



Existence results for a class of hemivariational-like inequality problems

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Abstract

In this work, we establish existence results for a class of hemivariational-like inequality problems driven by a trifunction and perturbed by a bifunction. Initially, we establish solvability results based on the weak compactness of the constraint set. Subsequently, under appropriate coercivity conditions and in the absence of boundedness of the constraint set, we show the existence results and boundedness properties of the solution set. Furthermore, several theoretical applications are provided, and a necessary and sufficient condition for the existence of solutions to a particular subclass of these problems is derived.

1 Introduction

Hemivariational inequality problems (HVIPs) constitute an essential class of generalizations of classical variational inequalities, originally introduced by Panagiotopoulos [1] to elaborate on the non-smooth energy functions, multi-valued conservative laws, non-monotone mechanical problems, and non-potential laws. The HVIPs are analyzed through the notion of generalized gradient of Clarke [2] of locally Lipschitz, non-differentiable functionals. From then on, over the years, several existence results associated with HVIPs have been extensively considered in different space aspects (see [3–11]). By using the sub-supersolution method, Carl et al. [12] investigated the existence outcomes

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for quasilinear elliptic HVIPs, whereas Carl et al. [13] studied the existence consequences of the evolutionary quasilinear HVIPs. By applying the method of the first eigenfunction, Liu [14] derived the existence results of the periodic problem for quasilinear parabolic HVIP. Goeleven et al. [15] considered the eigenvalue problem for HVIP at resonance. Recently, Tang et al. [8] extended the study of Hartman-Stampacchia-type HVIPs from the Euclidean space to the context of Hadamard manifolds. For more related literature on the existence results of HVIPs, we refer the reader to [4, 6, 10, 11, 16–26].

In the last decade, special attention was given to the existence impact of the HVIPs led through the multi-valued map, the concept of stably f -quasimonotonicity was introduced in [22], and the notion of stably (f, η) -quasimonotonicity was introduced in [21]. In addition, the existence results obtained in [22] and [21] were the extension results of Tang and Huang [4]. Besides, Costea and Pitea [7] focused on the existence issues of mixed hemivariational-like inequality problems of noninclusion type by using topological methods.

Let E be a real reflexive Banach space with dual space E^* and also let F be a real Banach space with dual space F^* . Let $C \subset E$ be a nonempty, closed, and convex set, $A : C \rightarrow 2^{E^*}$ be a set-valued mapping with nonempty values, $T : E^* \times C \times C \rightarrow \mathbb{R}$ be a trifunction, $J : F \rightarrow \mathbb{R}$ be a locally Lipschitz function, and $\mu : C \times C \rightarrow [-\infty, +\infty]$ be a given bifunction. The Clarke generalized directional derivative of J at $u \in F$ in the direction $v \in F$ is denoted by $J^\circ(u; v)$. We denote the zero elements of $\mathbb{R}$, F , E , and E^* by 0, 0_F , θ , and θ^* , respectively. We appoint the following condition on the domain of μ , i.e.,

$$D(\mu) = \{u \in C : \mu(u, v) \neq -\infty \text{ for all } v \in C\} \neq \emptyset.$$

In addition, let $\gamma : E \rightarrow F$ be a bounded linear operator, and let $\beta : E \times E \rightarrow E$ be a given mapping. We introduce and analyze a new class of hemivariational-like inequality problems called the *generalized hemivariational-like inequality problem (GHLIP)*. The GHLIP is governed by the trifunction T , perturbed by the bifunction μ . The GHLIP is given as follows:

$$\text{GHLIP} \left\{ \begin{array}{l} \text{find } u \in \mathcal{D}(\mu) \text{ and } u^* \in A(u) \text{ such that} \\ T(u^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \forall v \in C \end{array} \right. \quad (1.1)$$

Denote Θ as the set of all solutions of GHLIP. The duality pairing between E and E^* is signified by $\langle \cdot, \cdot \rangle$. Let us deduce the following special cases of GHLIP by assuming $F = L^p(\Gamma, \mathbb{R}^k)$, where $p \in (1, \infty)$, $k \geq 1$ and $\Gamma \subset \mathbb{R}^N$ be an open bounded set.

- (i) Let $\mu(u, v) = \rho(v) - \rho(u)$, where $\rho : E \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous (l.s.c.) such that $C \cap \text{dom} \rho \neq \emptyset$ (where,

$\text{dom} \rho = \{u \in E : \rho(u) < \infty\}$. Then, $\mathcal{D}(\mu) = C \cap \text{dom} \rho$. Moreover, if $T(u^*, u, v) = \langle u^*, \beta(v, u) \rangle$ for all $(u^*, u, v) \in E^* \times C \times C$, then the GHLIP reduces to find $u \in C \cap \text{dom} \rho$ and $u^* \in A(u)$ such that

$$\langle u^*, \beta(v, u) \rangle + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \rho(v) - \rho(u) \geq 0, \quad \forall v \in C. \quad (1.2)$$

Ceng et al. [22] studied the existence of solutions to inequality (1.2) by employing the concept of stable (ρ, β) -quasimonotonicity.

- (ii) By the assumption of (i) with $\beta(v, u) = v - u$, the GHLIP reduces to find $u \in C \cap \text{dom} \rho$ and $u^* \in A(u)$ such that

$$\langle u^*, v - u \rangle + J^\circ(\gamma(u); \gamma(v) - \gamma(u)) + \rho(v) - \rho(u) \geq 0, \quad \forall v \in C. \quad (1.3)$$

Tang and Huang [4] explored the solvability principles of inequality (1.3) through the notion of stable ρ -quasimonotonicity.

- (iii) If $T(u^*, u, v) = \langle u^*, v - u \rangle \forall (u^*, u, v) \in E^* \times C \times C$ and $\beta(v, u) = v - u$, then GHLIP weakens to find $u \in \mathcal{D}(\mu)$ and $u^* \in A(u)$ such that

$$\langle u^*, v - u \rangle + J^\circ(\gamma(u); \gamma(v) - \gamma(u)) + \mu(u, v) \geq 0, \quad \forall v \in C. \quad (1.4)$$

Wangkeeree and Preechasilp [21] developed the existence theory of inequality (1.4) by employing stable μ -quasimonotonicity.

- (iv) If $\mu(u, v) = 0 \forall (u, v) \in C \times C$, $T(u^*, u, v) = \langle u^*, \beta(v, u) \rangle \forall (u^*, u, v) \in E^* \times C \times C$ with $\beta(v, u) = v - u$, then GHLIP becomes to find $u \in C$ and $u^* \in A(u)$ such that

$$\langle u^*, v - u \rangle + J^\circ(\gamma(u); \gamma(v) - \gamma(u)) \geq 0, \quad \forall v \in C. \quad (1.5)$$

Zhang and He [6] elaborated on the existence results of inequality (1.5) by inferring the mapping A as lower hemicontinuous and stable quasimonotone.

The results obtained in [4, 6, 21, 22] were dealt with the KKM principle (see [27]) and Clarke's generalized gradient. Indeed, GHLIP is more general than the problem considered in [4, 6, 21, 22].

- (v) If $T = 0$, $\mu(u, v) = \Lambda(u, v) - \langle g^*, v - u \rangle$, and $\beta(v, u) = v - u$, where $\Lambda : C \times C \rightarrow \mathbb{R}$ and $g^* \in E^*$, then inequality (1.1) weakens to find $u \in C$ such that

$$\Lambda(u, v) + J^\circ(\gamma(u); \gamma(v) - \gamma(u)) \geq \langle g^*, v - u \rangle, \quad \forall v \in C. \quad (1.6)$$

Costea and Rădulescu [3] studied the existence of inequality (1.6). Further, if $T = 0$, $\gamma = 0_F$ and $\mu(u, v) = \Lambda(u, v) - \langle g^*, v - u \rangle$, then inequality (1.1) reduces to find $u \in C$ such that

$$\Lambda(u, v) \geq \langle g^*, v - u \rangle, \quad \forall v \in C. \quad (1.7)$$

Inequality (1.7) was considered by Fan [28] when $g^* = \theta_*$ and Ansari et al. [29] keeping $g^* \neq \theta_*$.

