



3-Parameter Generalized Quaternions and Dual Quaternions with Repunit Number Components

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Abstract

In this paper, our first aim is to combine the 3-parameter generalized quaternions, which are a general form of the quaternion algebra according to 3-parameters, and repunit numbers, which are created by integer numbers that are expressed in the decimal system as the repetition of the unit. That is, we determine a special new number system called 3-parameter generalized quaternions with repunit number components. Then, we construct a Maple code of this special number sequence. Additionally, we get some new and classical well-known equations such as Binet formulas, generating functions (usual, exponential, and Poisson), polar representation, and Cassini and Catalan identities. In addition to these, we express determinant property, characteristic polynomial, characteristic equation, eigenvalues, and eigenvectors concerning the matrix representation of repunit 3-parameter generalized quaternions. Next, the other purpose is to combine the dual quaternions with the repunit numbers. With the same manner of first aim and structure, we construct a new recurrence sequence, which is called dual quaternions with repunit number components.

Key Words: 3-parameter generalized quaternions, Repunit numbers.

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1 Introduction

Number theory is one of the most attractive frameworks in many concepts in mathematics. Several researchers interested in numbers and numbers system with lots of applications such as robotics, computer systems and others. One can say that quaternions are among the most eminent number systems, and this special number system is constructed as an extension of the complex numbers by W. R. Hamilton in 1843, [27, 28, 29]. Quaternion algebra is non-commutative, associative, and 4-dimensional Clifford algebra. The set of Hamilton's quaternions are defined by $\mathbb{H}$ and written as follows

$$\mathbb{H} := \{\rho | \rho = \rho_0 + \rho_1 i + \rho_2 j + \rho_3 k, \rho_0, \rho_1, \rho_2, \rho_3 \in \mathbb{R}\},$$

where i, j, k are quaternionic units satisfy the rules $i^2 = -1$, $j^2 = -1$, $k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Then, Cockle determined the split quaternions with the rules $i^2 = -1$, $j^2 = k^2 = 1$, $ijk = 1$ [9]. Also, the 2-parameter generalized quaternions were studied several papers by lots of researchers [37, 38, 39, 42, 45, 47, 56, 17, 22, 46]. For all $\lambda_1, \lambda_2 \in \mathbb{R}$, the set of the 2-parameter generalized quaternions is denoted by $\mathbb{H}_{2P}$ and expressed as

$$\mathbb{H}_{2P} := \{\rho = \rho_0 + \rho_1 i + \rho_2 j + \rho_3 k, \rho_0, \rho_1, \rho_2, \rho_3, \lambda_1, \lambda_2 \in \mathbb{R}\},$$

where $i^2 = -\lambda_1$, $j^2 = -\lambda_2$, $k^2 = -\lambda_1 \lambda_2$, $ij = -ji = k$, $jk = -kj = \lambda_2 i$, $ki = -ik = \lambda_1 j$.

- * For $\lambda_1 = 1, \lambda_2 = 1$; ρ is a real quaternion.
- * For $\lambda_1 = 1, \lambda_2 = -1$; ρ is a split quaternion.
- * For $\lambda_1 = 1, \lambda_2 = 0$; ρ is a semi-quaternion,
- * For $\lambda_1 = -1, \lambda_2 = 0$; ρ is a split semi-quaternion.
- * For $\lambda_1 = 0, \lambda_2 = 0$; ρ is a 1/4-quaternion [9, 37, 38, 39, 42, 45, 47, 56, 17, 22, 46, 43, 29].

Then, Şentürk and Ünal determined the 3-parameter generalized quaternions which are a new quaternion family type [52, 53]. They defined a general notion for the quaternion algebra according to the 3-parameters to get a generalization of real, split, and 2-parameter generalized quaternions. The set of 3-parameter generalized quaternions is defined by $\mathbb{H}_{3P}$ and written as

$$\mathbb{H}_{3P} := \{\rho = \rho_0 + \rho_1 i + \rho_2 j + \rho_3 k, \rho_0, \rho_1, \rho_2, \rho_3, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\},$$

where quaternionic units hold the rules written in Table 1.

Table 1: Multiplication Scheme [52, 53]

.	1	i	j	k
1	1	i	j	k
i	i	$-\lambda_1\lambda_2$	λ_1k	$-\lambda_2j$
j	j	$-\lambda_1k$	$-\lambda_1\lambda_3$	λ_3i
k	k	λ_2j	$-\lambda_3i$	$-\lambda_2\lambda_3$

Based on the values $\lambda_1, \lambda_2, \lambda_3$, some special cases of 3-parameter generalized quaternions are given in the following list [52, 53].

- * $\lambda_1 = 1, \lambda_2 = \lambda, \lambda_3 = \mu$, we get 2-parameter generalized quaternions [37, 38, 39, 42, 45, 47, 56, 17, 22, 46].
- * For $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1$, we get split quaternions [9].
- * For $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$, we get Hamilton's real quaternions [27, 28, 29].
- * For $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 0$, we get semi-quaternions [46, 43].
- * For $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0$, we get split semi-quaternions [46].
- * For $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0$, we get 1/4-quaternions [46, 29].

Furthermore, the other special types related to the $\lambda_1, \lambda_2, \lambda_3$ can be seen [52, 53].

One of the popular type of quaternions is dual quaternion, that is studied by Majernik [41]. Dual quaternions are defined by $\mathbb{H}_D = \{q|q = q_0 + q_1i + q_2j + q_3k, q_0, q_1, q_2, q_3 \in \mathbb{R}\}$ where i, j, k satisfy the following rules [20, 41]:

$$i^2 = j^2 = k^2 = ijk = 0, \quad ij = ji = jk = kj = ki = ik = 0. \quad (1)$$

On the other hand, several special recurrence numbers with different orders such as; second, third, and higher than three-order. In this paper, we are interested in second-order recurrence sequence which is called repunit number sequence. For all $n \geq 1$, the repunit sequence satisfies the recurrence relation

$$R_{n+1} = 11R_n - 10R_{n-1}, \quad n \geq 1 \quad (2)$$

with the initial values $R_0 = 0, R_1 = 1$ [49, 15, 50]. As for the repunit numbers (cf. OEIS, A002275), this number sequence is established by integers which

are expressed with the decimal system as the repetition of the unit, and by the set $\{0, 1, 11, 111, 1111, \dots\}$ [48, 49].

The characteristic equation of repunit numbers sequence is written as the Horadam-type characteristic equation of repunit numbers $r^2 - 11r + 10 = 0$, where $r_1 = 10$ and $r_2 = 1$. The Binet formula for repunit numbers is written as [49]:

$$R_n = \frac{10^n - 1}{9}, \quad (3)$$

For more detailed information with respect to the repunit sequence, see [49, 50, 15, 16, 58, 10, 12, 13, 14, 51, 23, 11].

Bringing together the quaternions and special recurrence sequences has been extensively studied working area, and also several researchers have been attracted due to their applications and various use areas. In the existing literature, 2-parameter generalized quaternions with some third-order recurrence sequences were examined. Cerda-Morales obtained the third-order Jacobsthal 2PGQs [6]. Padovan and Perrin 2PGQs [30], Pell-Padovan 2PGQs [31], 2PGQs with generalized Jacobsthal numbers components [26], 2PGQs with generalized 3-primes and generalized reverse 3-primes numbers components [32] are investigated by İşbilir and Gürses. Kızılateş et al. examined the bicomplex generalized Tribonacci quaternions in [40]. Also, Flaut and Shpakivskyi defined the generalized Fibonacci quaternions and Fibonacci-Narayana quaternions in [21]. Taşcı defined the Padovan and Pell-Padovan real quaternions [55]. Günay and Taşkara [25] studied some properties of Padovan real quaternions. Also, Günay defined the real quaternions with some generalized third-order recurrence numbers components [24]. Moreover, Dişkaya and Menken [18, 19] introduced the (s, t) -Padovan and (s, t) -Perrin real quaternions, and split (s, t) -Padovan and (s, t) -Perrin quaternions. Cerda-Morales studied the real quaternions with generalized Tribonacci numbers components [3], third-order Jacobsthal quaternions [4], third-order $\bar{h}$ -Jacobsthal and third-order $\bar{h}$ -Jacobsthal-Lucas sequences and related quaternions [5], and third-order Jacobsthal generalized quaternions [6]. Then, Bilgici defined the Fibonacci and Lucas 3PGQs [1] and 3PGQs with Jacobsthal and Jacobsthal-Lucas numbers components [2]. Chaker and Boua [8] studied some properties of generalized quaternions algebra with generalized Fibonacci quaternions. Also, Horadam 3PGQs were determined by İşbilir and Gürses in [33]. İşbilir et al. defined the 3-parameter generalized quaternions with generalized Tribonacci numbers components [36]. Dual third-order Jacobsthal quaternions were determined by Cerda-Morales [7].

In addition to these, generalized dual Fibonacci quaternions [59] and dual Fibonacci and Lucas quaternions [60] introduced by Yüce and Torunbalcı Aydın. Nurkan and Güven determined the quaternions with dual Fibonacci

and dual Lucas numbers components [44]. Dual numbers with Fibonacci and Lucas 3PGQs investigated by [57]. İsbilir and Gürses defined the Padovan, Perrin and Pell-Padovan dual quaternions [34] and generalized Tribonacci dual quaternions [35].

