int_A E(z, T) \{1, z, z^2\} dA \quad (4.3.7a)$$

$$C = \int_A \frac{E(z, T)}{1 + \nu(z, T)} dA \quad (4.3.7b)$$

The following equations are obtained by substituting Eqs. (4.3.6) into Eqs. (4.2.16):

$$\begin{aligned} &J \left(\partial_\theta w - \frac{u}{R} \right) - J \lambda (\partial_\theta^3 w - \partial_{R\theta}^2 u) + K \partial_\theta \phi - K \lambda \partial_\theta^3 \phi \\ &\quad + K_{\text{Shear}} C R (\partial_{R\theta} w + \partial_\theta^2 u - \partial_\theta \phi) - K_{\text{Shear}} C R \lambda (\partial_{R\theta}^3 w + \partial_\theta^4 u - \partial_\theta^3 \phi) \\ &= I_0 R (\partial_t^2 u - \mu^2 \Delta^\alpha u) \end{aligned} \quad (4.3.8a)$$

$$\begin{aligned}
 & JR(\partial_\theta^2 w - \partial_\theta u) - JR\lambda(\partial_\theta^4 w - \partial_{R\theta}^3 u) + KR\partial_\theta^2 \phi - KR\lambda\partial_\theta^4 \phi \\
 & - (\aleph \nabla^2 \theta + N_T) R \partial_\theta^2 w + \mu^2 (\aleph \nabla^2 \theta + N_T) R \Delta^\alpha w \\
 & - K_{\text{Shear}} C \left(\frac{w}{R} + \partial_\theta u - \phi \right) - K_{\text{Shear}} C \lambda (\partial_{R\theta}^2 w + \partial_\theta^3 u - \partial_\theta^2 \phi) \quad (4.3.8b)
 \end{aligned}$$

$$\begin{aligned}
 & = (I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2}) R \partial_t^2 w + (I_1 + \frac{I_2}{R}) R \partial_t^2 \phi \\
 & - \mu^2 (I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2}) R \Delta^\alpha w - \mu^2 (I_1 + \frac{I_2}{R}) R \Delta^\alpha \phi \\
 & K(\partial_\theta^2 w - \partial_{R\theta} u) - K\lambda(\partial_\theta^4 w - \partial_{R\theta}^3 u) + L\partial_\theta^2 \phi - L\lambda\partial_\theta^4 \phi \\
 & + K_{\text{Shear}} C \left(\frac{w}{R} + \partial_\theta u - \phi \right) - K_{\text{Shear}} C \lambda (\partial_{R\theta}^2 w + \partial_\theta^3 u - \partial_\theta^2 \phi) \quad (4.3.8c) \\
 & = (I_1 + \frac{I_2}{R}) \partial_t^2 w + I_2 \partial_t^2 \phi - \mu^2 (I_1 + \frac{I_2}{R}) \Delta^\alpha w - \mu^2 I_2 \Delta^\alpha \phi
 \end{aligned}$$

5 The Solution method

To determine the free vibration of a curved porous nanobeam with simple supports under a thermal environment, the governing equations of curved nanobeams are solved analytically using the Navier approach.. The displacement quantities are given as trigonometric functions in order to meet the simply supported boundary condition:

$$w(\theta, t) = \sum_{n=1}^{\infty} W_n \cos\left(\frac{n\pi}{L}\theta\right) e^{i\omega_n t}, \quad (5.1a)$$

$$u(\theta, t) = \sum_{n=1}^{\infty} U_n \sin\left(\frac{n\pi}{L}\theta\right) e^{i\omega_n t}, \quad (5.1b)$$

$$\varphi(\theta, t) = \sum_{n=1}^{\infty} \phi_n \cos\left(\frac{n\pi}{L}\theta\right) e^{i\omega_n t}, \quad (5.1c)$$

where, the coefficients W_n , U_n and ϕ_n are unknowns that are calculated for each specific value of n and the boundary condition is given by:

$$w(0) = 0, \quad \partial_\theta w(L) = 0, \quad u(0) = u(L) = 0, \quad \partial_\theta \phi(0) = \partial_\theta \phi(L) = 0. \quad (5.2)$$

Substituting Eqs. (5.1) into Eqs. (4.3.8) yields the resulting from presented in Eqs. (5.3):

$$\begin{aligned}
 & J\left(-\frac{U_n}{R} - \frac{n\pi}{L} W_n\right) + J\lambda\left(-\frac{U_n}{R}\left(\frac{n\pi}{L}\right)^2 - \left(\frac{n\pi}{L}\right)^3 W_n\right) - K\left(\frac{n\pi}{L}\right) \varphi_n \\
 & - \left(\frac{n\pi}{L}\right)^3 K\lambda\varphi + K_{\text{Shear}} CR\left[-R^{-1} W_n\left(\frac{n\pi}{L}\right) + \varphi_n\left(\frac{n\pi}{L}\right) - U_n\left(\frac{n\pi}{L}\right)^2\right] \\
 & + K_{\text{Shear}} CR\lambda\left[-\frac{1}{R}\left(\frac{n\pi}{L}\right)^3 W_n - \left(\frac{n\pi}{L}\right)^4 U_n + \left(\frac{n\pi}{L}\right)^3 \varphi_n\right] \\
 & = RI_0\left(-\omega_n^2 U_n - \mu^2 \Delta^\alpha \omega_n^2 \left(\frac{n\pi}{L}\right)^2 U_n\right), \quad (5.3a)
 \end{aligned}$$