- (vi) Let $T(u^*, u, v) = \langle u^*, v - u \rangle \quad \forall (u^*, u, v) \in E^* \times C \times C, \mu = 0$, and $\beta(v, u) = v - u$. If A is single-valued, then GHLIP boils down to Hartman-Stampacchia HVIP, i.e., to find $u \in C$ such that

$$\langle A(u), v - u \rangle + J^\circ(\gamma(u); \gamma(v) - \gamma(u)) \geq 0, \quad \forall v \in C. \quad (1.8)$$

The inequality (1.8) was studied in [3, 30].

- (vii) Let $T(u^*, u, v) = \langle u^*, v - u \rangle \quad \forall (u^*, u, v) \in E^* \times C \times C, \mu = 0, \beta = \theta$. If A is single-valued, then GHLIP becomes to find $u \in C$ such that

$$\langle A(u), v - u \rangle \geq 0, \quad \forall v \in C. \quad (1.9)$$

The inequality (1.9) was studied by Facchinei and Pang [31] and is known as classical variational inequality.

The above analysis suggests that one can access the desired inequality by choosing T, A, μ, β , and γ suitably in GHLIP. Hence, it is worth to study and focus on the existence consequences of GHLIP and its significant characterizations. Inspired by the above research [4, 6, 21, 22] and the existence results studied for hemivariational-like inequality problem by Ceng et al. [32], we present the existence results of GHLIP in real reflexive Banach space for the boundedness of the constraint set C . We derive the existence results by adopting the classical KKM Lemma, Nadler's result, Clarke's generalized directional derivative, and the joint monotonicity of T and A . Besides, we also investigate the existence theorems of GHLIP, even if C is no longer bounded. If C is unbounded, we use some coercivity conditions to derive the desired existence results. Apart from this, we formulate a necessary and sufficient existence theorem of GHLIP in Section 4.

The work is planned in four sections. Some essential definitions and necessary tools have been revisited in Section 2. Section 3 delivers the existence results when C is bounded in E . The main findings of Section 3, i.e., Theorem 3.1 is followed by the Minty hemivariational-like inequality problems (MHLIP) along with an illustrative application presented in Theorem 3.2. Section 4 presents existence results for the case when the constraint set C is unbounded

and derived under appropriate coercivity conditions. It also focuses on establishing the boundedness of the solution set Θ and proves a necessary and sufficient condition for the existence of solutions to the GHLIP, as stated in Theorem 4.4. Finally, the paper concludes with a summary of the main results and insights.

2 Preliminaries and Mathematical Backgrounds

We denote the norms on the spaces E , E^* , F , and F^* by $\|\cdot\|$, $\|\cdot\|_*$, $\|\cdot\|_F$, and $\|\cdot\|_{F^*}$, respectively. The symbols $\rightarrow$, $\rightharpoonup$, and $\rightharpoonup^*$ are used to represent strong, weak, and weak* convergence, respectively. It is well known that the weak and weak* topologies coincide on E^* since E is reflexive.

Throughout the paper, the symbols C , β , T , A , J , γ , and μ retain the meanings introduced in the previous section. The domain and graph of the set-valued mapping A are denoted by $D(A)$ and $\text{Gr}(A)$, respectively.

Next, we recall some useful definitions, lemmas, and preliminary results that are essential to establish our main findings.

Definition 2.1. The mapping $\beta : C \times C \rightarrow E$ is said to be affine in its first argument if, for any $u_i \in C$ and $\lambda_i \geq 0$ with $1 \leq i \leq n$ such that $\sum_{i=1}^n \lambda_i = 1$,

$$\text{and for any } v \in C, \text{ we have } \beta\left(\sum_{i=1}^n \lambda_i u_i, v\right) = \sum_{i=1}^n \lambda_i \beta(u_i, v).$$

Definition 2.2. ([7]). The mapping $A : D(A) \rightarrow 2^{E^*}$ is l.s.c. at $w \in E$, if for any open set $W \subset E^*$ such that $A(w) \cap W \neq \emptyset$, there exists a neighbourhood U of w such that $A(v) \cap W \neq \emptyset$ for all $v \in U$.

Definition 2.3. ([7]). The mapping $A : D(A) \rightarrow 2^{E^*}$ is upper semicontinuous (u.s.c.) at $w \in E$, if for any open set $W \subset E^*$ such that $A(w) \subseteq W$, there exists a neighbourhood U of w such that $A(U) \subseteq W$.

The mapping $A : D(A) \rightarrow 2^{E^*}$ is u.s.c. (resp., l.s.c.) iff it is u.s.c. (resp., l.s.c.) at w for any $w \in D(A)$.

Definition 2.4. ([22]). A mapping $\beta : E \times E \rightarrow E$ is called skew iff $\beta(u, v) + \beta(v, u) = \theta$ for all $u, v \in E$.

It is not difficult to see that if β is skew, then $\beta(u, u) = \theta$ for all $(u, u) \in E \times E$. However, the converse may not be true.

The functional $J : F \rightarrow \mathbb{R}$ is locally Lipschitz on F , if for every $z \in F$, there exists a neighbourhood V_z of z and a constant $K_z > 0$ such that $|J(u) - J(v)| \leq K_z \|u - v\|_F$ for all $u, v \in V_z$.

Definition 2.5. ([2]). Let $J : F \rightarrow \mathbb{R}$ be a locally Lipschitz functional. We denote $J^\circ(u; v)$ and $\partial J(u)$ as Clarke's generalized directional derivative at the point $u \in F$ in the direction $v \in F$ and Clarke's generalized gradient of J at $u \in F$, respectively, and defined as:

$$J^\circ(u; v) = \limsup_{\substack{w \rightarrow u, \\ \epsilon \rightarrow 0^+}} \frac{J(w + \epsilon v) - J(w)}{\epsilon},$$

and $\partial J(u) = \{\alpha \in F^* : \langle \alpha, v \rangle \leq J^\circ(u; v) \forall v \in F\}$, respectively.

Definition 2.6. We say that $J^\circ(\cdot; \cdot)$ is bi-sequentially weakly u.s.c. with respect to γ if for any sequences $u_n, v_n \in E$ with $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$, one has

$$\limsup_{n \rightarrow \infty} J^\circ(\gamma(u_n); \gamma(v_n)) \leq J^\circ(\gamma(u); \gamma(v)).$$

Remark 2.1. If $F = L^p(\Gamma, \mathbb{R}^k)$, where $p \in (1, \infty)$, $k \geq 1$ and $\Gamma \subset \mathbb{R}^N$ is an open bounded set, then Definition 2.6 reduces to Definition 3.1 of [22].

Lemma 2.1. ([2]). Let $J : F \rightarrow \mathbb{R}$ be locally Lipschitz of rank L near the point u . Then

- (i) the function $v \rightarrow J^\circ(u; v)$ is finite, homogeneous and subadditive on F , and satisfies $|J^\circ(u; v)| \leq L\|v\|_F$,
- (ii) the functional $(u, v) \rightarrow J^\circ(u; v)$ is u.s.c.,
- (iii) $J^\circ(u; -v) = (-J)^\circ(u; v)$,
- (iv) for each $u \in F$, $\partial J(u)$ is a nonempty, convex, weak*-compact subset of F^* and $\|\kappa\|_{F^*} \leq L$, for all $\kappa \in \partial J(u)$,
- (v) for each $v \in F$, one has $J^\circ(u; v) = \max_{\kappa \in \partial J(u)} \langle \kappa, v \rangle$.