In this study, firstly we introduce the 3-parameter generalized quaternions with repunit number components. Then, we give both classical and new properties of this type recurrence sequence. Secondly, we introduced the dual quaternions with repunit numbers sequence with same manner and structure of first purpose.

2 Preliminaries

In this part of this study, we remind some terminology.

2.1 3-parameter generalized quaternions

Let $\rho = \rho_0 + \rho_1 i + \rho_2 j + \rho_3 k$, $\varrho = \varrho_0 + \varrho_1 i + \varrho_2 j + \varrho_3 k \in \mathbb{H}_{3P}$, in the following list some basic algebraic properties are listed [52, 53]:

1. *Equality:* $\rho = \varrho \Leftrightarrow \rho_0 = \varrho_0, \quad \rho_1 = \varrho_1, \quad \rho_2 = \varrho_2, \quad \rho_3 = \varrho_3.$

2. *Addition and subtraction:*

$$\rho \pm \varrho = \rho_0 \pm \varrho_0 + (\rho_1 \pm \varrho_1) i + (\rho_2 \pm \varrho_2) j + (\rho_3 \pm \varrho_3) k.$$

3. *Multiplication by a scalar:* $\psi \rho = \psi \rho_0 + \psi \rho_1 i + \psi \rho_2 j + \psi \rho_3 k$ and $\psi \in \mathbb{R}.$

4. *Multiplication:*

$$\begin{aligned} \rho \varrho &= (\rho_0 + \rho_1 i + \rho_2 j + \rho_3 k) (\varrho_0 + \varrho_1 i + \varrho_2 j + \varrho_3 k) \\ &= S_\rho S_\varrho - f(V_\rho, V_\varrho) + S_\rho V_\varrho + S_\varrho V_\rho + V_\rho \wedge V_\varrho, \end{aligned}$$

where

$$f(V_\rho, V_\varrho) = \lambda_1 \lambda_2 \rho_1 \varrho_1 + \lambda_1 \lambda_3 \rho_2 \varrho_2 + \lambda_2 \lambda_3 \rho_3 \varrho_3$$

and

$$\begin{aligned} V_\rho \wedge V_\varrho &= \begin{vmatrix} \lambda_3 i & \lambda_2 j & \lambda_1 k \\ \rho_1 & \rho_2 & \rho_3 \\ \varrho_1 & \varrho_2 & \varrho_3 \end{vmatrix} \\ &= \lambda_3(\rho_2 \varrho_3 - \rho_3 \varrho_2) i + \lambda_2(\rho_3 \varrho_1 - \rho_1 \varrho_3) j + \lambda_1(\rho_1 \varrho_2 - \rho_2 \varrho_1) k. \end{aligned}$$

5. *Conjugation:* $\bar{\rho} = \rho_0 - \rho_1 i - \rho_2 j - \rho_3 k.$

6. *Inner product:* $\langle \rho, \varrho \rangle = \rho_0 \varrho_0 + \lambda_1 \lambda_2 \rho_1 \varrho_1 + \lambda_1 \lambda_3 \rho_2 \varrho_2 + \lambda_2 \lambda_3 \rho_3 \varrho_3$.

7. *Norm:* $N_\rho = \rho \bar{\rho} = \bar{\rho} \rho = \rho_0^2 + \lambda_1 \lambda_2 \rho_1^2 + \lambda_1 \lambda_3 \rho_2^2 + \lambda_2 \lambda_3 \rho_3^2$.

8. *Inverse:* $\rho^{-1} = \frac{\bar{\rho}}{N_\rho} = \frac{\rho_0 - \rho_1 i - \rho_2 j - \rho_3 k}{\rho_0^2 + \lambda_1 \lambda_2 \rho_1^2 + \lambda_1 \lambda_3 \rho_2^2 + \lambda_2 \lambda_3 \rho_3^2}$ where $N_\rho \neq 0$.

Also, $S_{\rho \pm \varrho} = \rho_0 \pm \varrho_0 = S_\rho \pm S_\varrho$, $V_{\rho \pm \varrho} = V_\rho \pm V_\varrho$, and $\bar{\rho} = S_\rho - V_\varrho$. If $N_\rho = 1$, then ρ is a 3-parameter generalized unit quaternion.

Further, for $N_\rho > 0$ and $\lambda_1 \lambda_2 \rho_1^2 + \lambda_1 \lambda_3 \rho_2^2 + \lambda_2 \lambda_3 \rho_3^2 \neq 0$, the 3-parameter generalized quaternion ρ is written in a polar form

$$\rho = \sqrt{N_\rho} (\cos \eta + \hat{\rho} \sin \eta),$$

where $\hat{\rho} = \frac{(\rho_1, \rho_2, \rho_3)}{\sqrt{\lambda_1 \lambda_2 \rho_1^2 + \lambda_1 \lambda_3 \rho_2^2 + \lambda_2 \lambda_3 \rho_3^2}}$. Also,

$$\begin{cases} \cos \eta = \frac{\rho_0}{\sqrt{N_\rho}}, \\ \sin \eta = \sqrt{\frac{\lambda_1 \lambda_2 \rho_1^2 + \lambda_1 \lambda_3 \rho_2^2 + \lambda_2 \lambda_3 \rho_3^2}{N_\rho}}, \end{cases}$$

and $\hat{\rho}$ is called 3-parameter generalized unit vector. Also, the following fundamental matrix $\mathcal{M}_\rho$ is obtained:

$$\mathcal{M}_\rho = \begin{pmatrix} \rho_0 & -\lambda_1 \lambda_2 \rho_1 & -\lambda_1 \lambda_3 \rho_2 & -\lambda_2 \lambda_3 \rho_3 \\ \rho_1 & \rho_0 & -\lambda_3 \rho_3 & \lambda_3 \rho_2 \\ \rho_2 & \lambda_2 \rho_3 & \rho_0 & -\lambda_2 \rho_1 \\ \rho_3 & -\lambda_1 \rho_2 & \lambda_1 \rho_1 & \rho_0 \end{pmatrix}.$$

It should be noted that, from beginning to end this study, this representation, which is the left matrix representation, will be used since the non-commutativity of 3-parameter generalized quaternion for 3-parameter generalized quaternion ρ . With the same logic, the right matrix representation can also be expressed. Based on the values $\lambda_1, \lambda_2, \lambda_3$, the matrix $\mathcal{M}_\rho$ is classified.

1. For $\lambda_1 = 1, \lambda_2, \lambda_3 \in \mathbb{R}$, $\mathcal{M}_\rho$ for 2-parameter generalized quaternions is attained.
2. For $\lambda_1 = \lambda_2 = 1, \lambda_3 = -1$ then, $\mathcal{M}_\rho$ for split quaternions is attained.
3. For $\lambda_1 = \lambda_2 = \lambda_3 = 1$ then, $\mathcal{M}_\rho$ for Hamilton quaternions is attained.

Then, the following list includes some algebraic calculations for $\mathcal{M}_\rho$.

* The determinant of $\mathcal{M}_\rho$ is $\det(\mathcal{M}_\rho) = N_\rho^2$.

* The characteristic polynomial of $\mathcal{M}_\rho$ is

$$\varrho_{\mathcal{M}_\rho}(\mathbf{v}) = (\mathbf{v}^2 - 2\mathbf{v}\rho_0 + \rho_0^2 + \lambda_1\lambda_2\rho_1^2 + \lambda_1\lambda_3\rho_2^2 + \lambda_2\lambda_3\rho_3^2)^2.$$

* The characteristic equation of $\mathcal{M}_\rho$ is

$$\det(\mathcal{M}_\rho - \mathbf{v}I_4) = 0$$

$$\Leftrightarrow \varrho_{\mathcal{M}_\rho}(\mathbf{v}) = (\mathbf{v}^2 - 2\mathbf{v}\rho_0 + \rho_0^2 + \lambda_1\lambda_2\rho_1^2 + \lambda_1\lambda_3\rho_2^2 + \lambda_2\lambda_3\rho_3^2)^2 = 0.$$

* The characteristic equation provides an opportunity to calculate the eigenvalues as follows.

$$\mathfrak{T}_{1,2} = \rho_0 + \sqrt{-\lambda_1\lambda_2\rho_1^2 - \lambda_1\lambda_3\rho_2^2 - \lambda_2\lambda_3\rho_3^2},$$

$$\mathfrak{T}_{3,4} = \rho_0 - \sqrt{-\lambda_1\lambda_2\rho_1^2 - \lambda_1\lambda_3\rho_2^2 - \lambda_2\lambda_3\rho_3^2}$$

* The following relation is written

$$\mathfrak{T}_{1,2}\mathfrak{T}_{3,4} = \rho_0^2 + \lambda_1\lambda_2\rho_1^2 + \lambda_1\lambda_3\rho_2^2 + \lambda_2\lambda_3\rho_3^2 = N_\rho.$$