$$\begin{aligned}
 & JR \left[-\frac{1}{R} \left(\frac{n\pi}{L} \right) U_n - \left(\frac{n\pi}{L} \right)^2 W_n \right] + JR\lambda \left[-\frac{1}{R} \left(\frac{n\pi}{L} \right)^3 U_n - \left(\frac{n\pi}{L} \right)^4 W_n \right] \\
 & - KR \left(\frac{n\pi}{L} \right)^2 \varphi_n - KR\lambda \left(\frac{n\pi}{L} \right)^4 \varphi_n - (\nabla^2 \theta + N_T) R \left(\frac{n\pi}{L} \right)^2 W_n \\
 & - \mu^2 \Delta^\alpha (\nabla^2 \theta + N_T) R \left(\frac{n\pi}{L} \right)^4 W_n - K_{\text{Shear}} C \left[\frac{1}{R} W_n + \left(\frac{n\pi}{L} \right) U_n - \varphi_n \right] \\
 & - K_{\text{Shear}} C \lambda \left[\frac{1}{R} \left(\frac{n\pi}{L} \right)^2 W_n + \left(\frac{n\pi}{L} \right)^3 U_n - \left(\frac{n\pi}{L} \right)^2 \varphi_n \right] \\
 & = - \left(I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2} \right) R \omega_n^2 W_n - \left(I_1 + \frac{I_2}{R} \right) R \omega_n^2 \varphi_n \\
 & - \mu^2 \Delta^\alpha R \left(\frac{n\pi}{L} \right)^2 \left(I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2} \right) \omega_n^2 W_n \\
 & - \mu^2 \Delta^\alpha \left(I_1 + \frac{I_2}{R} \right) R \left(\frac{n\pi}{L} \right)^2 \omega_n^2 \varphi_n, \tag{5.3b}
 \end{aligned}$$

$$\begin{aligned}
 & KR \left[-\frac{1}{R} \left(\frac{n\pi}{L} \right) U_n + \left(\frac{n\pi}{L} \right)^2 W_n \right] + KR\lambda \left[-\frac{1}{R} \left(\frac{n\pi}{L} \right)^3 U_n + \left(\frac{n\pi}{L} \right)^4 W_n \right] \\
 & - RL \left(\frac{n\pi}{L} \right)^2 \varphi_n - RL\lambda \left(\frac{n\pi}{L} \right)^4 \varphi_n + K_{\text{Shear}} CR \left[\frac{1}{R} W_n + \left(\frac{n\pi}{L} \right) U_n - \varphi_n \right] \\
 & + K_{\text{Shear}} CR\lambda \left[\frac{1}{R} \left(\frac{n\pi}{L} \right)^2 W_n + \left(\frac{n\pi}{L} \right)^3 U_n - \left(\frac{n\pi}{L} \right)^2 \varphi_n \right] \\
 & = -R \left(I_1 + \frac{I_2}{R} \right) \omega_n^2 W_n - RI_2 \omega_n^2 \varphi_n \\
 & - R\mu^2 \Delta^\alpha \left(I_1 + \frac{I_2}{R} \right) \left(\frac{n\pi}{L} \right)^2 \omega_n^2 W_n - R\mu^2 \Delta^\alpha I_2 \left(\frac{n\pi}{L} \right)^2 \omega_n^2 \varphi_n. \tag{5.3c}
 \end{aligned}$$

The natural frequencies ω_n are determined by computing the matrix determinant of the coefficients from given equations are used, and resulting polynomial is equated to zero.

$$\{[K] - \omega_n^2 [M]\} \begin{Bmatrix} U_n \\ W_n \\ \varphi_n \end{Bmatrix} = 0 \tag{5.4}$$

Here, K represents the stiffness matrix and M represents the mass matrix, which are given in the ‘‘Appendix’’.

6 Numerical results

Numerical comparison is made with existing literature for perfect curved FG nanobeams, as discussed by Hosseini and Rahmani [28] in Tables 2, across various gradient indices and opening angles, regardless of temperature rise. The results demonstrate excellent consistency with the reference data.

Table 1: Coefficients of density of mass, thermal conductivity, coefficient of thermal, and Young's modulus that depend on temperature for Si_3N_4 and $SUS304$ [36].

Material	Property	P_0	P_{-1}	P_1	P_2	P_3
Si_3N_4	E (Pa)	348.44×10^9	0	-3.07×10^{-4}	2.16×10^{-7}	-8.94×10^{-11}
	λ_1 (K^{-1})	5.872×10^{-6}	0	9.09×10^{-4}	0	0
	ρ (kg/m ³)	2371	0	0	0	0
	κ (W/mK)	13.723	0	-1.03×10^{-3}	5.46×10^{-7}	-7.87×10^{-11}
SUS304	E (Pa)	201.04×10^9	0	3.079×10^{-4}	-6.534×10^{-7}	0
	λ_1 (K^{-1})	12.330×10^{-6}	0	8.086×10^{-4}	0	0
	ρ (kg/m ³)	8166	0	0	0	0
	κ (W/mK)	15379	0	-1.26×10^{-3}	2.092×10^{-6}	-7.223×10^{-10}

Subsequently, the effects of nonlocal parameters, strain gradient terms, porosity levels, temperature variations, gradient index values, curvature, and aspect ratios on the natural frequencies under thermal loading are investigated. Eq.(4.3.8) offers a formula to compute the non-dimensional natural frequency (λ).

$$\lambda = \omega \frac{L^2}{h} \sqrt{\frac{\rho_u}{E_u}} \quad (6.1)$$

Table 2: Comparison of natural frequencies (ω_n) of nanobeams for different aspect ratios L/h and nonlocal parameters $(ea)^2$.