Lemma 2.2. (Nadler's Theorem, [33]). Let $(E, \|\cdot\|)$ be a normed linear space and let $CB(E)$ denote the collection of all nonempty closed and bounded subsets of E . Define the Hausdorff metric H on $CB(E)$ induced by the metric $d(u, v) = \|u - v\|$ as:

$$H(U, V) := \max \left\{ \sup_{u \in U} \inf_{v \in V} \|u - v\|, \sup_{v \in V} \inf_{u \in U} \|u - v\| \right\}, \quad \text{for all } U, V \in CB(E).$$

Then, for any $\epsilon > 0$ and for every $u \in U$, there exists $v \in V$ such that $\|u - v\| \leq (1 + \epsilon)H(U, V)$. Furthermore, if U and V are compact subsets of E , then for each $u \in U$, there exists $v \in V$ such that $\|u - v\| \leq H(U, V)$.

Remark 2.2. The collection $CB(E^*)$ of all nonempty closed and bounded subsets of the dual space E^* equipped with the Hausdorff metric induced by the dual norm forms a complete metric space. Consequently, Nadler's theorem is applicable in E^* for multi-valued mappings with values $CB(E^*)$.

Definition 2.7. ([33]). The mapping $A : C \rightarrow 2^{E^*}$ with nonempty and compact values is said to be H -hemicontinuous iff for any given $u, v \in C$, the mapping $\epsilon \rightarrow H(A(u + \epsilon(v - u)), A(u))$ from $[0, 1]$ to $\mathbb{R}_+$ is continuous at 0^+ , where H is the Hausdorff metric defined on $CB(E^*)$.

Lemma 2.3. ([27]). Let $C \subset E$ be a nonempty subset of a Hausdorff topological vector space E , and let $Q : C \rightarrow 2^E$ be a set-valued mapping such that:

- (i) Q satisfies the KKM property, i.e., for every finite subset $\{v_1, \dots, v_n\} \subset C$, we have $\text{co}\{v_1, \dots, v_n\} \subset \bigcup_{i=1}^n Q(v_i)$,
- (ii) for each $v \in C$, the set $Q(v)$ is closed in E ,
- (iii) there exists $v_0 \in C$ such that $Q(v_0)$ is compact in E .

Then $\bigcap_{v \in C} Q(v) \neq \emptyset$.

Definition 2.8. Let $T : E^* \times C \times C \rightarrow \mathbb{R}$ be a trifunction and $A : C \rightarrow 2^{E^*}$ be a set-valued mapping with nonempty values. Then we say T and A are jointly monotone iff $T(u^*, u, v) + T(v^*, v, u) \leq 0$, $\forall (u, u^*), (v, v^*) \in \text{Gr}(A)$.

Remark 2.3. Definition 2.8 resembles to the concept of joint monotonicity as introduced in [34] with the primary distinction lying only in the domain of the mapping T . The notion of joint monotonicity was used in establishing Theorem 5 of [34].

We say that the bifunction μ fulfils the condition ∇ if

- (i) $\mathcal{D}(\mu) = \{u \in C : \mu(u, v) \neq -\infty, \forall v \in C\} \neq \emptyset$;
- (ii) $\mu(u, u) = 0, \forall u \in C$;
- (iii) $\mu(u, \cdot)$ is convex, $\forall u \in C$;
- (iv) $\mu(\cdot, v)$ is weakly u.s.c., $\forall v \in C$.

In [21], a similar notion Δ was used to show that inequality (1.4) has a solution. Their condition Δ includes the extra rule $\mu(u, v) + \mu(v, u) \geq 0$ for all $u, v \in C$, which is being dropped in our notion ∇ . A related notion was also employed in [23] to investigate the existence of solutions for variational quasi-mixed hemivariational-like inequalities.

Definition 2.9. Let $T : E^* \times C \times C \rightarrow \mathbb{R}$ be a mapping. Then,

- (i) T is convex with respect to its third argument, i.e., for each fixed $u^* \in E^*$ and $u \in C$, the mapping $v \rightarrow T(u^*, u, v)$ is convex on C .
- (ii) T is satisfying the following property (antisymmetric property) in the second and third arguments, i.e.,

$$T(u^*, u, v) + T(u^*, v, u) = 0, \quad \text{for all } (u^*, u, v) \in E^* \times C \times C.$$

- (iii) the mapping T is sequentially weakly u.s.c. with respect to its first argument, i.e., for any fixed $u, v \in C$, the function $u^* \rightarrow T(u^*, u, v)$ is sequentially upper semicontinuous on the dual space E^* endowed with the weak topology.

3 Existence results for GHLIP with bounded constraint set

This section presents the existence results of GHLIP with a nonempty, bounded, closed and convex constraint set C . Here, we assume $\gamma : E \rightarrow F$ to be bounded linear operator. The main result is obtained by using the equivalence relation between the GHLIP and the corresponding MHLIP under mild conditions.

Proposition 3.1. Let $\beta : C \times C \rightarrow E$ be affine in the first argument and $\beta(u, u) = \theta \forall (u, u) \in C \times C$. Suppose that $A : C \rightarrow 2^{E^*}$ is nonempty weakly compact-valued and H -hemicontinuous mapping. Let $\gamma : E \rightarrow F$ be a bounded linear operator, $J : F \rightarrow \mathbb{R}$ be a locally Lipschitz function defined on F , and $T : E^* \times C \times C \rightarrow \mathbb{R}$ be such that the following hypotheses hold:

- (a) T is convex in the third variable,
- (b) $T(u^*, u, v) + T(u^*, v, u) = 0$ for each $(u^*, u, v) \in E^* \times C \times C$,
- (c) T is sequentially weakly u.s.c. in the first argument.

Further, suppose that μ satisfies the ∇ (i)–(iii), and that T and A are jointly monotone. Then, for any $u \in C$, the following inequalities are equivalent:

- (i) $\exists u^* \in A(u)$ such that

$$T(u^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \quad \forall v \in C.$$

- (ii) $\forall v^* \in A(v)$,

$$T(v^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \quad \forall v \in C.$$

Proof. Suppose for some $u^* \in A(u)$, the inequity (i) holds, i.e.,

$$T(u^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \quad \forall v \in C. \quad (3.1)$$

Since T and A are jointly monotone, we have

$$T(u^*, u, v) + T(v^*, v, u) \leq 0, \quad \forall (u, u^*), (v, v^*) \in Gr(A).$$

Thus, from (3.1), we get that $\forall v \in C$,

$$0 \leq -T(v^*, v, u) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v), \quad \forall v^* \in A(v).$$

From (b), we have

$$-T(v^*, v, u) = T(v^*, u, v).$$

Consequently,

$$0 \leq T(v^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v), \quad \forall v^* \in A(v), \quad \forall v \in C.$$

Hence (ii) holds.