* The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{1,2}$ are given

$$\begin{pmatrix} \frac{\lambda_1\rho_2\sqrt{-\lambda_1\lambda_2\rho_1^2 - \lambda_1\lambda_3\rho_2^2 - \lambda_2\lambda_3\rho_3^2} - \lambda_1\lambda_2\rho_1\rho_3}{\lambda_1\rho_2^2 + \lambda_2\rho_3^2} \\ \frac{\rho_3\sqrt{-\lambda_1\lambda_2\rho_1^2 - \lambda_1\lambda_3\rho_2^2 - \lambda_2\lambda_3\rho_3^2} + \lambda_1\rho_1\rho_2}{\lambda_1\rho_2^2 + \lambda_2\rho_3^2} \\ 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{\lambda_2\rho_3\sqrt{-\lambda_1\lambda_2\rho_1^2 - \lambda_1\lambda_3\rho_2^2 - \lambda_2\lambda_3\rho_3^2} + \lambda_1\lambda_2\rho_1\rho_2}{\lambda_1\rho_2^2 + \lambda_2\rho_3^2} \\ -\frac{\rho_2\sqrt{-\lambda_1\lambda_2\rho_1^2 - \lambda_1\lambda_3\rho_2^2 - \lambda_2\lambda_3\rho_3^2} - \lambda_2\rho_1\rho_3}{\lambda_1\rho_2^2 + \lambda_2\rho_3^2} \\ 0 \\ 1 \end{pmatrix}.$$

* The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{3,4}$ are given

$$\begin{pmatrix} \frac{\lambda_1\rho_2\sqrt{-\lambda_1\lambda_2\rho_1^2 - \lambda_1\lambda_3\rho_2^2 - \lambda_2\lambda_3\rho_3^2} + \lambda_1\lambda_2\rho_1\rho_3}{\lambda_1\rho_2^2 + \lambda_2\rho_3^2} \\ -\frac{\rho_3\sqrt{-\lambda_1\lambda_2\rho_1^2 - \lambda_1\lambda_3\rho_2^2 - \lambda_2\lambda_3\rho_3^2} - \lambda_1\rho_1\rho_2}{\lambda_1\rho_2^2 + \lambda_2\rho_3^2} \\ 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -\frac{\lambda_2 \rho_3 \sqrt{-\lambda_1 \lambda_2 \rho_1^2 - \lambda_1 \lambda_3 \rho_2^2 - \lambda_2 \lambda_3 \rho_3^2} - \lambda_1 \lambda_2 \rho_1 \rho_2}{\lambda_1 \rho_2^2 + \lambda_2 \rho_3^2} \\ \frac{\rho_2 \sqrt{-\lambda_1 \lambda_2 \rho_1^2 - \lambda_1 \lambda_3 \rho_2^2 - \lambda_2 \lambda_3 \rho_3^2} + \lambda_2 \rho_1 \rho_3}{\lambda_1 \rho_2^2 + \lambda_2 \rho_3^2} \\ 0 \\ 1 \end{pmatrix}.$$

2.2 Dual quaternions

A dual quaternion $\rho \in \mathbb{H}_D$ is expressed as $\rho = S_\rho + \vec{V}_\rho$, where $S_\rho = \rho_0$ is scalar part and $\vec{V}_\rho = \rho_1 i + \rho_2 j + \rho_3 k$ is vector part. For $\rho, \varrho \in \mathbb{H}_D$, regarding (1), the following operations are written:

- $\rho = \varrho \Leftrightarrow \rho_0 = \varrho_0, \rho_1 = \varrho_1, \rho_2 = \varrho_2, \rho_3 = \varrho_3,$
- $\rho \pm \varrho = \rho_0 \pm \varrho_0 + (\rho_1 \pm \varrho_1)i + (\rho_2 \pm \varrho_2)j + (\rho_3 \pm \varrho_3)k,$
- $\omega \rho = \omega \rho_0 + \omega \rho_1 i + \omega \rho_2 j + \omega \rho_3 k, \lambda \in \mathbb{R},$
- $\rho \varrho = (\rho_0 + \rho_1 i + \rho_2 j + \rho_3 k)(\varrho_0 + \varrho_1 i + \varrho_2 j + \varrho_3 k).$

Then, $S_{\rho \pm \varrho} = \rho_0 \pm \varrho_0 = S_\rho \pm S_\varrho$ and $\vec{V}_{\rho \pm \varrho} = \vec{V}_\rho \pm \vec{V}_\varrho$. The conjugate of dual quaternion ρ is written as $\bar{\rho} = \rho_0 - \rho_1 i - \rho_2 j - \rho_3 k$. Also, $\bar{\rho} = S_\rho - \vec{V}_\rho$ and the norm of q is expressed as $N_\rho = \rho \bar{\rho} = \bar{\rho} \rho = \rho_0^2$ [41, 35].

3 Repunit 3-Parameter Generalized Quaternions

In this part of this study, we introduce the 3-parameter generalized quaternions with repunit numbers sequence.

Definition 1. Let $\mathcal{R}_n$ be the n th repunit 3-parameter generalized quaternion. Then, it is defined as follows:

$$\mathcal{R}_n = R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k \quad \text{for every } n \in \mathbb{N}, \quad (4)$$

where R_n is the n th repunit number and i, j, k are satisfied rules written in Table 1. The initial conditions of repunit 3-parameter generalized quaternions can be written as follows:

$$\begin{cases} \mathcal{R}_0 = i + 11j + 111k, \\ \mathcal{R}_1 = 1 + 11i + 111j + 1111k. \end{cases}$$

The classification of the repunit 3-parameter generalized quaternions can be examined in the Table 2.

Table 2: Classification of Repunit 3-Parameter Generalized Quaternions

For	Type
$\lambda_1 = 1, \lambda_2, \lambda_3 \in \mathbb{R}$	Repunit 2-parameter generalized quaternions
$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1$	Repunit split quaternions
$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$	Repunit Hamilton quaternions [15]
$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 0$	Repunit semi-quaternions
$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0$	Repunit split semi-quaternions
$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0$	Repunit 1/4-quaternions

Let us obtain some algebraic operations over 3PGQs with repunit number components. For every $n, m \in \mathbb{N}$, let $\mathcal{R}_n$ and $\mathcal{R}_m$ be the n th and m th 3PGQ with repunit number components, respectively. Equality, addition/subtraction, and multiplication by scalar, scalar and vector parts, multiplication of any elements of this family, conjugation, inner product, norm and inverse are given as follows:

1. **Equality:**

$$\mathcal{R}_n = \mathcal{R}_m \Leftrightarrow R_n = R_m, \quad R_{n+1} = R_{m+1}, \quad R_{n+2} = R_{m+2}, \quad R_{n+3} = R_{m+3},$$

2. **Addition/Subtraction:**

$$\mathcal{R}_n \pm \mathcal{R}_m = (R_n \pm R_m) + (R_{n+1} \pm R_{m+1})i + (R_{n+2} \pm R_{m+2})j + (R_{n+3} \pm R_{m+3})k,$$

3. **Multiplication by a scalar:**

$$\psi \mathcal{R}_n = \psi R_n + \psi R_{n+1}i + \psi R_{n+2}j + \psi R_{n+3}k \quad \text{where } \psi \in \mathbb{R},$$

4. **Multiplication:**

$$\begin{aligned} \mathcal{R}_n \mathcal{R}_m &= (R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k)(R_m + R_{m+1}i + R_{m+2}j + R_{m+3}k) \\ &= S_{\mathcal{R}_n} S_{\mathcal{R}_m} - f(V_{\mathcal{R}_n}, V_{\mathcal{R}_m}) + S_{\mathcal{R}_n} V_{\mathcal{R}_m} + S_{\mathcal{R}_m} V_{\mathcal{R}_n} + V_{\mathcal{R}_n} \wedge V_{\mathcal{R}_m}, \end{aligned}$$

where

$$f(V_{\mathcal{R}_n}, V_{\mathcal{R}_m}) = \lambda_1 \lambda_2 R_{n+1} R_{m+1} + \lambda_1 \lambda_3 R_{n+2} R_{m+2} + \lambda_2 \lambda_3 R_{n+3} R_{m+3}$$

and

$$\begin{aligned}
 V_{\mathcal{R}_n} \wedge V_{\mathcal{R}_m} &= \begin{vmatrix} \lambda_3 i & \lambda_2 j & \lambda_1 k \\ R_{n+1} & R_{n+2} & R_{n+3} \\ R_{m+1} & R_{m+2} & R_{m+3} \end{vmatrix} \\
 &= \lambda_3 (R_{n+2}R_{m+3} - R_{n+3}R_{m+2}) i \\
 &\quad + \lambda_2 (R_{n+3}R_{m+1} - R_{n+1}R_{m+3}) j \\
 &\quad + \lambda_1 (R_{n+1}R_{m+2} - R_{n+2}R_{m+1}) k.
 \end{aligned}$$

From Table 1, the multiplication is expressed as follows, as well:

$$\begin{aligned}
 \mathcal{R}_n \mathcal{R}_m &= R_n R_m - \lambda_1 \lambda_2 R_{n+1} R_{m+1} - \lambda_1 \lambda_3 R_{n+2} R_{m+2} - \lambda_2 \lambda_3 R_{n+3} R_{m+3} \\
 &\quad + (R_n R_{m+1} + R_m R_{n+1} + \lambda_3 (R_{n+2} R_{m+3} - R_{n+3} R_{m+2})) i \\
 &\quad + (R_n R_{m+2} + R_m R_{n+2} + \lambda_2 (R_{n+3} R_{m+1} - R_{n+1} R_{m+3})) j \\
 &\quad + (R_n R_{m+3} + R_m R_{n+3} + \lambda_1 (R_{n+1} R_{m+2} - R_{n+2} R_{m+1})) k.
 \end{aligned}$$

5. **Scalar and Vector Parts:** The scalar part of $\mathcal{R}_n$ is represented as $S_{\mathcal{R}_n}$ and $S_{\mathcal{R}_n} = R_n$. The vector part of $\mathcal{R}_n$ is represented by $V_{\mathcal{R}_n}$ and $V_{\mathcal{R}_n} = R_{n+1}i + R_{n+2}j + R_{n+3}k$. So, $S_{\mathcal{R}_n \pm \mathcal{R}_m} = R_n \pm R_m = S_{\mathcal{R}_n} \pm S_{\mathcal{R}_m}$ and $V_{\mathcal{R}_n \pm \mathcal{R}_m} = V_{\mathcal{R}_n} \pm V_{\mathcal{R}_m}$.