$\frac{L}{h}$	ω_n	$(ea)^2 = 0$		$(ea)^2 = 1$		$(ea)^2 = 2$		$(ea)^2 = 3$		$(ea)^2 = 4$	
		Ref. [30]	Article	Ref. [30]	Article	Ref. [30]	Article	Ref. [30]	Article	Ref. [30]	Article
10	$n = 1$	8.199	8.199	7.822	7.822	7.493	7.493	7.202	7.202	6.943	6.943
	$n = 2$	35.745	35.745	30.267	30.267	26.720	26.720	24.186	24.186	22.258	22.258
	$n = 3$	77.399	77.399	56.326	56.326	46.450	46.450	40.431	40.431	36.273	36.273
20	$n = 1$	8.292	8.292	7.910	7.910	7.577	7.577	7.283	7.283	7.020	7.020
	$n = 2$	37.288	37.288	31.573	31.573	27.873	27.873	25.229	25.229	23.218	23.218
	$n = 3$	84.319	84.319	61.361	61.361	50.602	50.602	44.045	44.045	39.516	39.516
50	$n = 1$	8.318	8.318	7.935	7.935	7.601	7.601	7.306	7.306	7.043	7.043
	$n = 2$	37.766	37.766	31.978	31.978	28.231	28.231	25.553	25.553	23.516	23.516
	$n = 3$	86.708	86.708	63.100	63.100	52.037	52.037	45.294	45.294	40.636	40.636

Table 3: Influence of P and ΔT on the fundamental frequency of FG curved nanobeams with varying strain gradient, porosity, curvature, and thermal load $\frac{L}{h} = 30, (ea)^2 = 2$

θ	Load type	$\Delta T = 30 [K]$			$\Delta T = 60 [K]$			$\Delta T = 90 [K]$		
		$P = 0$	$P = 0.5$	$P = 1$	$P = 0$	$P = 0.5$	$P = 1$	$P = 0$	$P = 0.5$	$P = 1$
$\alpha = \frac{\pi}{6}$										
0	$\lambda = 0$	6.5399	5.5380	4.8444	6.4502	5.4505	4.7589	6.3603	5.3568	4.6666
	$\lambda = 2$	7.7179	6.0753	5.3160	7.0904	5.9960	5.2387	6.4698	5.9111	5.1552
0.1	$\lambda = 0$	6.8107	5.6266	4.8418	6.7297	5.5482	4.7655	6.6436	5.4644	4.6831
	$\lambda = 2$	7.4668	6.1708	5.3117	7.3961	6.0972	5.2407	7.3149	6.0235	5.1681
0.2	$\lambda = 0$	7.1850	5.7388	4.8351	7.1153	5.6694	4.7680	7.0343	5.5950	4.6954
	$\lambda = 2$	7.8722	6.2919	5.3028	7.8008	6.2291	5.2422	7.7388	6.1616	5.1765
$\alpha = \frac{\pi}{4}$										
0	$\lambda = 0$	5.9831	5.0550	4.4132	5.8217	4.8901	4.2513	5.6444	4.7113	4.0747
	$\lambda = 2$	6.5759	5.5597	4.8570	6.4234	5.4366	4.7042	6.1655	5.2498	4.5522
0.1	$\lambda = 0$	6.2414	5.1446	4.4183	6.0898	4.9917	4.2741	5.9256	4.8788	4.1172
	$\lambda = 2$	6.8557	5.6549	4.8597	6.7182	5.5214	4.7294	6.5715	5.3778	4.5883
0.2	$\lambda = 0$	6.5953	5.2559	4.4196	6.4600	5.1257	4.2929	6.3157	4.9855	4.1554
	$\lambda = 2$	7.2420	5.7738	4.8582	7.1113	5.6559	4.7437	6.9867	5.5293	4.6198
$\alpha = \frac{\pi}{3}$										
0	$\lambda = 0$	4.4773	3.7430	3.2362	4.0290	3.2754	2.7688	3.4896	2.7039	2.1776
	$\lambda = 2$	4.9577	4.1712	3.6128	4.5595	3.7577	3.2047	4.0812	3.2717	2.7107
0.1	$\lambda = 0$	4.7116	3.8440	3.2762	4.2823	3.4126	2.8607	3.8410	2.9344	2.3593
	$\lambda = 2$	5.2240	4.2735	3.6446	4.8397	4.0052	3.4560	4.4321	3.4862	2.8566
0.2	$\lambda = 0$	5.0212	3.9619	3.3008	4.6564	3.6042	2.9466	4.2434	3.1890	2.5248
	$\lambda = 2$	5.5424	4.3943	3.6698	5.2279	4.0751	3.5533	4.8637	3.7190	2.9919

Table 3 presents the frequency results of porous FG curved nanobeams under varying thermal conditions, considering different power-law indices, porosity levels and opening angles. The dimensionless frequency decreases when the material gradient index rises because of the greater flexibility. Similarly, temperature rise leads to a decrease in frequency, whereas higher porosity values result in increased frequencies for all opening angles and temperatures. The power-law index's impact is reversed when $p > 1$, showing a drop in frequency. Additionally, a higher length scale parameter increases the natural frequency, and frequency tends to reduce as the opening angle increased.

Table 4: Effect of nonlocal parameter and gradient index on fundamental frequency of S-S FG curved nanobeams under different porosity, strain gradient and thermal conditions, $\frac{L}{h} = 30, \alpha = \frac{\pi}{2}$.