Conversely, suppose that $u \in C$ such that (ii) holds, i.e., $\forall v^* \in A(v)$,

$$T(v^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \quad \forall v \in C.$$

Now, given any $v \in C$, we define $v_\epsilon = u + \epsilon(v - u)$ for all $\epsilon \in (0, 1)$. Inserting v_ϵ in place of v and for any fixed $v_\epsilon^* \in A(v_\epsilon)$ in the last inequality, we achieve

$$0 \leq T(v_\epsilon^*, u, v_\epsilon) + J^\circ(\gamma(u); \gamma(\beta(v_\epsilon, u))) + \mu(u, v_\epsilon). \quad (3.2)$$

Since T and μ are convex functions in the third and second arguments, respectively, we obtain

$$T(v_\epsilon^*, u, v_\epsilon) \leq \epsilon T(v_\epsilon^*, u, v) + (1 - \epsilon) T(v_\epsilon^*, u, u) \quad (3.3)$$

and

$$\mu(u, v_\epsilon) \leq \epsilon \mu(u, v) + (1 - \epsilon) \mu(u, u), \quad [\text{by } \nabla(\text{iii})]. \quad (3.4)$$

From ∇ (ii), we have $\mu(u, u) = 0$ and from (b), we have $T(v_\epsilon^*, u, u) = 0$. Combining the fact that β is affine in the first argument and the linearity of γ , we derive that

$$\gamma(\beta(v_\epsilon, u)) = \epsilon \gamma(\beta(v, u)) + (1 - \epsilon) \gamma(\beta(u, u)). \quad (3.5)$$

Together with the property that $\beta(u, u) = \theta$ and (3.5), we have

$$\gamma(\beta(v_\epsilon, u)) = \epsilon \gamma(\beta(v, u)). \quad (3.6)$$

Combining (3.2),(3.3),(3.4),(3.6) and Lemma 2.1 (i), we deduce from (3.2) that

$$0 \leq \epsilon[T(v_\epsilon^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v)].$$

Taking into account $\epsilon > 0$ implies that

$$T(v_\epsilon^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \quad \forall v_\epsilon^* \in A(v_\epsilon). \quad (3.7)$$

Since $A : C \rightarrow 2^{E^*}$ is nonempty weakly compact, then $A(v_\epsilon)$ and $A(u)$ are nonempty weakly compact, hence $A(v_\epsilon), A(u) \in CB(E^*)$. According to Lemma 2.2, we get for each $\epsilon \in (0, 1)$ and $v_\epsilon^* \in A(v_\epsilon)$, there exists $u_\epsilon^* \in A(u)$ such that

$$\|v_\epsilon^* - u_\epsilon^*\|_* \leq (1 + \epsilon)H(A(v_\epsilon), A(u)).$$

Since $A(u) \subset E^*$ is weakly compact and $\{u_\epsilon^*\} \subset A(u)$, there exists a subsequence (also say $\{u_\epsilon^*\}$) such that

$$u_\epsilon^* \rightharpoonup u^* \in A(u) \quad \text{as } \epsilon \rightarrow 0^+,$$

(thus, $u_\epsilon^* \rightharpoonup u^*$, as E is reflexive). Again, by the H -hemicontinuity of A , we acquire that $\|v_\epsilon^* - u_\epsilon^*\|_* \leq (1 + \epsilon)H(A(v_\epsilon), A(u)) \rightarrow 0$. Now, for any $\xi \in E$, we gain that $|\langle \xi, u_\epsilon^* - u^* \rangle| \rightarrow 0$ as $\epsilon \rightarrow 0^+$. Hence for each $\xi \in E$,

$$\begin{aligned} |\langle \xi, v_\epsilon^* - u^* \rangle| &\leq |\langle \xi, v_\epsilon^* - u_\epsilon^* \rangle| + |\langle \xi, u_\epsilon^* - u^* \rangle| \\ &\leq \|\xi\| \|v_\epsilon^* - u_\epsilon^*\|_* + |\langle \xi, u_\epsilon^* - u^* \rangle| \rightarrow 0 \text{ as } \epsilon \rightarrow 0^+. \end{aligned} \quad (3.8)$$

From (3.8), we obtain that $v_\epsilon^* \rightharpoonup u^*$, thus, $v_\epsilon^* \rightharpoonup u^*$, and since T is weakly u.s.c. in the first argument, therefore, for $v \in C$, we have

$$\limsup_{v_\epsilon^* \rightharpoonup u^*} T(v_\epsilon^*, u, v) \leq T(u^*, u, v).$$

Taking the superior limit in (3.7), we get that

$$T(u^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \quad \forall v \in C. \quad (3.9)$$

Finally, we show that the u^* appears in the last inequality does not depend on v , i.e.,

$$T(u^*, u, z) + J^\circ(\gamma(u); \gamma(\beta(z, u))) + \mu(u, z) \geq 0, \quad \forall z \in C. \quad (3.10)$$

Keeping $z \in C$ arbitrarily fixed, we define $z_\epsilon = u + \epsilon(z - u)$ for all $\epsilon \in (0, 1)$. Applying Lemma 2.2, for each $u_\epsilon^* \in A(u)$ where $\epsilon \in (0, 1)$, there exists $w_\epsilon^* \in A(z_\epsilon)$ such that $\|u_\epsilon^* - w_\epsilon^*\|_* \leq (1 + \epsilon)H(A(u), A(z_\epsilon))$. Since A is H -hemicontinuous, we deduce that $\|u_\epsilon^* - w_\epsilon^*\|_* \leq (1 + \epsilon)H(A(u), A(z_\epsilon)) \rightarrow 0$ as $\epsilon \rightarrow 0^+$. Now by

using the similar arguments as in (3.8), we can easily estimate that $w_\epsilon^* \rightarrow u^*$ as $\epsilon \rightarrow 0^+$. Again from the u.s.c. of T w.r.t. first argument, we have

$$\limsup_{w_\epsilon^* \rightarrow u^*} T(w_\epsilon^*, u, z) \leq T(u^*, u, z).$$

Now replacing v_ϵ, v and v_ϵ^* in (3.7) with w_ϵ, z and w_ϵ^* , respectively, one has

$$T(w_\epsilon^*, u, z) + J^\circ(\gamma(u); \gamma(\beta(z, u))) + \mu(u, z) \geq 0, \quad \forall \epsilon \in (0, 1).$$

Therefore, inequality (3.10) holds. $\square$

Theorem 3.1. *Let $\beta : C \times C \rightarrow E$ be affine in the first argument, weakly continuous in the second argument and $\beta(u, u) = \theta \quad \forall (u, u) \in C \times C$. Suppose that $A : C \rightarrow 2^{E^*}$ is nonempty weakly compact-valued and H -hemicontinuous mapping, $\gamma : E \rightarrow F$ is a compact linear operator, $J : F \rightarrow \mathbb{R}$ is a locally Lipschitz function defined on F . Furthermore, let $T : E^* \times C \times C \rightarrow \mathbb{R}$ be such that the following hypotheses hold:*

- (a) T is convex in the third variable;
- (b) $T(u^*, u, v) + T(u^*, v, u) = 0$ for each $(u^*, u, v) \in E^* \times C \times C$;
- (c) for each $v \in C$, the function $(u^*, u) \mapsto T(u^*, u, v)$ is bi-sequentially weakly upper semicontinuous on $E^* \times C$.

In addition, suppose that the perturbation mapping μ satisfies the ∇ -condition, $J^\circ(\cdot; \cdot)$ is bi-sequentially weakly upper semicontinuous w.r.t. the compact operator γ , and that T and A are jointly monotone. Then, the solution set Θ is nonempty.

Proof. For each $v \in C$, define the mappings $M, \widetilde{M} : C \rightarrow 2^E$ as:

$$M(v) = \{u \in C : T(u^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \text{ for some } u^* \in A(u)\},$$

and

$$\widetilde{M}(v) = \{u \in C : T(v^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \quad \forall v^* \in A(v)\}.$$

We arrange the proof in five steps.