6. **Conjugation:** $\overline{\mathcal{R}_n} = R_n - R_{n+1}i - R_{n+2}j - R_{n+3}k$ in which $\overline{\mathcal{R}_n}$ represents the conjugation of $\mathcal{R}_n$.

7. **Inner product:**

$$\langle \mathcal{R}_n, \mathcal{R}_m \rangle = R_n R_m + \lambda_1 \lambda_2 R_{n+1} R_{m+1} + \lambda_1 \lambda_3 R_{n+2} R_{m+2} + \lambda_2 \lambda_3 R_{n+3} R_{m+3}.$$

8. **Norm:** $N_{\mathcal{R}_n} = \mathcal{R}_n \overline{\mathcal{R}_n} = \overline{\mathcal{R}_n} \mathcal{R}_n = R_n^2 + \lambda_1 \lambda_2 R_{n+1}^2 + \lambda_1 \lambda_3 R_{n+2}^2 + \lambda_2 \lambda_3 R_{n+3}^2$.

9. **Inverse:** $\mathcal{R}_n^{-1} = \frac{\overline{\mathcal{R}_n}}{N_{\mathcal{R}_n}} = \frac{R_n - R_{n+1}i - R_{n+2}j - R_{n+3}k}{R_n^2 + \lambda_1 \lambda_2 R_{n+1}^2 + \lambda_1 \lambda_3 R_{n+2}^2 + \lambda_2 \lambda_3 R_{n+3}^2}$ where $N_{\mathcal{R}_n} \neq 0$.

Theorem 1 (Recurrence Relation). *Let $\mathcal{R}_n$ be the n th repunit 3-parameter generalized quaternion. The following recurrence relation is satisfied as follows:*

$$\mathcal{R}_{n+2} = 11\mathcal{R}_{n+1} - 10\mathcal{R}_n \quad \text{for every } n \geq 0. \quad (5)$$

Proof. From the equations (2) and (4), we have

$$\begin{aligned}
 11\mathcal{R}_{n+1} - 10\mathcal{R}_n &= 11(R_{n+1} + R_{n+2}i + R_{n+3}j + R_{n+4}k) \\
 &\quad - 10(R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k) \\
 &= 11R_{n+1} - 10R_n + (11R_{n+2} - 10R_{n+1})i \\
 &\quad + (11R_{n+3} - 10R_{n+2})j + (11R_{n+4} - 10R_{n+3})k \\
 &= R_{n+2} + R_{n+3}i + R_{n+4}j + R_{n+5}k \\
 &= \mathcal{R}_{n+2}.
 \end{aligned}$$

□

Let us establish the Maple 12 code to calculate repunit 3-parameter generalized quaternions:

```

> restart: with(LinearAlgebra):with(linalg):
> R(n):
> R:=proc(n)
> if n=0 then return 0:
> elif n=1 then return 1:
> elif n=2 then return 11:
> else return ((10^n)-1)*(1/9)
> end if;
> end proc;
> R_star(n):
> R_star:=proc(n)
> if n=0 then return R(0)+R(1)*i+R(2)*j+R(3)*k:
> elif n=1 then return R(1)+R(2)*i+R(3)*j+R(4)*k:
> elif n=2 then return 11*(R(1)+R(2)*i+R(3)*j+R(4)*k)-10*(R(0)+R
    ↪ (1)*i+R(2)*j+R(3)*k):
> else return 11*R_star(n-1)-10*R_star(n-2)
> end if;
> end proc;

```

*The notations R and R_star represent, repunit number and repunit 3-parameter generalized quaternion, respectively.

Theorem 2. For every $n, m \in \mathbb{N}$, the following properties hold for repunit 3-parameter generalized quaternions.

$$\begin{aligned}
 (1) \quad & \mathcal{R}_n + \mathcal{R}_{n+1}i + \mathcal{R}_{n+2}j + \mathcal{R}_{n+3}k \\
 &= 2\mathcal{R}_n - (R_n + \lambda_1\lambda_2R_{n+2} + \lambda_1\lambda_3R_{n+4} + \lambda_2\lambda_3R_{n+6})
 \end{aligned}$$

$$(2) \quad \overline{\mathcal{R}}_n = 2R_n - \mathcal{R}_n$$

$$(3) \quad \begin{cases} \mathcal{R}_n^2 + \overline{\mathcal{R}}_n^2 = 2(R_n^2 - \lambda_1\lambda_2 R_{n+1}^2 - \lambda_1\lambda_3 R_{n+2}^2 - \lambda_2\lambda_3 R_{n+3}^2) \\ \mathcal{R}_n^2 - \overline{\mathcal{R}}_n^2 = 4R_n\mathcal{R}_n - 4R_n^2 \end{cases}$$

$$(4) \quad \overline{\mathcal{R}}_n\overline{\mathcal{R}}_m - \mathcal{R}_n\mathcal{R}_m = -2 \begin{bmatrix} (R_n\mathcal{R}_{m+1} + R_{n+1}\mathcal{R}_m) i \\ + (R_n\mathcal{R}_{m+2} + R_{n+2}\mathcal{R}_m) j \\ + (R_n\mathcal{R}_{m+3} + R_{n+3}\mathcal{R}_m) k \end{bmatrix}$$

$$(5) \quad \overline{\mathcal{R}}_n\overline{\mathcal{R}}_m + \mathcal{R}_n\mathcal{R}_m = 2 \begin{bmatrix} R_n R_m - \lambda_1\lambda_2 R_{n+1} R_{m+1} - \lambda_1\lambda_3 R_{n+2} R_{m+2} \\ - \lambda_2\lambda_3 R_{n+3} R_{m+3} \\ + \lambda_3 (R_{n+2} R_{m+3} - R_{n+3} R_{m+2}) i \\ + \lambda_2 (R_{n+3} R_{m+1} - R_{n+1} R_{m+3}) j \\ + \lambda_1 (R_{n+1} R_{m+2} - R_{n+2} R_{m+1}) k \end{bmatrix}$$

$$(6) \quad \overline{\mathcal{R}}_n\mathcal{R}_m - \mathcal{R}_n\overline{\mathcal{R}}_m = -2 \begin{bmatrix} (R_{n+1}R_m - R_n R_{m+1}) i \\ + (R_{n+2}R_m - R_n R_{m+2}) j \\ + (R_{n+3}R_m - R_n R_{m+3}) k \end{bmatrix}$$

$$(7) \quad \overline{\mathcal{R}}_n\mathcal{R}_m + \mathcal{R}_n\overline{\mathcal{R}}_m = 2 \begin{bmatrix} R_n R_m + \lambda_1\lambda_2 R_{n+1} R_{m+1} \\ + \lambda_1\lambda_3 R_{n+2} R_{m+2} + \lambda_2\lambda_3 R_{n+3} R_{m+3} \\ + \lambda_3 (R_{n+3} R_{m+2} - R_{n+2} R_{m+3}) i \\ + \lambda_2 (R_{n+1} R_{m+3} - R_{n+3} R_{m+1}) j \\ + \lambda_1 (R_{n+2} R_{m+1} - R_{n+1} R_{m+2}) k \end{bmatrix}.$$

Proof. (1) Based on Table 1 and equation (4), we have

$$\begin{aligned} \mathcal{R}_n + \mathcal{R}_{n+1}i + \mathcal{R}_{n+2}j + \mathcal{R}_{n+3}k &= R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k \\ &+ (R_{n+1} + R_{n+2}i + R_{n+3}j + R_{n+4}k) i \\ &+ (R_{n+2} + R_{n+3}i + R_{n+4}j + R_{n+5}k) j \\ &+ (R_{n+3} + R_{n+4}i + R_{n+5}j + R_{n+6}k) k \\ &= 2\mathcal{R}_n - (R_n + \lambda_1\lambda_2 R_{n+2} + \lambda_1\lambda_3 R_{n+4} + \lambda_2\lambda_3 R_{n+6}). \end{aligned}$$

Then, we obtain

$$R_n + \lambda_1\lambda_2 R_{n+2} + \lambda_1\lambda_3 R_{n+4} + \lambda_2\lambda_3 R_{n+6} = \mathcal{R}_n - \mathcal{R}_{n+1}i - \mathcal{R}_{n+2}j - \mathcal{R}_{n+3}k.$$