μ^2	P	λ	Fundamental frequency							
			$\vartheta = 0$				$\vartheta = 0.1$			
			$\Delta T = 30$	$\Delta T = 60$	$\Delta T = 90$	$\Delta T = 120$	$\Delta T = 30$	$\Delta T = 60$	$\Delta T = 90$	$\Delta T = 120$
1	0.2	0	5.234	4.982	4.756	4.556	5.156	4.908	4.684	4.487
		2	4.872	4.632	4.416	4.224	4.801	4.565	4.352	4.163
	0.5	0	4.754	4.525	4.318	4.134	4.688	4.463	4.259	4.078
		2	4.431	4.213	4.016	3.840	4.369	4.154	3.960	3.787
	1	0	4.214	4.006	3.818	3.650	4.158	3.953	3.768	3.603
		2	3.927	3.729	3.550	3.390	3.873	3.678	3.501	3.343
2	0.2	0	4.682	4.456	4.250	4.064	4.620	4.397	4.194	4.011
		2	4.361	4.146	3.950	3.773	4.302	4.090	3.897	3.722
	0.5	0	4.254	4.048	3.860	3.690	4.198	3.994	3.808	3.641
		2	3.981	3.784	3.606	3.445	3.928	3.733	3.557	3.399
	1	0	3.782	3.594	3.423	3.270	3.732	3.546	3.378	3.227
		2	3.542	3.361	3.197	3.050	3.494	3.315	3.152	3.006
3	0.2	0	4.264	4.062	3.878	3.711	4.213	4.013	3.830	3.665
		2	3.978	3.784	3.607	3.446	3.930	3.738	3.563	3.405
	0.5	0	3.874	3.690	3.522	3.370	3.829	3.646	3.479	3.329
		2	3.629	3.452	3.291	3.146	3.586	3.410	3.250	3.107
	1	0	3.449	3.272	3.111	2.965	3.407	3.232	3.073	2.931
		2	3.234	3.064	2.910	2.770	3.194	3.025	2.872	2.734

Table 4 shows that the fundamental frequency decreases with increasing temperature and nonlocal parameter, indicating a stiffness-softening response due to nonlocal effects. Conversely, an increase in the length scale parameter raises the frequency, reflecting a stiffness-hardening behavior. This trend holds across various porosity levels and material gradation profiles. The results emphasize that both nonlocality and length scale significantly affect vibrational characteristics, and their combined influence must be considered for accurate modeling of FG nanobeams under thermal environments and size-dependent effects.

Figures 2–4 illustrate the variation of normal stress with respect to the gradient index for different strain gradient parameters ($\lambda = 0, 1, 2, 3$) under increasing porosity conditions ($\vartheta = 0, 0.1, 0.2$). In all cases, the normal stress decreases sharply as the gradient index increases, indicating a stress-relieving effect of material gradation.

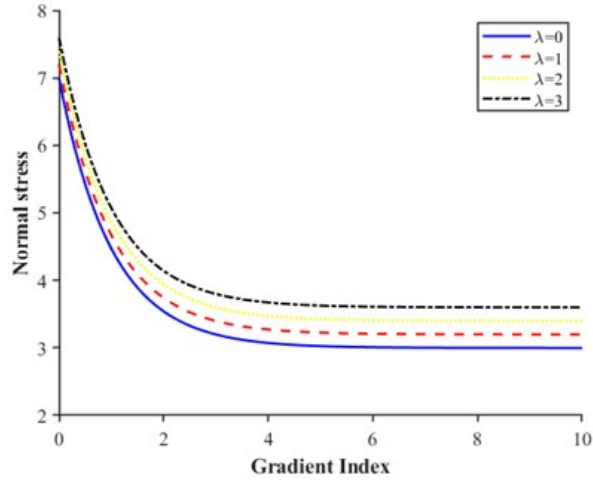


Figure 2: Normal stress along Gradient Index for different values of strain gradient parameters at $\vartheta = 0$.

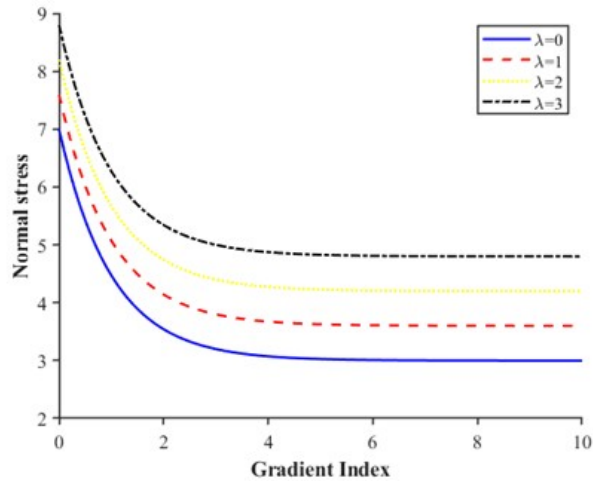


Figure 3: Normal stress along Gradient Index for different values of strain gradient parameters at $\vartheta = 0.1$.

At lower values of the gradient index (near 0), the difference in stress values

for various λ is more prominent, showing the strong influence of strain gradient effects. As ϑ increases from 0 to 0.2, the overall stress level rises slightly, and the stress curves become steeper initially, but flatten out faster for higher λ values, emphasizing the combined influence of thermal loading and gradient elasticity.

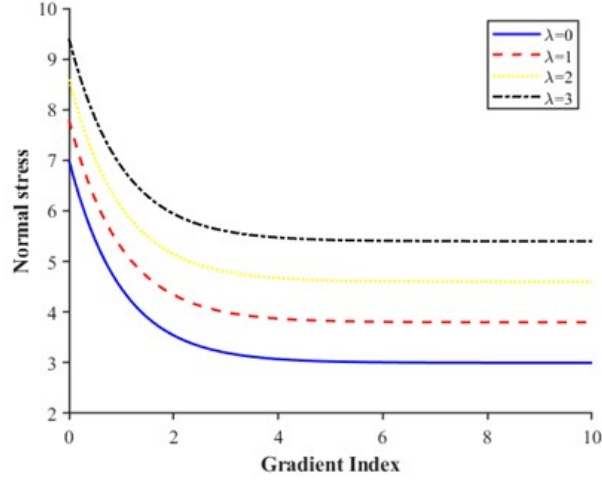


Figure 4: Normal stress along Gradient Index for different values of strain gradient parameters at $\vartheta = 0.2$.