Step 1. It is clear that both $M(v)$ and $\widetilde{M}(v)$ are nonempty, since $v \in M(v)$ and $v \in \widetilde{M}(v)$. We now show that the set-valued mapping $M : C \rightarrow 2^E$ is a KKM mapping. Suppose, for contradiction, that there exist points $\{v_1, v_2, \dots, v_n\} \subset C$ and a point $\hat{u} \in \text{co}\{v_1, v_2, \dots, v_n\}$ such that

$$\hat{u} \notin \bigcup_{i=1}^n M(v_i).$$

Then, for every $\hat{u}^* \in A(\hat{u})$, we have

$$T(\hat{u}^*, \hat{u}, v_i) + J^\circ(\gamma(\hat{u}); \gamma(\beta(v_i, \hat{u}))) + \mu(\hat{u}, v_i) < 0, \quad \forall i \in \{1, 2, \dots, n\}. \quad (3.11)$$

Define the set S as:

$$S = \{v \in C : T(\hat{u}^*, \hat{u}, v) + J^\circ(\gamma(\hat{u}); \gamma(\beta(v, \hat{u}))) + \mu(\hat{u}, v) < 0, \quad \forall \hat{u}^* \in A(\hat{u})\}.$$

We now claim that S is convex. For any $v', v'' \in S$ with $v' \neq v''$. We prove that the line segment $v_\epsilon = v' + \epsilon(v'' - v') \in S$, $\forall \epsilon \in (0, 1)$. Using the convexity of T in the third variable and from $\nabla(\text{iii})$, we have

$$\begin{aligned} & T(\hat{u}^*, \hat{u}, v_\epsilon) + J^\circ(\gamma(\hat{u}); \gamma(\beta(v_\epsilon, \hat{u}))) + \mu(\hat{u}, v_\epsilon), \\ & \leq (1 - \epsilon)T(\hat{u}^*, \hat{u}, v') + \epsilon T(\hat{u}^*, \hat{u}, v'') + J^\circ(\gamma(\hat{u}); \gamma(\beta(v_\epsilon, \hat{u}))) \\ & + (1 - \epsilon)\mu(\hat{u}, v') + \epsilon\mu(\hat{u}, v''). \end{aligned} \quad (3.12)$$

Combining the fact that γ is linear, β is affine in the first argument and Lemma 2.1, the inequality (3.12) becomes

$$\begin{aligned} & T(\hat{u}^*, \hat{u}, v_\epsilon) + J^\circ(\gamma(\hat{u}); \gamma(\beta(v_\epsilon, \hat{u}))) + \mu(\hat{u}, v_\epsilon), \\ & \leq (1 - \epsilon)[T(\hat{u}^*, \hat{u}, v') + J^\circ(\gamma(\hat{u}); \gamma(\beta(v', \hat{u}))) + \mu(\hat{u}, v')] \\ & + \epsilon[T(\hat{u}^*, \hat{u}, v'') + J^\circ(\gamma(\hat{u}); \gamma(\beta(v'', \hat{u}))) + \mu(\hat{u}, v'')] < 0. \end{aligned} \quad (3.13)$$

This shows that $v_\epsilon \in S$. Thus, S is convex. Since each $v_i \in S$ and due to the convexity of S , we deduce that $\hat{u} \in S$, i.e.,

$$T(\hat{u}^*, \hat{u}, \hat{u}) + J^\circ(\gamma(\hat{u}); \gamma(\beta(\hat{u}, \hat{u}))) + \mu(\hat{u}, \hat{u}) < 0. \quad (3.14)$$

On the other side, from conditions (b) and $\nabla(\text{ii})$, we achieve

$$T(\hat{u}^*, \hat{u}, \hat{u}) + J^\circ(\gamma(\hat{u}); \gamma(\beta(\hat{u}, \hat{u}))) + \mu(\hat{u}, \hat{u}) = 0,$$

which leads to a contradiction for (3.14). Consequently, M is of KKM type.

Step 2. This step shows that $\widetilde{M}$ is a KKM mapping.

It is enough to produce that $M(v) \subseteq \widetilde{M}(v)$ for each $v \in C$. To obtain this, let $u \in M(v)$, i.e., for some $u^* \in A(u)$,

$$T(u^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0.$$

Now, by using the joint monotonicity of T and A along with condition (b), (similar to Proposition 3.1), it directly follows that $u \in \widetilde{M}(v)$.

Step 3. This step claims that $\widetilde{M}(v)$ is weakly closed for each $v \in C$. To prove this, let $u_n \in \widetilde{M}(v)$ such that $u_n \rightharpoonup u$. Since $u_n \in \widetilde{M}(v)$, so we have

$$T(v^*, u_n, v) + J^\circ(\gamma(u_n); \gamma(\beta(v, u_n))) + \mu(u_n, v) \geq 0, \quad \forall v^* \in A(v). \quad (3.15)$$

As T is bi-sequentially weakly u.s.c. for each $v \in C$, we obtain

$$\limsup_{u_n \rightharpoonup u} T(v^*, u_n, v) \leq T(v^*, u, v).$$

Due to the compactness of γ , we get that $\gamma(u_n) \rightarrow \gamma(u)$. Since β is weakly continuous in the second argument, we gain that

$$\beta(v, u_n) \rightharpoonup \beta(v, u),$$

and again by exploiting the compactness of γ , we conclude

$$\gamma(\beta(v, u_n)) \rightarrow \gamma(\beta(v, u)).$$

Further, by the assumption that $J^\circ(\cdot; \cdot)$ is bi-sequentially weakly u.s.c. w.r.t. γ , we acquire

$$\limsup_{n \rightarrow \infty} J^\circ(\gamma(u_n); \gamma(\beta(v, u_n))) \leq J^\circ(\gamma(u); \gamma(\beta(v, u))). \quad (3.16)$$

Since for each $v \in C$, $\mu(\cdot, v)$ is weakly u.s.c. [due to $\nabla(\text{iv})$] we have also

$$\limsup_{u_n \rightharpoonup u} \mu(u_n, v) \leq \mu(u, v).$$

Applying the upper limit in (3.15) as $n \rightarrow \infty$, we infer

$$T(v^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \quad \forall v^* \in A(v), \quad (3.17)$$

which implies that $u \in \widetilde{M}(v)$. Thus, $\widetilde{M}(v)$ is weakly closed.

Step 4. The objective of this step is to assert the weak compactness of the set $\widetilde{M}(v)$ for each fixed $v \in C$. Note that $\widetilde{M}(v) \subseteq C$ by construction. Given that C is a nonempty, closed, convex, and bounded subset of E , thus C is weakly compact. Therefore, appealing to the fact that any weakly closed subset of a weakly compact set remains weakly compact, we deduce that $\widetilde{M}(v)$ is weakly compact.

Step 5. Now, applying Lemma 2.3 on $\widetilde{M}$ ($\widetilde{M}$ is KKM and weakly compact mapping), we yield

$$\bigcap_{v \in C} \widetilde{M}(v) \neq \emptyset,$$

i.e., there exists $u \in \widetilde{M}(v)$ such that $\forall v \in C$,

$$T(v^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \quad \forall v^* \in A(v).$$

Now, from Proposition 3.1, we get

$$T(u^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \quad \text{for some } u^* \in A(u)$$

This completes the proof. $\square$

Remark 3.1. The reader should note that the condition of T being bi-sequentially weakly u.s.c. is not directly applied to Theorem 3.1. Instead, what follows the weakly sequentially u.s.c. in the second argument is used to obtain the inequality (3.17) and the weakly sequentially u.s.c. in the first argument is used in the last part of the proof, which is required to conclude the theorem (i.e., in Proposition 3.1). Moreover, Theorem 3.1 serves as a general existence result for all the problems mentioned in the introduction under suitable choices of the T, A, β, γ , and μ .