(3) From Table 1 and equation (4) and conjugation property of repunit 3-parameter generalized quaternions, we get

$$\begin{aligned} \mathcal{R}_n^2 &= (R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k)(R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k) \\ &= 2R_n\mathcal{R}_n - (R_n^2 + \lambda_1\lambda_2 R_{n+1}^2 + \lambda_1\lambda_3 R_{n+2}^2 + \lambda_2\lambda_3 R_{n+3}^2). \end{aligned}$$

and

$$\begin{aligned}\overline{\mathcal{R}}_n^2 &= (R_n - R_{n+1}i - R_{n+2}j - R_{n+3}k)(R_n - R_{n+1}i - R_{n+2}j - R_{n+3}k) \\ &= -2R_n\mathcal{R}_n - (-3R_n^2 + \lambda_1\lambda_2R_{n+1}^2 + \lambda_1\lambda_3R_{n+2}^2 + \lambda_2\lambda_3R_{n+3}^2).\end{aligned}$$

Then, we attain the followings

$$\begin{aligned}\mathcal{R}_n^2 + \overline{\mathcal{R}}_n^2 &= 2R_n\mathcal{R}_n - (R_n^2 + \lambda_1\lambda_2R_{n+1}^2 + \lambda_1\lambda_3R_{n+2}^2 + \lambda_2\lambda_3R_{n+3}^2) \\ &\quad - 2R_n\mathcal{R}_n - (-3R_n^2 - \lambda_1\lambda_2R_{n+1}^2 - \lambda_1\lambda_3R_{n+2}^2 - \lambda_2\lambda_3R_{n+3}^2) \\ &= 2(R_n^2 - \lambda_1\lambda_2R_{n+1}^2 - \lambda_1\lambda_3R_{n+2}^2 - \lambda_2\lambda_3R_{n+3}^2)\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_n^2 - \overline{\mathcal{R}}_n^2 &= 2R_n\mathcal{R}_n - (R_n^2 + \lambda_1\lambda_2R_{n+1}^2 + \lambda_1\lambda_3R_{n+2}^2 + \lambda_2\lambda_3R_{n+3}^2) \\ &\quad + 2R_n\mathcal{R}_n + (-3R_n^2 + \lambda_1\lambda_2R_{n+1}^2 + \lambda_1\lambda_3R_{n+2}^2 + \lambda_2\lambda_3R_{n+3}^2) \\ &= 4R_n\mathcal{R}_n - 4R_n^2.\end{aligned}$$

- (5) According to Table 1, equation (4), conjugation and multiplication properties of repunit 3-parameter generalized quaternions, we obtain

$$\begin{aligned}\overline{\mathcal{R}}_n\overline{\mathcal{R}}_m + \mathcal{R}_n\mathcal{R}_m &= (R_n - R_{n+1}i - R_{n+2}j - R_{n+3}k)(R_m - R_{m+1}i - R_{m+2}j - R_{m+3}k) \\ &\quad + (R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k)(R_m + R_{m+1}i + R_{m+2}j + R_{m+3}k) \\ &= 2[R_nR_m - \lambda_1\lambda_2R_{n+1}R_{m+1} - \lambda_1\lambda_3R_{n+2}R_{m+2} - \lambda_2\lambda_3R_{n+3}R_{m+3} \\ &\quad + \lambda_3(R_{n+2}R_{m+3} - R_{n+3}R_{m+2})i + \lambda_2(R_{n+3}R_{m+1} - R_{n+1}R_{m+3})j \\ &\quad + \lambda_1(R_{n+1}R_{m+2} - R_{n+2}R_{m+1})k].\end{aligned}$$

- (7) With the help of Table 1, equation (4), conjugation and multiplication properties of repunit 3-parameter generalized quaternions, we attain

$$\begin{aligned}\overline{\mathcal{R}}_n\mathcal{R}_m + \mathcal{R}_n\overline{\mathcal{R}}_m &= (R_n - R_{n+1}i - R_{n+2}j - R_{n+3}k)(R_m + R_{m+1}i + R_{m+2}j + R_{m+3}k) \\ &\quad + (R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k)(R_m - R_{m+1}i - R_{m+2}j - R_{m+3}k) \\ &= 2[R_nR_m + \lambda_1\lambda_2R_{n+1}R_{m+1} + \lambda_1\lambda_3R_{n+2}R_{m+2} + \lambda_2\lambda_3R_{n+3}R_{m+3} \\ &\quad + \lambda_3(R_{n+3}R_{m+2} - R_{n+2}R_{m+3})i + \lambda_2(R_{n+1}R_{m+3} - R_{n+3}R_{m+1})j \\ &\quad + \lambda_1(R_{n+2}R_{m+1} - R_{n+1}R_{m+2})k].\end{aligned}$$

Hence, we completed the proof. $\square$

Theorem 3. *For every $n \in \mathbb{N}$, Binet formula for repunit 3-parameter generalized quaternions is given*

$$\mathcal{R}_n = \frac{10^n \phi_1 - \phi_2}{9}, \quad (6)$$

where $\phi_1 = 1 + 10i + 100j + 1000k$ and $\phi_2 = 1 + i + j + k$.

Proof. According to the equations (3) and (4), the proof is done

$$\begin{aligned} \mathcal{R}_n &= R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k \\ &= \frac{10^n - 1}{9} + \frac{10^{n+1} - 1}{9}i + \frac{10^{n+2} - 1}{9}j + \frac{10^{n+3} - 1}{9}k \\ &= \frac{10^n}{9} (1 + 10i + 100j + 1000k) - \frac{1}{9} (1 + i + j + k) \\ &= \frac{10^n \phi_1 - \phi_2}{9}. \end{aligned}$$

□

Theorem 4. *For every $n \in \mathbb{N}$, the generating function for repunit 3-parameter generalized quaternions is written as follows:*

$$\sum_{n=0}^{\infty} \mathcal{R}_n x^n = \frac{\mathcal{R}_0 + (\mathcal{R}_1 - 11\mathcal{R}_0)x}{1 - 11x + 10x^2}. \quad (7)$$

Proof. Let $\sum_{n=0}^{\infty} \mathcal{R}_n x^n = \mathcal{R}_0 + \mathcal{R}_1 x + \mathcal{R}_2 x^2 + \dots + \mathcal{R}_n x^n + \dots$ be generating function of repunit 3-parameter generalized quaternions Then, let us multiply both sides of this equality by $-11x$ and $10x^2$ as follows

$$\begin{aligned} -11x \sum_{n=0}^{\infty} \mathcal{R}_n x^n &= -11\mathcal{R}_0 x - 11\mathcal{R}_1 x^2 - 11\mathcal{R}_2 x^3 + \dots - 11\mathcal{R}_n x^{n+1} + \dots \\ 10x^2 \sum_{n=0}^{\infty} \mathcal{R}_n x^n &= 10\mathcal{R}_0 x^2 + 10\mathcal{R}_1 x^3 + 10\mathcal{R}_2 x^4 + \dots + 10\mathcal{R}_n x^{n+2} + \dots \end{aligned}$$

After these, by using equation (5), we have

$$(1 - 11x + 10x^2) \sum_{n=0}^{\infty} \mathcal{R}_n x^n = \mathcal{R}_0 + (\mathcal{R}_1 - 11\mathcal{R}_0)x.$$

Finally, we attain equation (7). □

The following theorems are given without proofs since they are straightforward.

Theorem 5. *The following Catalan identities are satisfied.*

$$\begin{aligned} 1. \quad & \mathcal{R}_m^2 - \mathcal{R}_{m-n}\mathcal{R}_{m+n} = \frac{10^{m-n}}{9}\mathcal{R}_n \\ & - \left[-1109\mathcal{R}_n + \left(\frac{10^n 911 + 889}{9} \right) i - \left(\frac{10^n 889 + 1091}{9} \right) j + \left(\frac{10^n 1091 + 911}{9} \right) k \right] \\ 2. \quad & \mathcal{R}_m^2 - \mathcal{R}_{m+n}\mathcal{R}_{m-n} = \frac{10^{m-n}}{9}\mathcal{R}_n \\ & + \left[-1109\mathcal{R}_n - \left(\frac{10^n 889 + 911}{9} \right) i + \left(\frac{1091 10^n + 889}{9} \right) j + \left(\frac{911 \cdot 10^n + 1091}{9} \right) k \right] \end{aligned}$$

Theorem 6. *The following Cassini identities are satisfied.*

$$\begin{aligned} 1. \quad & \mathcal{R}_m^2 - \mathcal{R}_{m-1}\mathcal{R}_{m+1} = 10^{m-1} \left[-1109 + 1111i - 1109j + \frac{11821}{9}k \right] \\ 2. \quad & \mathcal{R}_m^2 - \mathcal{R}_{m+1}\mathcal{R}_{m-1} = 10^{m-1} \left[-1109 + \frac{9881}{9}i + 1311j + 891k \right] \end{aligned}$$

Theorem 7. *The exponential generating function of repunit 3-parameter generalized quaternions is written as*

$$\sum_{n=0}^{\infty} \mathcal{R}_n \frac{y^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{10^n \phi_1 - \phi_2}{9} \right) \frac{y^n}{n!}. \quad (8)$$