Figures 5–7 present the normal strain distribution with respect to porosity for various strain gradient parameters at increasing values of the power-law index ($P = 1, 2, 3$). As porosity increases, the normal strain also increases, suggesting that higher void content leads to greater material deformation. The effect of strain gradient parameter λ becomes more noticeable at higher porosity values, with $\lambda = 3$ consistently showing the highest strain values. Furthermore, as the power-law index increases, the overall strain values also increase across all porosity values, highlighting the combined effect of porosity and material gradation on the structural response.

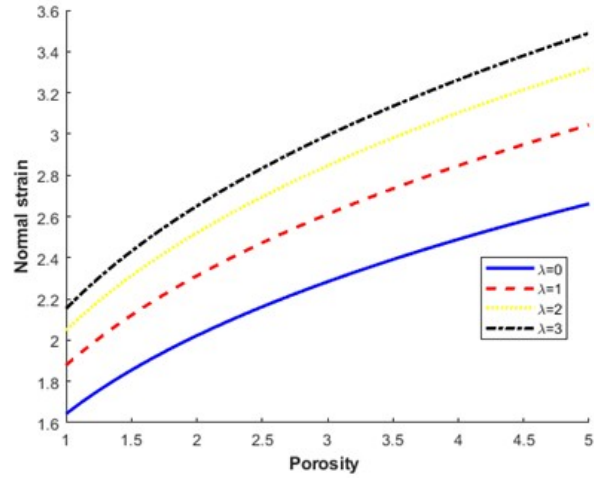


Figure 5: Normal strain along porosity for different values of strain gradients at $\mathbf{P} = 1$.

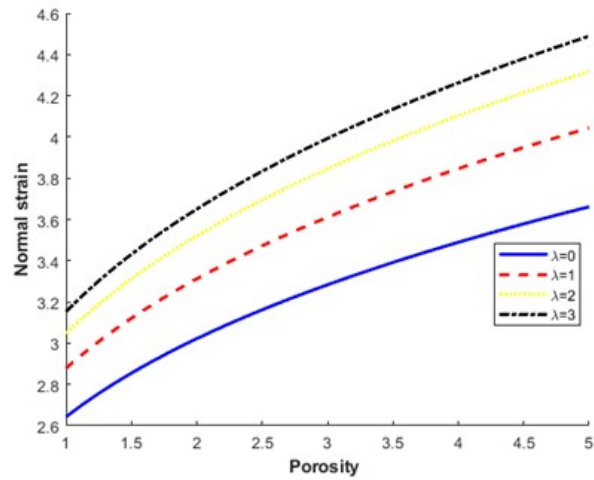


Figure 6: Normal strain along porosity for different values of strain gradients at $\mathbf{P} = 2$.

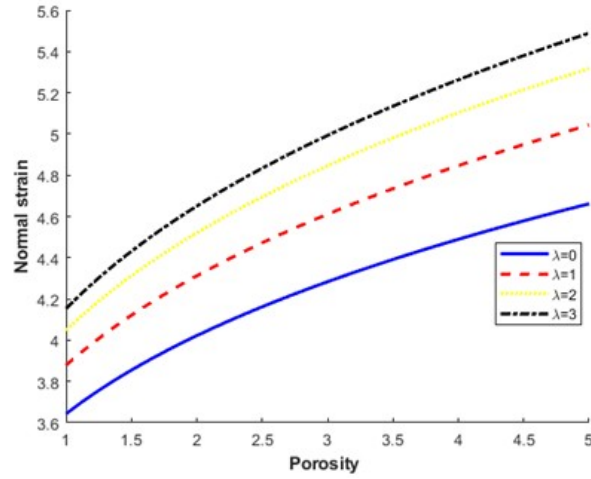


Figure 7: Normal strain along porosity for different values of strain gradients at $\mathbf{P} = 3$.

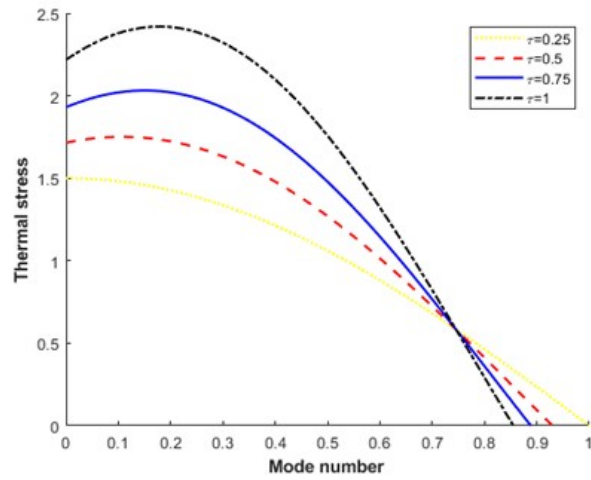


Figure 8: Thermal stress along mode number for different values of time delay factors at $\mathbf{L/H} = 10$.