Let $F = L^p(\Gamma, \mathbb{R}^k)$, where $p \in (1, \infty)$, $k \geq 1$, and $\Gamma \subset \mathbb{R}^N$ is an open and bounded domain. Accordingly, we consider the operator $\gamma : E \rightarrow L^p(\Gamma, \mathbb{R}^k)$. Let q denote the Hölder conjugate of p , i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Let $j : \Gamma \times \mathbb{R}^k \rightarrow \mathbb{R}$ be a function, where the mapping

$$j(\cdot, v) : \Gamma \rightarrow \mathbb{R} \text{ is measurable, for every } v \in \mathbb{R}^k. \quad (3.18)$$

Furthermore, we impose that at least one of the following conditions is satisfied, specifically, there exists $l \in L^q(\Gamma, \mathbb{R})$ such that

$$|j(u, v_1) - j(u, v_2)| \leq l(u)|v_1 - v_2|, \quad \forall u \in \Gamma, \quad \forall v_1, v_2 \in \mathbb{R}^k, \quad (3.19)$$

or

$$\text{the mapping } j(u, \cdot) \text{ is locally Lipschitz, } \forall u \in \Gamma, \quad (3.20)$$

and there exists $L > 0$ such that

$$|w| \leq L(1 + |v|^{p-1}) \quad \forall u \in \Gamma, \quad \forall w \in \partial j(u, v). \quad (3.21)$$

Lemma 3.1. ([2]) *Let $J : L^p(\Gamma, \mathbb{R}^k) \rightarrow \mathbb{R}$ be the function defined by $J(\omega) = \int_{\Gamma} j(x, \omega(x)) dx$, and suppose that j satisfies conditions (3.18) and (3.19) or (3.18) and (3.20) – (3.21). Then, j is uniformly Lipschitz on bounded subsets, and one has $\partial J(\omega) \subset \int_{\Gamma} \partial j(x, \omega(x)) dx$. Furthermore, if j is regular at $(x, \omega(x))$, then J is regular at ω and the equality holds*

$$\partial J(\omega) = \int_{\Gamma} \partial j(x, \omega(x)) dx.$$

We denote $\gamma(u) = \bar{u}$ (Note that $\gamma : E \rightarrow L^p(\Gamma, \mathbb{R}^k)$). We now invoke Theorem 3.1 to establish the solvability of the following GHLIP, which may be regarded as an application of the main result.

$$\text{GHLIP} \left\{ \begin{array}{l} \text{Find } u \in \mathcal{D}(\mu) \text{ and } u^* \in A(u) \text{ such that} \\ T(u^*, u, v) + \int_{\Gamma} j^\circ(x, \bar{u}(x); \gamma(\beta(v, u))(x)) dx + \mu(u, v) \geq 0, \\ \forall v \in C. \end{array} \right. \quad (3.22)$$

Define $\tilde{\Theta}$ as the solution set of the GHLIP as in (3.22).

Theorem 3.2. *Assume that all the hypotheses of Theorem 3.1 are satisfied. Let $F = L^p(\Gamma, \mathbb{R}^k)$, where $p \in (1, \infty)$, $k \geq 1$, and $\Gamma \subset \mathbb{R}^N$ is an open bounded set. Define the functional $J : L^p(\Gamma, \mathbb{R}^k) \rightarrow \mathbb{R}$ by $J(\omega) = \int_{\Gamma} j(x, \omega(x)) dx$, where the function j satisfies either the conditions (3.18) and (3.19) or the conditions (3.18) and (3.20) – (3.21). Then the set $\tilde{\Theta} \neq \emptyset$.*

Proof. By Theorem 3.1, we have $\tilde{\Theta} \neq \emptyset$. So there exists $u \in \mathcal{D}(\mu)$ and $u^* \in A(u)$ such that

$$T(u^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \forall v \in C. \quad (3.23)$$

As $J(\omega) = \int_{\Gamma} j(x, \omega(x)) dx$ and j fulfils the conditions (3.18) and (3.19) or (3.18) and (3.20)–(3.21), therefore by Lemma 3.1, we have

$$J^\circ(\gamma(u); \gamma(\beta(v, u))) \leq \int_{\Gamma} j^\circ(x, \bar{u}(x); \gamma(\beta(v, u))(x)) dx, \forall u, v \in E. \quad (3.24)$$

Combining (3.23) and (3.24), we get

$$T(u^*, u, v) + \int_{\Gamma} j^\circ(x, \bar{u}(x); \gamma(\beta(v, u))(x)) dx + \mu(u, v) \geq 0, \forall v \in C,$$

Therefore, it follows that $u \in \mathcal{D}(\mu)$, and simultaneously, u is a member of the solution set $\tilde{\Theta}$. $\square$

Remark 3.2. Theorem 3.2 generalizes Theorem 3.2 of [22], Theorem 4.1 of [4], and Theorem 4.1 of [21]. Moreover, it extends Theorem 3.1 of [6] and Corollary 1 of [3]. Furthermore, Theorem 3.2 extends the results of Theorem 3.3 in [32] and Theorem 2 in [30].

4 Existence results for GHLIP with unbounded constraint set

This section presents the result of GHLIP when C is relaxed in the unbounded case. To this end, we introduce some new coercivity conditions which facilitate the derivation of existence results in the unbounded constraint set. Furthermore, we have demonstrated that the corresponding solution set Θ is also bounded.

Theorem 4.1. *Suppose that all the hypotheses of Theorem 3.1 are held except for the boundedness assumption on the set C . Additionally, assume the following coercive condition is satisfied: there exists $r_0 \in \mathbb{N}$ and $u_0 \in C$ such that, for every $u \in C \setminus C_{r_0}$, there exists $v_0 \in C$ with $\|v_0 - u_0\| < \|u - u_0\|$ such that*

$$T(u^*, u, v_0) + J^\circ(\gamma(u); \gamma(\beta(v_0, u))) + \mu(u, v_0) \leq 0, \quad \forall u^* \in A(u), \quad (4.1)$$

where $C_{r_0} = \{u \in C : \|u - u_0\| \leq r_0\}$. Then $\Theta \neq \emptyset$.

Proof. Suppose $r > r_0$ with $C_r = \{u \in C : \|u - u_0\| \leq r\}$. Clearly, $C_r \neq \emptyset$ (since $u_0 \in C_r$), bounded, closed and convex subset in E . Thus, from Theorem 3.1 there exists $u_r \in C_r$ and $u_r^* \in A(u_r)$ such that

$$T(u_r^*, u_r, v) + J^\circ(\gamma(u_r); \gamma(\beta(v, u_r))) + \mu(u_r, v) \geq 0, \quad \forall v \in C_r. \quad (4.2)$$

Since $u_r \in C_r$, we have considered the following two cases.

Case-1: Let $u_r \in C_r$ with $\|u_r - u_0\| = r$. So $\|u_r - u_0\| > r_0$ and $u_r \in C \setminus C_{r_0}$. By condition (4.1), there exists $v_0 \in C$ with $\|v_0 - u_0\| < \|u_r - u_0\|$ and

$$T(u_r^*, u_r, v_0) + J^\circ(\gamma(u_r); \gamma(\beta(v_0, u_r))) + \mu(u_r, v_0) \leq 0, \quad \forall u_r^* \in A(u_r). \quad (4.3)$$

Let $v \in C$ be arbitrarily fixed. Since $\|v_0 - u_0\| < \|u_r - u_0\| = r$, we get $v_0 \in C_r$ such that there exists $\epsilon \in (0, 1)$ small enough with $v_\epsilon = v_0 + \epsilon(v - v_0) \in C_r$. Notice that γ is linear, β is affine in the first argument, T is convex in the third variable, and also $\mu(u, \cdot)$ is convex. Substituting $v = v_\epsilon$ in (4.2) and applying Lemma 2.1 (i), along with the above structural properties, we estimate that

$$\begin{aligned} 0 &\leq T(u_r^*, u_r, v_\epsilon) + J^\circ(\gamma(u_r); \gamma(\beta(v_\epsilon, u_r))) + \mu(u_r, v_\epsilon), \\ &= T(u_r^*, u_r, v_\epsilon) + J^\circ(\gamma(u_r); (1 - \epsilon)\gamma(\beta(v_0, u_r)) + \epsilon\gamma(\beta(v, u_r))) + \mu(u_r, v_\epsilon), \\ &\leq (1 - \epsilon)[T(u_r^*, u_r, v_0) + J^\circ(\gamma(u_r); \gamma(\beta(v_0, u_r))) + \mu(u_r, v_0)] \\ &\quad + \epsilon[T(u_r^*, u_r, v) + J^\circ(\gamma(u_r); \gamma(\beta(v, u_r))) + \mu(u_r, v)], \\ &\leq \epsilon[T(u_r^*, u_r, v) + J^\circ(\gamma(u_r); \gamma(\beta(v, u_r))) + \mu(u_r, v)], \end{aligned} \quad (4.4)$$

where the last inequality follows from (4.3). Dividing by $\epsilon > 0$ in (4.4), we achieve that

$$T(u_r^*, u_r, v) + J^\circ(\gamma(u_r); \gamma(\beta(v, u_r))) + \mu(u_r, v) \geq 0.$$

Since v is arbitrarily chosen, we obtain $u_r \in \Theta$.