Proof. By equation (6), we get the desired result. $\square$

Theorem 8. *The Poisson generating function of repunit 3-parameter generalized quaternions is as follows.*

$$e^{-y} \sum_{n=0}^{\infty} \mathcal{R}_n \frac{y^n}{n!} = e^{-y} \sum_{n=0}^{\infty} \left(\frac{10^n \phi_1 - \phi_2}{9} \right) \frac{y^n}{n!}$$

Proof. Based on equation (8), we get the desired results, since Poisson generating function is written as multiplying the exponential generating function by e^{-y} (see [54]). $\square$

Definition 2. *For $N_{\mathcal{R}_n} > 0$ and $\lambda_1 \lambda_2 R_{n+1}^2 + \lambda_1 \lambda_3 R_{n+2}^2 + \lambda_2 \lambda_3 R_{n+3}^2 \neq 0$, polar representation of repunit 3-parameter generalized quaternions is written*

$$\mathcal{R}_n = \sqrt{N_{\mathcal{R}_n}} \left(\cos \eta + \hat{\mathcal{R}}_n \sin \eta \right),$$

where

$$\hat{\mathcal{R}}_n = \frac{(R_{n+1}, R_{n+2}, R_{n+3})}{\sqrt{\lambda_1 \lambda_2 R_{n+1}^2 + \lambda_1 \lambda_3 R_{n+2}^2 + \lambda_2 \lambda_3 R_{n+3}^2}}$$

and

$$\cos \eta = \frac{R_n}{\sqrt{N_{\mathcal{R}_n}}}, \quad \sin \eta = \sqrt{\frac{\lambda_1 \lambda_2 R_{n+1}^2 + \lambda_1 \lambda_3 R_{n+2}^2 + \lambda_2 \lambda_3 R_{n+3}^2}{N_{\mathcal{R}_n}}},$$

where $\hat{\mathcal{R}}_n$ is called generalized unit vector of repunit 3-parameter generalized quaternions.

Theorem 9. The matrix representation of n th 3-parameter generalized quaternion is expressed as

$$\mathcal{M}_{\mathcal{R}_n} = \begin{pmatrix} R_n & -\lambda_1 \lambda_2 R_{n+1} & -\lambda_1 \lambda_3 R_{n+2} & -\lambda_2 \lambda_3 R_{n+3} \\ R_{n+1} & R_n & -\lambda_3 R_{n+3} & \lambda_3 R_{n+2} \\ R_{n+2} & \lambda_2 R_{n+3} & R_n & -\lambda_2 R_{n+1} \\ R_{n+3} & -\lambda_1 R_{n+2} & \lambda_1 R_{n+1} & R_n \end{pmatrix},$$

where the matrix $\mathcal{M}_{\mathcal{R}_n}$ is called fundamental matrix for n th repunit 3-parameter generalized quaternion.

Proof. Multiplying $\mathcal{R}_n = R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k$ with $1, i, j, k$ from the left side and using Table 1, we attain

$$\begin{cases} \mathcal{R}_n 1 = R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k, \\ \mathcal{R}_n i = -\lambda_1 \lambda_2 R_{n+1} + R_n i + \lambda_2 R_{n+3}j - \lambda_1 R_{n+2}k, \\ \mathcal{R}_n j = -\lambda_1 \lambda_3 R_{n+2} - \lambda_3 R_{n+3}i + R_n j + \lambda_1 R_{n+1}k, \\ \mathcal{R}_n k = -\lambda_2 \lambda_3 R_{n+3} + \lambda_3 R_{n+2}i - \lambda_2 R_{n+1}j + R_n k. \end{cases}$$

Then, writing the coefficients of $\{1, i, j, k\}$ of the above equations as columns constructs the matrix $\mathcal{M}_{\mathcal{R}_n}$. $\square$

From the values of $\lambda_{i \in \{1,2,3\}}$, $\mathcal{M}_{\mathcal{R}_n}$ can be classified. For $\lambda_1 = 1, \lambda_2, \lambda_3 \in \mathbb{R}$, the fundamental matrix for repunit 2-parameter generalized quaternions is obtained. For $\lambda_1 = \lambda_2 = 1, \lambda_3 = -1$, then the fundamental matrix for repunit split quaternions is given. Additionally, for $\lambda_1 = \lambda_2 = \lambda_3 = 1$, then the fundamental matrix for repunit Hamilton quaternion is written.

Remark 1. For every $m, n \in \mathbb{N}$, let $\mathcal{R}_n$ and $\mathcal{R}_m$ be n th and m th repunit 3-parameter generalized quaternions. Then, the following matrix computation can be given.

$$\mathcal{M}_{\mathcal{R}_n} (\mathcal{R}_m^*)^t = \mathcal{M}_{\mathcal{R}_m} (\mathcal{R}_n^*)^t = (\mathcal{R}_n \mathcal{R}_m)^*,$$

where the superscript $*$ represents column matrix forms. So,

$$\begin{aligned}\mathcal{R}_n^* &= \begin{pmatrix} R_n & R_{n+1} & R_{n+2} & R_{n+3} \end{pmatrix}^t, \\ \mathcal{R}_m^* &= \begin{pmatrix} R_m & R_{m+1} & R_{m+2} & R_{m+3} \end{pmatrix}^t,\end{aligned}$$

and

$$(\mathcal{R}_n \mathcal{R}_m)^* = \begin{pmatrix} R_n R_m - \lambda_1 \lambda_2 R_{n+1} R_{m+1} - \lambda_1 \lambda_3 R_{n+2} R_{m+2} - \lambda_2 \lambda_3 R_{n+3} R_{m+3} \\ R_n R_{m+1} + R_{n+1} R_m + \lambda_3 R_{n+2} R_{m+3} - \lambda_3 R_{n+3} R_{m+2} \\ R_n R_{m+2} + R_{n+2} R_m - \lambda_2 R_{n+1} R_{m+3} + \lambda_2 R_{n+3} R_{m+1} \\ R_n R_{m+3} + R_{n+3} R_m + \lambda_1 R_{n+1} R_{m+2} - \lambda_1 R_{n+2} R_{m+1} \end{pmatrix}.$$

Definition 3. Let $\mathcal{R}_n$ be the n th repunit 3-parameter generalized quaternions for every $n \in \mathbb{N}$. Then, the following mathematical equations hold

1. The determinant of $\mathcal{M}_{\mathcal{R}_n}$: $\det(\mathcal{M}_{\mathcal{R}_n}) = N_{\mathcal{R}_n}^2$.

2. The characteristic polynomial of $\mathcal{M}_{\mathcal{R}_n}$:

$$\varrho_{\mathcal{M}_{\mathcal{R}_n}}(\mathbf{v}) = (\mathbf{v}^2 - 2\mathbf{v}R_n + R_n^2 + \lambda_1 \lambda_2 R_{n+1}^2 + \lambda_1 \lambda_3 R_{n+2}^2 + \lambda_2 \lambda_3 R_{n+3}^2)^2.$$

3. The characteristic equation of $\mathcal{M}_{\mathcal{R}_n}$:

$$\begin{aligned}\det(\mathcal{M}_{\mathcal{R}_n} - \mathbf{v}I_4) &= 0 \\ \Leftrightarrow \varrho_{\mathcal{M}_{\mathcal{R}_n}}(\mathbf{v}) &= (\mathbf{v}^2 - 2\mathbf{v}R_n + R_n^2 + \lambda_1 \lambda_2 R_{n+1}^2 + \lambda_1 \lambda_3 R_{n+2}^2 + \lambda_2 \lambda_3 R_{n+3}^2)^2 \\ &= 0.\end{aligned}$$

4. The eigenvalues of $\mathcal{M}_{\mathcal{R}_n}$:

$$\begin{aligned}\mathfrak{R}_{1,2} &= R_n + \sqrt{-\lambda_1 \lambda_2 R_{n+1}^2 - \lambda_1 \lambda_3 R_{n+2}^2 - \lambda_2 \lambda_3 R_{n+3}^2}, \\ \mathfrak{R}_{3,4} &= R_n - \sqrt{-\lambda_1 \lambda_2 R_{n+1}^2 - \lambda_1 \lambda_3 R_{n+2}^2 - \lambda_2 \lambda_3 R_{n+3}^2}.\end{aligned}$$

5. Multiplication of eigenvalues of $\mathcal{M}_{\mathcal{R}_n}$:

$$\mathfrak{R}_{1,2} \mathfrak{R}_{3,4} = R_n^2 + \lambda_1 \lambda_2 R_{n+1}^2 + \lambda_1 \lambda_3 R_{n+2}^2 + \lambda_2 \lambda_3 R_{n+3}^2 = N_{\mathcal{R}_n}.$$