Figures 8–10 depict thermal stress behavior along normalized mode number

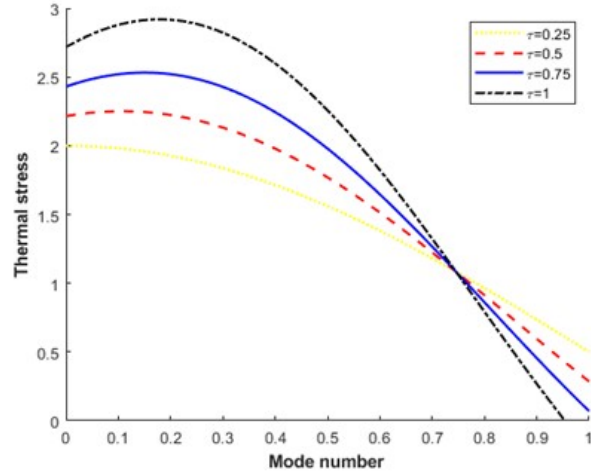


Figure 9: Thermal stress along mode number for different values of time delay factors at $\mathbf{L/H = 20}$.

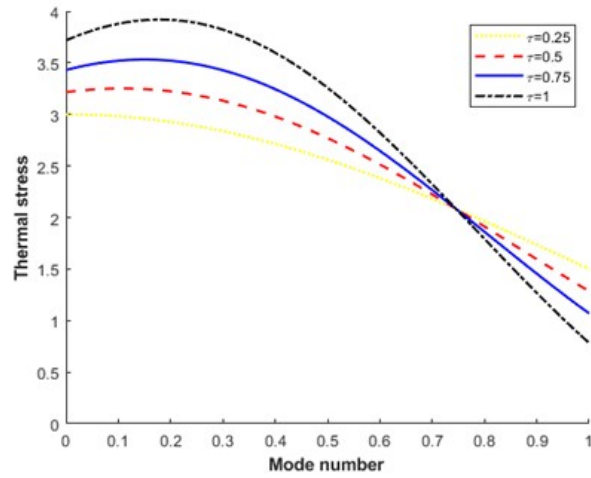


Figure 10: Thermal stress along mode number for different values of time delay factors at $\mathbf{L/H = 30}$.

for different time delay factors ($\tau = 0.25, 0.5, 0.75, 1$) at increasing thickness-

to-length ratios ($L/H = 10, 20, 30$). Thermal stress initially rises with mode number but then declines sharply after reaching a peak. This peak shifts slightly based on the time delay parameter. Higher values of τ reduce the peak thermal stress, indicating that delayed thermal responses help dissipate stress. Also, as L/H increases from 10 to 30, the magnitude of peak thermal stress increases, which shows the role of geometric scaling in intensifying thermal effects in such structures. Figures 11–13 show the variation of frequency parameter with gradient index for different porosity parameters ($\vartheta = 0$ to 0.3) under rising thermal gradients ($\Delta T = 30, 60, 90$).

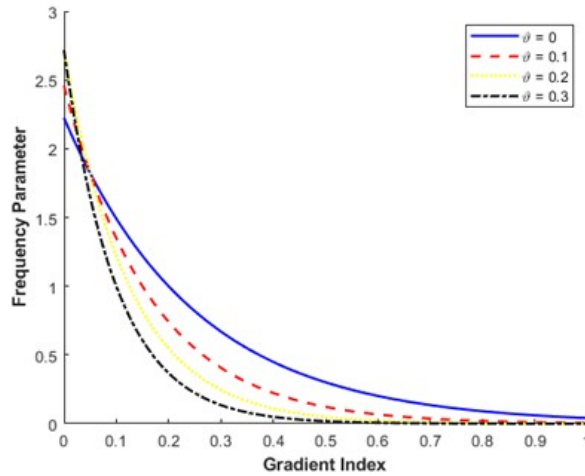


Figure 11: Frequency parameter along gradient index for different values of porosity parameters at $\Delta T = 30$.

Across all temperature conditions, the frequency parameter decreases rapidly with increasing gradient index, particularly for higher porosity levels. This reduction becomes more significant as the porosity parameter increases, implying that the structural stiffness diminishes with both material gradation and void content. Moreover, the frequency values increase overall with higher ΔT , which suggests that thermal stiffening slightly offsets the softening induced by porosity and gradient index.

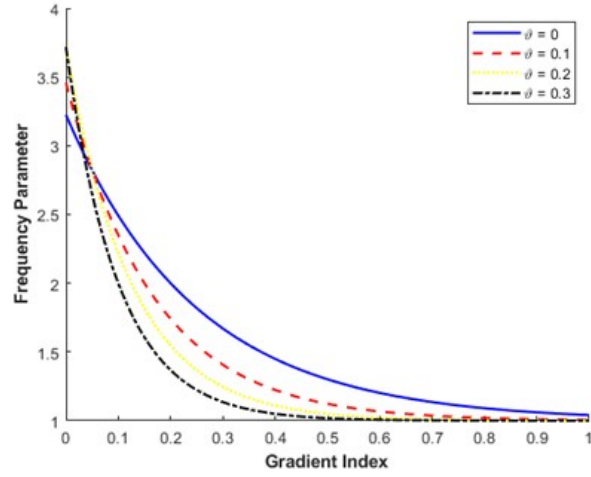


Figure 12: Frequency parameter along gradient index for different values of porosity parameters at $\Delta T = 60$.