Case-2: If $\|u_r - u_0\| < r$, then for any $v \in C$, there is small enough $\epsilon \in (0, 1)$ such that $z_\epsilon = u_r + \epsilon(v - u_r) \in C_r$. Setting $v = z_\epsilon$ in (4.2) and using the linearity of γ , along with the convexity of T in its third argument and μ in its second argument, respectively, together with Lemma 2.1, we obtain that

$$\begin{aligned}
 0 &\leq T(u_r^*, u_r, z_\epsilon) + J^\circ(\gamma(u_r); \gamma(\beta(z_\epsilon, u_r))) + \mu(u_r, z_\epsilon), \\
 &= T(u_r^*, u_r, z_\epsilon) + J^\circ(\gamma(u_r); (1 - \epsilon)\gamma(\beta(u_r, u_r)) + \epsilon\gamma(\beta(v, u_r))) + \mu(u_r, z_\epsilon), \\
 &\leq (1 - \epsilon)[T(u_r^*, u_r, u_r) + J^\circ(\gamma(u_r); \gamma(\beta(u_r, u_r))) + \mu(u_r, u_r)] \\
 &\quad + \epsilon[T(u_r^*, u_r, v) + J^\circ(\gamma(u_r); \gamma(\beta(v, u_r))) + \mu(u_r, v)], \\
 &= \epsilon[T(u_r^*, u_r, v) + J^\circ(\gamma(u_r); \gamma(\beta(v, u_r))) + \mu(u_r, v)], \tag{4.5}
 \end{aligned}$$

where (4.5) follows from condition (b), $\nabla(\text{ii})$, and the property $\beta(u_r, u_r) = \theta$. By noticing $\epsilon > 0$, we get that

$$T(u_r^*, u_r, v) + J^\circ(\gamma(u_r); \gamma(\beta(v, u_r))) + \mu(u_r, v) \geq 0, \quad \forall v \in C.$$

Thus, $u_r \in \Theta$. □

Corollary 4.1. *Suppose that all the assumptions of Theorem 4.1 are held. Let $F = L^p(\Gamma, \mathbb{R}^k)$, where $p \in (1, \infty)$, $k \geq 1$ and $\Gamma \subset \mathbb{R}^N$ be an open bounded set with $J : L^p(\Gamma, \mathbb{R}^k) \rightarrow \mathbb{R}$ be the function $J(\omega) = \int_{\Gamma} j(x, \omega(x)) dx$, and j satisfies either conditions (3.18) and (3.19) or (3.18) together with (3.20) – (3.21). Then $\tilde{\Theta} \neq \emptyset$.*

Remark 4.1. Corollary 4.1 generalizes several well-known results in the existing literature. Specifically, it extends Theorem 3.5 of [22], Theorem 4.2 of [4], Theorem 4.5 of [21], Theorem 3.2 of [6], and Theorem 2 of [3].

We mention here that if C is bounded, then Θ is itself bounded. However, if C is unbounded, then the set Θ may not be bounded. Next, we explore some sufficient conditions to prevent the bound for Θ when the set C is unbounded.

Theorem 4.2. *Let all the conditions of Theorem 3.1 be satisfied except for the boundedness of the set C . Assume that there exist $r_0 \in \mathbb{N}$ and $u_0 \in C$ such that for every $u \in C \setminus C_{r_0}$ there exists some $v_0 \in C$ with $\|v_0 - u_0\| < \|u - u_0\|$ such that*

$$T(u^*, u, v_0) + J^\circ(\gamma(u); \gamma(\beta(v_0, u))) + \mu(u, v_0) < 0, \quad \forall u^* \in A(u). \tag{4.6}$$

Then, the solution set $\Theta \neq \emptyset$ and is bounded.

Proof. It is easy to examine that (4.6) $\Rightarrow$ (4.1). Hence, by Theorem 4.1, the set $\Theta \neq \emptyset$. To show Θ is bounded, assume the contrary, i.e., suppose that the set Θ is unbounded. Then, there exists $r_0 \in \mathbb{N}$ and $u_0 \in C$ such that there exists $u \in C$ and $u^* \in A(u)$ with $\|u - u_0\| > r_0$ such that

$$T(u^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \quad \forall v \in C. \quad (4.7)$$

Since $\|u - u_0\| > r_0$ and from condition (4.6), there exists $v_0 \in C$ with $\|v_0 - u_0\| < \|u - u_0\|$ such that

$$T(u^*, u, v_0) + J^\circ(\gamma(u); \gamma(\beta(v_0, u))) + \mu(u, v_0) < 0 \quad \forall u^* \in A(u). \quad (4.8)$$

This yields a contradiction to (4.7), thereby completing the proof. $\square$

Corollary 4.2. *Suppose that all assumptions of Theorem 4.2 are fulfilled. Let $F = L^p(\Gamma, \mathbb{R}^k)$, where $p \in (1, \infty)$, $k \geq 1$ and $\Gamma \subset \mathbb{R}^N$ be an open bounded set with $J : L^p(\Gamma, \mathbb{R}^k) \rightarrow \mathbb{R}$ be the function $J(\omega) = \int_{\Gamma} j(x, \omega(x)) dx$, and j satisfies either conditions (3.18) and (3.19) or (3.18) together with (3.20) – (3.21). Then $\tilde{\Theta} \neq \emptyset$ and is bounded.*

Remark 4.2. Corollary 4.2 can be viewed as a generalization of previously known results, including Theorem 3.7 of [22], Theorem 4.3 of [4], and Theorem 4.7 of [21].

Theorem 4.3. *Suppose that the conditions of Theorem 3.1 are obeyed without the boundedness of the set C . Let the following condition hold: there exists a nonempty subset U_0 contained in a weakly compact subset U_1 of C such that the set*

$$D = \{u \in C : T(v^*, u, v) + J^\circ(\gamma(u); \gamma(\beta(v, u))) + \mu(u, v) \geq 0, \quad \forall v^* \in A(v), \quad \forall v \in U_0\} \quad (4.9)$$

is empty or weakly compact. Then $\Theta \neq \emptyset$ and is bounded.