6. The eigenvectors corresponding to the eigenvalue $\mathfrak{R}_{1,2}$ of $\mathcal{M}_{\mathcal{R}_n}$:

$$\begin{pmatrix} \frac{\lambda_1 R_{n+2} \sqrt{-\lambda_1 \lambda_2 R_{n+1}^2 - \lambda_1 \lambda_3 R_{n+2}^2 - \lambda_2 \lambda_3 R_{n+3}^2} - \lambda_1 \lambda_2 R_{n+1} R_{n+3}}{\lambda_1 R_{n+2}^2 + \lambda_2 R_{n+3}^2} \\ \frac{R_{n+3} \sqrt{-\lambda_1 \lambda_2 R_{n+1}^2 - \lambda_1 \lambda_3 R_{n+2}^2 - \lambda_2 \lambda_3 R_{n+3}^2} + \lambda_1 R_{n+1} R_{n+2}}{\lambda_1 R_{n+2}^2 + \lambda_2 R_{n+3}^2} \\ 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{\lambda_2 R_{n+3} \sqrt{-\lambda_1 \lambda_2 R_{n+1}^2 - \lambda_1 \lambda_3 R_{n+2}^2 - \lambda_2 \lambda_3 R_{n+3}^2} + \lambda_1 \lambda_2 R_{n+1} R_{n+2}}{\lambda_1 R_{n+2}^2 + \lambda_2 R_{n+3}^2} \\ - \frac{R_{n+2} \sqrt{-\lambda_1 \lambda_2 R_{n+1}^2 - \lambda_1 \lambda_3 R_{n+2}^2 - \lambda_2 \lambda_3 R_{n+3}^2} - \lambda_2 R_{n+1} R_{n+3}}{\lambda_1 R_{n+2}^2 + \lambda_2 R_{n+3}^2} \\ 0 \\ 1 \end{pmatrix}.$$

7. The eigenvectors corresponding to the eigenvalue $\mathfrak{R}_{3,4}$ of $\mathcal{M}_{\mathcal{R}_n}$:

$$\begin{pmatrix} \frac{\lambda_1 R_{n+2} \sqrt{-\lambda_1 \lambda_2 R_{n+1}^2 - \lambda_1 \lambda_3 R_{n+2}^2 - \lambda_2 \lambda_3 R_{n+3}^2} + \lambda_1 \lambda_2 R_{n+1} R_{n+3}}{\lambda_1 R_{n+2}^2 + \lambda_2 R_{n+3}^2} \\ - \frac{R_{n+3} \sqrt{-\lambda_1 \lambda_2 R_{n+1}^2 - \lambda_1 \lambda_3 R_{n+2}^2 - \lambda_2 \lambda_3 R_{n+3}^2} - \lambda_1 R_{n+1} R_{n+2}}{\lambda_1 R_{n+2}^2 + \lambda_2 R_{n+3}^2} \\ 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} - \frac{\lambda_2 R_{n+3} \sqrt{-\lambda_1 \lambda_2 R_{n+1}^2 - \lambda_1 \lambda_3 R_{n+2}^2 - \lambda_2 \lambda_3 R_{n+3}^2} - \lambda_1 \lambda_2 R_{n+1} R_{n+2}}{\lambda_1 R_{n+2}^2 + \lambda_2 R_{n+3}^2} \\ \frac{R_{n+2} \sqrt{-\lambda_1 \lambda_2 R_{n+1}^2 - \lambda_1 \lambda_3 R_{n+2}^2 - \lambda_2 \lambda_3 R_{n+3}^2} + \lambda_2 R_{n+1} R_{n+3}}{\lambda_1 R_{n+2}^2 + \lambda_2 R_{n+3}^2} \\ 0 \\ 1 \end{pmatrix}.$$

Let us construct a numerical example.

Example 1. Let $\mathcal{R}_5$ be the 5th repunit 3-parameter generalized quaternions.

* According to equation (2), polar representation of $\tilde{u}_6$ is as follows:

$$\mathcal{R}_5 = \sqrt{123454321 + 12345654321\lambda_1\lambda_2 + 1234567654321\lambda_1\lambda_3 + 123456787654321\lambda_2\lambda_3} \left(\cos \eta + \hat{\mathcal{R}}_5 \sin \eta \right),$$

where

$$\hat{\mathcal{R}}_5 = \frac{(111111, 111111, 111111)}{\sqrt{12345654321\lambda_1\lambda_2 + 1234567654321\lambda_1\lambda_3 + 123456787654321\lambda_2\lambda_3}}$$

is 3-parameter generalized unit vector of $\mathcal{R}_5$ and

$$\begin{cases} \cos \eta = \frac{11111}{\sqrt{123454321 + 12345654321\lambda_1\lambda_2 \\ + 1234567654321\lambda_1\lambda_3 + 123456787654321\lambda_2\lambda_3}}, \\ \sin \eta = \frac{\frac{1111111\lambda_1\lambda_2 + 1111111\lambda_1\lambda_3 + 1111111\lambda_2\lambda_3}{123454321 + 12345654321\lambda_1\lambda_2}}{\sqrt{+ 1234567654321\lambda_1\lambda_3 + 123456787654321\lambda_2\lambda_3}}. \end{cases}$$

* The following matrix is written as follows

$$\mathcal{M}_{\mathcal{R}_5} = \begin{pmatrix} 11111 & -111111\lambda_1\lambda_2 & -1111111\lambda_1\lambda_3 & -11111111\lambda_2\lambda_3 \\ 111111 & 11111 & 11111111\lambda_3 & 1111111\lambda_3 \\ 1111111 & 11111111\lambda_2 & 11111 & -111111\lambda_2 \\ 11111111 & -1111111\lambda_1 & 111111\lambda_1 & 11111 \end{pmatrix}.$$

* The determinant of $\mathcal{M}_{\mathcal{R}_5}$ is determined as

$$\det(\mathcal{M}_{\mathcal{R}_5}) = \left(\frac{123454321 + 12345654321\lambda_1\lambda_2}{+ 1234567654321\lambda_1\lambda_3 + 123456787654321\lambda_2\lambda_3} \right)^2 = (N_{\mathcal{R}_5})^2.$$

* The characteristic polynomial of $\mathcal{M}_{\mathcal{R}_5}$ is

$$\varrho_{\mathcal{R}_5}(\mathbf{v}) = \left(\mathbf{v}^2 - 246908642\mathbf{v} + \frac{123454321 + 12345654321\lambda_1\lambda_2}{+ 1234567654321\lambda_1\lambda_3 + 123456787654321\lambda_2\lambda_3} \right)^2.$$

* The eigenvalues of $\mathcal{M}_{\mathcal{R}_5}$ are written as follows

$$\mathcal{T}_{1,2} = 11111 + \sqrt{-12345654321\lambda_1\lambda_2 - 1234567654321\lambda_1\lambda_3 - 123456787654321\lambda_2\lambda_3}$$

and

$$\mathcal{T}_{3,4} = 11111 - \sqrt{-12345654321\lambda_1\lambda_2 - 1234567654321\lambda_1\lambda_3 - 123456787654321\lambda_2\lambda_3}.$$

* The multiplication of the eigenvalues of $\mathcal{M}_{\mathcal{R}_5}$ is:

$$\begin{aligned} \mathcal{T}_{1,2}\mathcal{T}_{3,4} &= 123454321 + 12345654321\lambda_1\lambda_2 + 1234567654321\lambda_1\lambda_3 \\ &\quad + 123456787654321\lambda_2\lambda_3 \\ &= N_{\mathcal{R}_5}. \end{aligned}$$

* The eigenvectors corresponding to the eigenvalue $\mathcal{T}_{1,2}$ of $\mathcal{M}_{\mathcal{R}_5}$ are

$$\begin{pmatrix} 1111111\lambda_1 \sqrt{\begin{matrix} -12345654321\lambda_1\lambda_2 \\ -1234567654321\lambda_1\lambda_3 \\ -123456787654321\lambda_2\lambda_3 \end{matrix}} - \lambda_1\lambda_2 1.23456665 \times 10^{12} \\ \hline 1234567654321\lambda_1 + 123456787654321\lambda_2 \\ 1111111\lambda_1 \sqrt{\begin{matrix} -12345654321\lambda_1\lambda_2 \\ -1234567654321\lambda_1\lambda_3 \\ -123456787654321\lambda_2\lambda_3 \end{matrix}} + \lambda_1 1.23456654 \times 10^{11} \\ \hline 1234567654321\lambda_1 + 123456787654321\lambda_2 \\ 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1111111\lambda_2 \sqrt{\begin{matrix} -12345654321\lambda_1\lambda_2 \\ -1234567654321\lambda_1\lambda_3 \\ -123456787654321\lambda_2\lambda_3 \end{matrix}} + \lambda_1\lambda_2 1.23456654 \times 10^{11} \\ \hline 1234567654321\lambda_1 + 123456787654321\lambda_2 \\ 1111111 \sqrt{\begin{matrix} -12345654321\lambda_1\lambda_2 \\ -1234567654321\lambda_1\lambda_3 \\ -123456787654321\lambda_2\lambda_3 \end{matrix}} - \lambda_2 = 1.23456654 \times 10^{12} \\ \hline 1234567654321\lambda_1 + 123456787654321\lambda_2 \\ 0 \\ 1 \end{pmatrix}$$