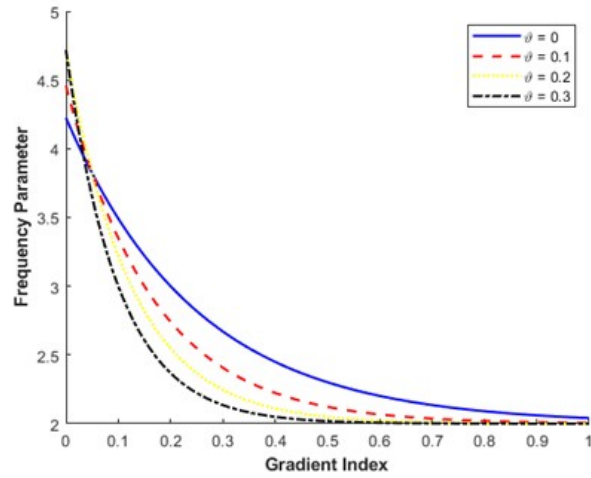


Figure 13: Frequency parameter along gradient index for different values of porosity parameters at $\Delta T = 90$.

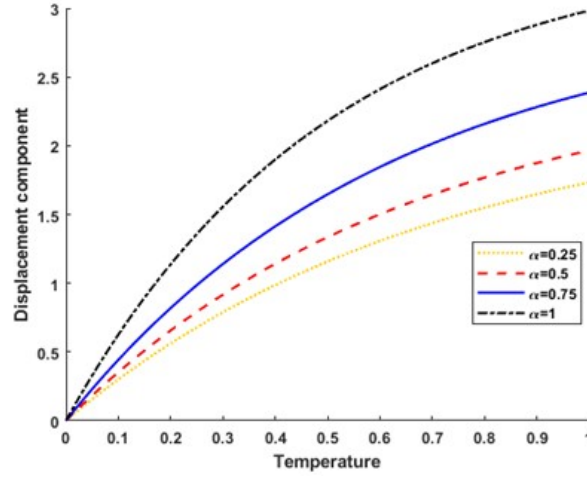


Figure 14: Displacement component along temperature for different values of fractal dimensions at $\mu^2 = 1$.

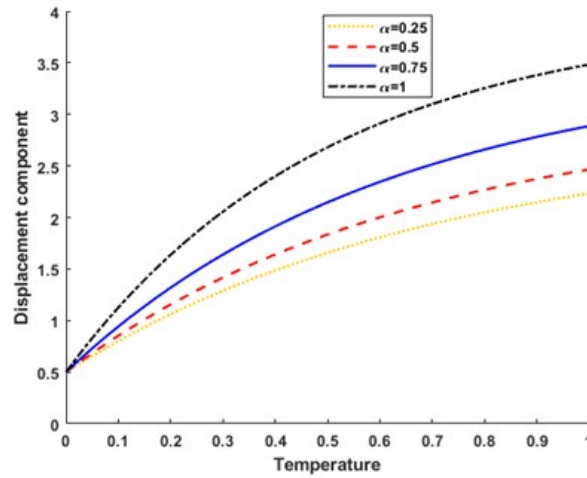


Figure 15: Displacement component along temperature for different values of fractal dimensions at $\mu^2 = 2$.

Figures 14 to 16 illustrate the effect of fractal dimensions on the dis-

placement component with increasing temperature for different values of the nonlocal parameter. In all cases, displacement increases monotonically with temperature, and higher fractal dimensions ($\alpha = 1$) consistently yield larger displacement values compared to lower dimensions ($\alpha = 0.25$).

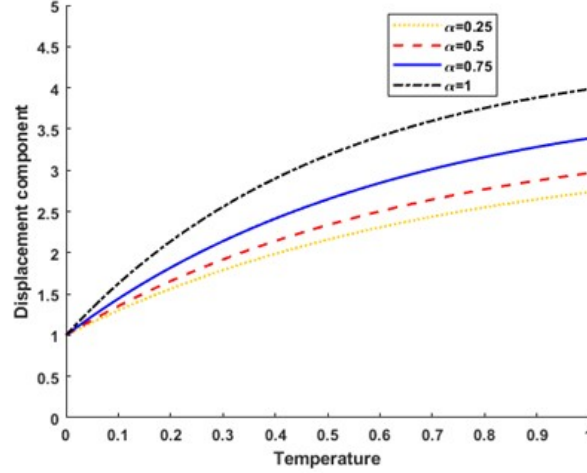


Figure 16: Displacement component along temperature for different values of fractal dimensions at $\mu^2 = 3$.

This behavior reflects the sensitivity of displacement to the microstructural irregularity captured by the fractal dimension. As the nonlocal parameter increases from 1 to 3, the magnitude of displacement also increases across all temperature values, showing the role of nonlocal effects in enhancing thermal deformation. The gap between curves for different α widens with higher temperature and larger nonlocal parameter values, indicating a more pronounced influence of the fractal dimension in such regimes.

7 Conclusion

This study examines the vibrational behavior of porous functionally graded nanobeams subjected to various loading conditions. The analysis is conducted using the nonlocal strain gradient elasticity theory, which simultaneously accounts for nonlocal effects and material length scale parameters. Material properties are temperature-dependent and graded radially via a modified power-law model assuming uniform porosity. To capture time-dependent

material behavior, a memory-dependent derivative is included, accounting for hereditary effects in the thermo-mechanical response. Energy dissipation due to viscoelastic behavior and wave dispersion influenced by curvature and porosity are also evaluated in the dynamic model. Additionally, fractal-based modeling is employed to better represent the complexity of the beam's geometry and its dynamic behavior across multiple scales. Results, validated with existing literature, reveal that vibration characteristics are significantly affected by material gradation index, porosity level, temperature environment, length scale, nonlocality, curvature angle, fractal influence, gradient index, and memory effect parameters. The numerical findings indicate that:

- Increasing the gradient index significantly lowers normal stress and frequency, confirming the stress-relieving and softening effect of material gradation under thermal and porous conditions.
- Higher strain gradient parameters lead to increased stress and strain responses, especially at low gradient index and high porosity, highlighting size-dependent mechanical behavior.
- As porosity and power-law index increase, normal strain rises sharply, indicating enhanced material deformation due to void content and gradation effects.
- Thermal stress increases with mode number up to a peak, then drops; this behavior is strongly influenced by time delay and geometry, showing their role in thermal wave response.
- Displacement grows with temperature, and further rises with higher fractal dimension and nonlocal parameters, emphasizing the influence of microstructural irregularities and long-range interactions.