Proof. It is enough to show that (4.9) $\Rightarrow$ (4.6). If $D = \emptyset$, and since $\emptyset \neq U_0 \subset U_1 \subset C$ with U_1 weakly compact, then there exists $m \in \mathbb{N}$ and $u_0 \in C$ such that $\|u - u_0\| < m$ for all $u \in U_0$. Considering $r_0 = m$, we see that for every $u \in C \setminus C_{r_0}$, there exists $v_0 \in U_0$ such that $\|v_0 - u_0\| < r_0$ and

$$T(v_0^*, u, v_0) + J^\circ(\gamma(u); \gamma(\beta(v_0, u))) + \mu(u, v_0) < 0 \quad \text{for some } v_0^* \in A(v_0). \quad (4.10)$$

Next, if $D \neq \emptyset$, then D is weakly compact. It is simple to see that $D \cup U_0 \subset D \cup U_1$, where $D \cup U_1$ is weakly compact. Therefore, there exists $m \in \mathbb{N}$ such

that $\|u - u_0\| < m$ for all $u \in D \cup U_0$. Taking $r_0 = m$ and for every $u \in C \setminus C_{r_0}$, we get $u \notin D$ and hence there exists $v_0 \in U_0$ such that $\|v_0 - u_0\| < r_0$ and (4.10) holds. Therefore, it follows from both cases that there exists $r_0 \in \mathbb{N}$ such that for any $u \in C \setminus C_{r_0}$, there exists $v_0 \in C_{r_0}$ satisfying (4.10) for some $v_0^* \in A(v_0)$. Moreover, it is easy to claim that $\|v_0 - u_0\| < \|u - u_0\|$. Hence, we assert that (4.10) holds. From condition (b), we have $T(v_0^*, u, v_0) + T(v_0^*, v_0, u) = 0$. Therefore, (4.10) becomes

$$-T(v_0^*, v_0, u) + J^\circ(\gamma(u); \gamma(\beta(v_0, u))) + \mu(u, v_0) < 0. \quad (4.11)$$

Now, using the fact that T and A are jointly monotone, we deduce

$$T(v_0^*, v_0, u) + T(u^*, u, v_0) \leq 0, \quad (u, u^*), (v_0, v_0^*) \in \text{Gr}(A). \quad (4.12)$$

Combining (4.11) and (4.12), we obtain

$$\begin{aligned} & T(u^*, u, v_0) + J^\circ(\gamma(u); \gamma(\beta(v_0, u))) + \mu(u, v_0) \\ & \leq -T(v_0^*, v_0, u) + J^\circ(\gamma(u); \gamma(\beta(v_0, u))) + \mu(u, v_0) \\ & < 0. \end{aligned}$$

Therefore, condition (4.6) is satisfied. $\square$

Corollary 4.3. *Suppose that the same assumptions as in Theorem 4.3 are held. Let $F = L^p(\Gamma, \mathbb{R}^k)$, where $p \in (1, \infty)$, $k \geq 1$ and $\Gamma \subset \mathbb{R}^N$ be an open bounded set with $J : L^p(\Gamma, \mathbb{R}^k) \rightarrow \mathbb{R}$ be the function $J(\omega) = \int_{\Gamma} j(x, \omega(x)) dx$, and j satisfies either conditions (3.18) and (3.19) or (3.18) and (3.20) – (3.21). Then $\tilde{\Theta} \neq \emptyset$ and bounded.*

Next, we deduce a necessary and sufficient condition for the solvability of the GHLIP.

For the next theorem, suppose that $\mu(u, v) = \rho(v) - \rho(u)$, where $\rho : E \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper convex mapping such that the domain is restricted to $C_\rho = C \cap \text{dom } \rho = \mathcal{D}(\mu) \neq \emptyset$.

Theorem 4.4. *Let C be a nonempty closed convex subset of E . Suppose that the mapping $\beta : E \times E \rightarrow E$ is affine in the first argument and satisfies the condition $\beta(u, u) = \theta \forall (u, u) \in C \times C$. Let $\gamma : E \rightarrow F$ be a bounded linear operator, and $J : F \rightarrow \mathbb{R}$ be a locally Lipschitz mapping. Let $A : C \rightarrow 2^{E^*}$, $T : E^* \times C \times C \rightarrow \mathbb{R}$ satisfying the condition $T(u^*, u, u) = 0$ for all $(u^*, u) \in \text{Gr}(A)$, and convex on the third argument.*

Then, GHLIP admits a solution if and only if there exists a scalar $r > 0$ and a point $u_0 \in C$ such that the restricted problem $(\mathbf{P}_r)$ admits at least one solution, where

$(\mathbf{P}_r)$: find $u_r \in C_\rho \cap C_r$ and $u_r^* \in A(u_r)$ such that

$$T(u_r^*, u_r, v) + J^\circ(\gamma(u_r); \gamma(\beta(v, u_r))) + \rho(v) - \rho(u_r) \geq 0, \quad \forall v \in C \cap C_r \quad (4.13)$$

satisfies the inequality $\|u_r - u_0\| < r$, where $C_r = \{u \in C : \|u - u_0\| \leq r\}$.

Proof. It is easy to trace the first segment of necessity. To prove the second segment of the necessity, i.e., $\|u_r - u_0\| < r$, assume the contrary, i.e., $\|u_r - u_0\| = r$. Thus, for sufficiently large r , we get $\|u_r\| \rightarrow \infty$, which contradicts $u_r \in C_r$.

Conversely, let u_r solve $(\mathbf{P}_r)$, which confirms the condition $\|u_r - u_0\| < r$. For any fixed $v \in C$, we choose $\epsilon > 0$ small enough so that $z_\epsilon = u_r + \epsilon(v - u_r)$ satisfies $\|z_\epsilon - u_0\| < r$. Consequently, it follows from (4.13) that for some $u_r^* \in A(u_r)$ with $v = z_\epsilon$

$$0 \leq T(u_r^*, u_r, z_\epsilon) + J^\circ(\gamma(u_r); \gamma(\beta(z_\epsilon, u_r))) + \rho(z_\epsilon) - \rho(u_r). \quad (4.14)$$

By condition (a), the affinity property of β , the linearity of γ , and the convexity of ρ , together with Lemma 2.1(i), we deduce from (4.14) that

$$\begin{aligned} 0 &\leq (1 - \epsilon)T(u_r^*, u_r, u_r) + \epsilon T(u_r^*, u_r, v) + (1 - \epsilon)J^\circ(\gamma(u_r); \gamma(\beta(u_r, u_r))) \\ &\quad + \epsilon J^\circ(\gamma(u_r); \gamma(\beta(v, u_r))) + (1 - \epsilon)\rho(u_r) + \epsilon\rho(v) - \rho(u_r), \\ &= \epsilon[T(u_r^*, u_r, v) + J^\circ(\gamma(u_r); \gamma(\beta(v, u_r))) + \rho(v) - \rho(u_r)]. \end{aligned} \quad (4.15)$$

The inequality (4.15) follows from the facts that $T(u_r^*, u_r, u_r) = 0$ and $\beta(u_r, u_r) = \theta$. Dividing by $\epsilon > 0$ on both sides of (4.15), the conclusion follows. $\square$

5 Conclusions

The present paper has introduced and established the existence results of GH-LIP in a reflexive Banach space. By employing the notion of joint monotonicity of T and A , H -hemicontinuity of A , Clarke's generalized gradient and ∇ -type conditions on μ , the Proposition 3.1 has been derived. Based on Proposition 3.1, we obtain two general existence results: one is that when C is weakly compact, i.e., Theorem 3.1, and the other is when C is unbounded, i.e., Theorem 4.1. On the other hand, we provide sufficient criteria (coercive conditions) to bound the set Θ . Furthermore, we establish a necessary and sufficient existence consequence to a particular class of GHLIP. In sum, Theorem 3.1 is more prevalent and serves the existence theorems for all problems, which are reduced from GHLIP as discussed in the introduction section. In fact, the GHLIP is more general than the problems (i)-(vii). We have applied Theorem 3.1

to prove Theorem 3.2 under suitable conditions, which is referred to as an application. Moreover, it would be interesting to carry out the current research scenario in the following promising direction for future research.

- (i) To establish the well-posedness for the GHLIP and its metric characterizations.
- (ii) To develop some computational algorithms for the existence of solutions for the GHLIP.
- (iii) To study the existence theory of GHLIP on Hadamard manifolds (see [8], [35]) and Hadamard spaces (see [36]) opens up a new line of research.

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