* The eigenvectors corresponding to the eigenvalue $\mathcal{T}_{3,4}$ of $\mathcal{M}_{\mathcal{R}_5}$ are

$$\begin{pmatrix} 1111111\lambda_1 \sqrt{\begin{matrix} -12345654321\lambda_1\lambda_2 \\ -1234567654321\lambda_1\lambda_3 \\ -123456787654321\lambda_2\lambda_3 \end{matrix}} + \lambda_1\lambda_2 1.23456665 \times 10^{12} \\ \hline 1234567654321\lambda_1 + 123456787654321\lambda_2 \\ 1111111\lambda_1 \sqrt{\begin{matrix} -12345654321\lambda_1\lambda_2 \\ -1234567654321\lambda_1\lambda_3 \\ -123456787654321\lambda_2\lambda_3 \end{matrix}} - \lambda_1 1.23456654 \times 10^{11} \\ \hline 1234567654321\lambda_1 + 123456787654321\lambda_2 \\ 1 \\ 0 \end{pmatrix}$$

and

$$\left(\begin{array}{c} 11111111\lambda_2 \sqrt{\begin{array}{l} -12345654321\lambda_1\lambda_2 \\ -1234567654321\lambda_1\lambda_3 \\ -123456787654321\lambda_2\lambda_3 \end{array}} - \lambda_1\lambda_2 1.23456654 \times 10^{11} \\ \hline 1234567654321\lambda_1 + 123456787654321\lambda_2 \\ 1111111 \sqrt{\begin{array}{l} -12345654321\lambda_1\lambda_2 \\ -1234567654321\lambda_1\lambda_3 \\ -123456787654321\lambda_2\lambda_3 \end{array}} + \lambda_2 1.23456654 \times 10^{12} \\ \hline 1234567654321\lambda_1 + 123456787654321\lambda_2 \\ 0 \\ 1 \end{array} \right)$$

4 Repunit Dual Quaternions

In this section, we investigate and examine the dual quaternions with repunit number components.

This section is given without proof for the sake of brevity.

Definition 4. Let $\mathcal{R}_n$ be the n th repunit dual quaternion. Then, it is determined as follows:

$$\mathcal{R}_n = R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k \quad \text{for every } n \in \mathbb{N}, \quad (9)$$

where R_n is the n th repunit number and i, j, k are satisfied rules written in equation (1). The initial conditions of repunit dual quaternions can be expressed as follows:

$$\begin{cases} \mathcal{R}_0 = i + 11j + 111k, \\ \mathcal{R}_1 = 1 + 11i + 111j + 1111k. \end{cases}$$

Let us give some fundamental algebraic properties of $\mathcal{R}_n$. Let $\mathcal{R}_n = R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k$ and $\mathcal{R}_m = R_m + R_{m+1}i + R_{m+2}j + R_{m+3}k$ be the n th and m th repunit dual quaternion for every $n, m \in \mathbb{N}$, respectively. The followings are written:

- * $\mathcal{R}_n = \mathcal{R}_m \Leftrightarrow R_n = R_m, R_{n+1} = R_{m+1}, R_{n+2} = R_{m+2}, R_{n+3} = R_{m+3},$
- * $\mathcal{R}_n \pm \mathcal{R}_m = (R_n \pm R_m) + (R_{n+1} \pm R_{m+1})i + (R_{n+2} \pm R_{m+2})j + (R_{n+3} \pm R_{m+3})k,$
- * $\omega \mathcal{R}_n = \omega R_n + \omega R_{n+1}i + \omega R_{n+2}j + \omega R_{n+3}k, \quad \omega \in \mathbb{R},$

$$* \mathcal{R}_n \mathcal{R}_m = R_n R_m + (R_n R_{m+1} + R_{n+1} R_m) i + (R_n R_{m+2} + R_{n+2} R_m) j + (R_n R_{m+3} + R_{n+3} R_m) k,$$

where (1) is satisfied.

$$* \overline{\mathcal{R}}_n = R_n - R_{n+1} i - R_{n+2} j - R_{n+3} k,$$

where $\overline{\mathcal{R}}_n$ denotes the conjugate of $\mathcal{R}_n$. Also, the scalar part of $\mathcal{R}_n$ is $S_{\mathcal{R}_n} = R_n$. The vector part of $\mathcal{R}_n$ is $\vec{V}_{\mathcal{R}_n} = R_{n+1} i + R_{n+2} j + R_{n+3} k$. Then: $S_{\mathcal{R}_n \pm \mathcal{R}_m} = R_n \pm R_m = S_{\mathcal{R}_n} \pm S_{\mathcal{R}_m}$ and $\vec{V}_{\mathcal{R}_n \pm \mathcal{R}_m} = \vec{V}_{\mathcal{R}_n} \pm \vec{V}_{\mathcal{R}_m}$.

Theorem 10 (Recurrence Relation). *Let $\mathcal{R}_n$ be the n th repunit dual quaternion. The following recurrence relation is satisfied as follows:*

$$\mathcal{R}_{n+2} = 11\mathcal{R}_{n+1} - 10\mathcal{R}_n \quad \text{for every } n \geq 0. \quad (10)$$

Let us establish the Maple 12 code to calculate repunit 3-parameter generalized quaternions:

```
> restart: with(LinearAlgebra):with(linalg):
> R(n):
> R:=proc(n)
> if n=0 then return 0:
> elif n=1 then return 1:
> elif n=2 then return 11:
> else return ((10^n)-1)*(1/9)
> end if;
> end proc;
> R_ast(n):
> R_ast:=proc(n)
> if n=0 then return R(0)+R(1)*i+R(2)*j+R(3)*k:
> elif n=1 then return R(1)+R(2)*i+R(3)*j+R(4)*k:
> elif n=2 then return 11*(R(1)+R(2)*i+R(3)*j+R(4)*k)-10*(R(0)+R
  ↪ (1)*i+R(2)*j+R(3)*k):
> else return 11*R_ast(n-1)-10*R_ast(n-2)
> end if;
> end proc;
```

*The notations R and R_ast represent, repunit number and repunit dual quaternion, respectively.

Theorem 11. *For every $n \in \mathbb{N}$, the followings are satisfied:*

$$* \mathcal{R}_n - \mathcal{R}_{n+1} i - \mathcal{R}_{n+2} j - \mathcal{R}_{n+3} k = R_n,$$

- * $\mathcal{R}_n + \mathcal{R}_{n+1}i + \mathcal{R}_{n+2}j + \mathcal{R}_{n+3}k = 2\mathcal{R}_n - R_n,$
- * $\mathcal{R}_n \overline{\mathcal{R}_n} = R_n^2,$
- * $\mathcal{R}_n + \overline{\mathcal{R}_n} = 2R_n,$
- * $\mathcal{R}_n - \overline{\mathcal{R}_n} = 2\mathcal{R}_n - 2R_n,$
- * $\mathcal{R}_n^2 = R_n^2 + 2R_n R_{n+1}i + 2R_n R_{n+2}j + 2R_n R_{n+3}k = 2R_n \mathcal{R}_n - R_n^2,$
- * $\overline{\mathcal{R}_n}^2 = R_n^2 - 2R_n R_{n+1}i - 2R_n R_{n+2}j - 2R_n R_{n+3}k = 3R_n^2 - 2R_n \mathcal{R}_n.$

Theorem 12. For all $n \in \mathbb{N}$, the followings are satisfied:

- * $\mathcal{R}_n \mathcal{R}_m - \overline{\mathcal{R}_n} \overline{\mathcal{R}_m} = 2(R_n \mathcal{R}_m - R_m \mathcal{R}_n - 2R_n R_m),$
- * $\mathcal{R}_n \mathcal{R}_m + \overline{\mathcal{R}_n} \overline{\mathcal{R}_m} = 2R_n R_m,$
- * $\mathcal{R}_n \overline{\mathcal{R}_m} - \overline{\mathcal{R}_n} \mathcal{R}_m = 2(R_m \mathcal{R}_n - R_n \mathcal{R}_m),$
- * $\mathcal{R}_n \overline{\mathcal{R}_m} + \overline{\mathcal{R}_n} \mathcal{R}_m = 2R_n R_m.$

Theorem 13. For every $n \in \mathbb{N}$, Binet formula for repunit dual quaternions is written

$$\mathcal{R}_n = \frac{10^n \phi_1 - \phi_2}{9}, \quad (11)$$

where $\check{\phi}_1 = 1 + 10i + 100j + 1000k$ and $\check{\phi}_2 = 1 + i + j + k.$

Theorem 14. For all $n \in \mathbb{N}$, the generating function for repunit dual quaternions is expressed as follows:

$$\sum_{n=0}^{\infty} \mathcal{R}_n x^n = \frac{\mathcal{R}_0 + (\mathcal{R}_1 - 11\mathcal{R}_0)x}{1 - 11x + 10x^2}. \quad (12)$$

Theorem 15. The exponential generating function of repunit dual quaternions is expressed as

$$\sum_{n=0}^{\infty} \mathcal{R}_n \frac{y^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{10^n \check{\phi}_1 - \check{\phi}_2}{9} \right) \frac{y^n}{n!}. \quad (13)$$

Theorem 16. The Poisson generating function of repunit dual quaternions is as follows.

$$e^{-y} \sum_{n=0}^{\infty} \mathcal{R}_n \frac{y^n}{n!} = e^{-y} \sum_{n=0}^{\infty} \left(\frac{10^n \check{\phi}_1 - \check{\phi}_2}{9} \right) \frac{y^n}{n!}$$

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