Appendix

The stiffness matrix K and mass matrix M used in Equation (37) are presented below:

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}, \quad M = \begin{bmatrix} M_{11} & 0 & 0 \\ 0 & M_{22} & M_{23} \\ 0 & M_{32} & M_{33} \end{bmatrix}$$

where

$$\begin{aligned}
 K_{11} &= -J \frac{1}{R} \left(\frac{n\pi}{L} \right) + J \lambda \left(\frac{n\pi}{L} \right)^2 - K_{\text{Shear}} C R \left(\frac{n\pi}{L} \right)^2 - K_{\text{Shear}} C R \lambda \left(\frac{n\pi}{L} \right)^4, \\
 K_{12} &= -J \left(\frac{n\pi}{L} \right) - J \lambda \left(\frac{n\pi}{L} \right)^3 - K_{\text{Shear}} C \left(\frac{n\pi}{L} \right) - K_{\text{Shear}} C \lambda \left(\frac{n\pi}{L} \right)^3, \\
 K_{13} &= -K \left(\frac{n\pi}{L} \right) - K \lambda \left(\frac{n\pi}{L} \right)^3 + K_{\text{Shear}} C R \left(\frac{n\pi}{L} \right) + K_{\text{Shear}} C R \lambda \left(\frac{n\pi}{L} \right)^3, \\
 K_{21} &= - \left[J \left(\frac{n\pi}{L} \right) + J \lambda \left(\frac{n\pi}{L} \right)^3 + K_{\text{Shear}} C \left(\frac{n\pi}{L} \right) + K_{\text{Shear}} C \lambda \left(\frac{n\pi}{L} \right)^3 \right], \\
 K_{22} &= -J R \left(\frac{n\pi}{L} \right)^2 - J R \lambda \left(\frac{n\pi}{L} \right)^4 - (\aleph \nabla^2 \theta + N_T) R \left(\frac{n\pi}{L} \right)^2 \\
 &\quad - \mu^2 \Delta^\alpha (\aleph \nabla^2 \theta + N_T) R \left(\frac{n\pi}{L} \right)^4 - K_{\text{Shear}} \frac{C}{R} - K_{\text{Shear}} \frac{C}{R} \lambda \left(\frac{n\pi}{L} \right)^2, \\
 K_{23} &= -K R \left(\frac{n\pi}{L} \right)^2 - K R \lambda \left(\frac{n\pi}{L} \right)^4 + K_{\text{Shear}} C + K_{\text{Shear}} C \lambda \left(\frac{n\pi}{L} \right)^2, \\
 K_{31} &= -K \left(\frac{n\pi}{L} \right) - K \lambda \left(\frac{n\pi}{L} \right)^3 + K_{\text{Shear}} C R \left(\frac{n\pi}{L} \right) + K_{\text{Shear}} C R \lambda \left(\frac{n\pi}{L} \right)^3, \\
 K_{32} &= K R \left(\frac{n\pi}{L} \right)^2 + K R \lambda \left(\frac{n\pi}{L} \right)^4 + K_{\text{Shear}} C + K_{\text{Shear}} C \lambda \left(\frac{n\pi}{L} \right)^2, \\
 K_{33} &= - \left[R L \left(\frac{n\pi}{L} \right)^2 + R L \lambda \left(\frac{n\pi}{L} \right)^4 + K_{\text{Shear}} C R + K_{\text{Shear}} C R \lambda \left(\frac{n\pi}{L} \right)^2 \right].
 \end{aligned}$$

and

$$\begin{aligned}
 M_{11} &= -RI_0 \left(1 + \mu^2 \Delta^\alpha \left(\frac{n\pi}{L} \right)^2 \right), \\
 M_{22} &= -R \left(I_0 + \frac{2I_1}{R} + \frac{I_2}{R^2} \right) \left(1 + \mu^2 \Delta^\alpha \left(\frac{n\pi}{L} \right)^2 \right), \\
 M_{23} &= -R \left(I_1 + \frac{I_2}{R} \right) \left(1 + \mu^2 \Delta^\alpha \left(\frac{n\pi}{L} \right)^2 \right), \\
 M_{32} &= -R \left(I_1 + \frac{I_2}{R} \right) \omega_n^2 - R\mu^2 \Delta^\alpha \left(I_1 + \frac{I_2}{R} \right) \left(\frac{n\pi}{L} \right)^2 \omega_n^2, \\
 M_{33} &= -RI_2 - R\mu^2 \Delta^\alpha I_2 \left(\frac{n\pi}{L} \right)^2.
 \end{aligned}$$

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Rajendran Selvamani,
Department of Mathematics,
Karunya Institute of Technology and Sciences,
Karunya Nagar, Coimbatore, 641114 Tamilnadu, India.
Email: selvam1729@gmail.com

Thangamuni Prabhakaran,
Department of Mathematics,
Karunya Institute of Technology and Sciences,
Karunya Nagar, Coimbatore, 641114 Tamilnadu, India.
Email: prabha251999@gmail.com

Marin Marin,
Department of Mathematics and Computer Science,
Transilvania University of Brasov,
Brasov 500036, Romanian
Email: m.marin@unitbv.ro
Academy of Romanian Scientists,
Iifov street, 3, 050045 Bucharest,
Romania