



Convergence analysis of an iterative algorithm for generalized nonexpansive mappings in Hadamard manifolds

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Abstract

This paper presents a novel iterative scheme for approximating fixed points of generalized nonexpansive mappings in Hadamard manifolds. Under mild and standard conditions on the control parameters, we establish comprehensive convergence results for the algorithm. We further introduce an iterative framework for addressing inclusion problems associated with these mappings. Numerical experiments are included to highlight the efficiency and improved performance of the proposed method relative to existing algorithms.

1 Introduction

Let C be a nonempty closed convex subset of a Hilbert space H , and let $A: C \rightarrow 2^H$ be a maximal monotone operator. The inclusion problem was first studied by Rockafellar [19] and is commonly formulated as follows:

$$\text{Find } x \in C \text{ such that } 0 \in A(x). \quad (1)$$

The proximal point method was first introduced by Martinet [15] for solving the inclusion problem (1) and was later generalized by Rockafellar. Many

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problems in nonlinear analysis, optimization, and variational inequalities can be reduced to finding the solution of an inclusion problem.

Let $A : C \rightarrow 2^H$ be a multivalued monotone operator and let $\lambda > 0$. The resolvent of A is defined by

$$J_\lambda^B(x) = (I + \lambda A)^{-1}(x).$$

It is well known that $\text{Fix}(J_\lambda^A) = A^{-1}(\mathbf{0})$, where $\text{Fix}(J_\lambda^A)$ is the set of all fixed points and $A^{-1}(\mathbf{0}) = \{x \in C \mid \mathbf{0} \in A(x)\}$. Many problems in linear spaces can be formulated within the framework of nonlinear spaces such as CAT(0) spaces and Hadamard manifolds. In this direction, the inclusion problem (1) was extended by Li *et al.* [12] for the vector field $A : C \rightarrow TM$ defined on Hadamard manifold M and $\mathbf{0}$ is the zero vector of tangent bundle TM .

Let $T : C \rightarrow C$ be a self-mapping defined on a nonempty subset C . The fixed point problem is formulated as follows:

$$\text{Find } x \in C \text{ such that } x = Tx. \quad (2)$$

Many researchers have shown great interest in establishing algorithms for approximating the solution of the fixed point problem. For instance, the well-known Picard iteration, defined by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$, converges to the solution of (2) when T is a contraction. A key motivation for developing new algorithms arises from the fact that the Picard iteration may fail to converge for a nonexpansive mapping. To overcome this limitation, Mann [14] introduced the following iterative scheme defined by:

$$x_{n+1} = (1 - a_n)x_n + a_nTx_n, \quad (3)$$

for all $n \in \mathbb{N}$, where $\{a_n\}$ is a real sequence in $(0, 1)$. However, Mann's iteration converges only weakly to a solution of (2). On the other hand, Sahu *et al.* [1] introduced a two-step iterative algorithms as:

$$\begin{cases} y_n = (1 - b_n)x_n + b_nTx_n, \\ x_{n+1} = (1 - a_n)Tx_n + a_nTy_n, \end{cases} \quad (4)$$

for all $n \in \mathbb{N}$, where $\{a_n\}$ and $\{b_n\}$ are real sequences in $(0, 1)$. The following S-iterative algorithm was studied by Sahu *et al.* [20] to approximate fixed points of nonexpansive mappings in the setting of uniformly convex Banach spaces. The iterative sequence $\{x_n\}$ is defined as:

$$\begin{cases} z_n = (1 - c_n)x_n + c_nTx_n \\ y_n = (1 - b_n)Tx_n + b_nTz_n \\ x_{n+1} = (1 - a_n)Tx_n + a_nTy_n, \end{cases} \quad (5)$$

for all $n \in \mathbb{N}$, where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $(0, 1)$. Padcharoen and Sukprasert [17] proposed an iterative method to find common solutions of accretive operators, convex constrained least squares problems, convex minimization problems, and signal processing problems in Banach spaces. The iterative sequence $\{x_n\}$ is defined as:

$$\begin{cases} z_n = (1 - c_n)x_n + c_nTx_n \\ y_n = (1 - b_n)z_n + b_nTz_n \\ x_{n+1} = (1 - a_n)Tz_n + a_nTy_n, \end{cases} \quad (6)$$

for all $n \in \mathbb{N}$, where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $(0, 1)$. In recent work, we proposed an iterative scheme [10] (referred to as the PJ iteration, with the remark that it was an adaptation of an iterative process introduced in [8]) to establish several convergence results in the framework of $CAT(0)$ spaces and showed that it is more efficient than many other methods in the literature. The iterative sequence $\{x_n\}$ is defined as:

$$\begin{cases} w_n = T[(1 - c_n)x_n + c_nTx_n] \\ z_n = T[(1 - b_n)w_n + b_nTw_n] \\ y_n = T[(1 - a_n)z_n + b_nTz_n] \\ x_{n+1} = Ty_n, \end{cases} \quad (7)$$

for all $n \in \mathbb{N}$, where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $(0, 1)$.

In recent years, numerous extensions have been developed to approximate the solution of problem (1) in the setting of nonlinear spaces. Chong *et al.* [6] investigated Mann's iteration (3) in the context of Hadamard manifolds. The S-iterative scheme (5) for approximating fixed points of nonexpansive mappings was recently presented by Sahu *et al.* [21] in the setting of Hadamard manifolds. Khammahawong *et al.* [9] discussed the algorithm (6) to approximate common fixed points of nonexpansive mappings in the framework of Hadamard manifolds. Moreover, Khatibzadeh and Pouladi [11] established mean ergodic theorems for affine nonexpansive mappings in nonpositively curved metric spaces, further emphasizing the relevance of fixed point theory in Hadamard settings.

Iterative processes play a fundamental role in the study of nonlinear problems on Hadamard manifolds, providing constructive tools for approximating fixed points, equilibrium states, and solutions of inclusion problems in curved spaces. In recent years, the development of iterative algorithms adapted to the non-Euclidean geometry of Hadamard manifolds has attracted significant attention. For instance, Ansari *et al.* [2] proposed iterative schemes for systems of variational inclusions, extending classical Euclidean results to the manifold setting. Chaipunya *et al.* [5] introduced an efficient iterative algorithm for handling singularities arising in inclusion problems, while Babu *et al.* [3] developed

an extragradient method suitable for non-monotone equilibrium problems, demonstrating its numerical stability and fast convergence on Hadamard manifolds. More recently, Sahu *et al.* [22] designed a self-adaptive iterative method that improves convergence by dynamically adjusting algorithmic parameters. Furthermore, Nguyen *et al.* [16] studied the existence and solution methods for strongly pseudomonotone equilibrium problems, while Ranjbar [18] established strong convergence results for inclusion problems involving families of monotone operators in Hadamard spaces, highlighting the versatility of iterative frameworks in manifold optimization. Together, these contributions underscore the central importance of iterative methods as a unifying approach for analyzing nonlinear phenomena in the rich geometric setting of Hadamard manifolds.

Motivated by this literature, the goal of this paper is to introduce the iterative process (7) in terms of exponential maps on Hadamard manifolds to approximate common fixed points of nonexpansive mappings. We establish a convergence theorem for our proposed method, which generalizes many existing results in the literature. Additionally, we present numerical experiments to illustrate that our proposed algorithm is more efficient. Finally, we demonstrate the applicability of the method to solve inclusion problems.

2 Preliminaries

Let M be a finite-dimensional manifold with a smooth structure. The tangent bundle of M is defined by $TM = \bigcup_{p \in M} T_p M$, where $T_p M$ is the tangent space

of M at p . An inner product $\langle \cdot, \cdot \rangle_p$ on $T_p M$ is a Riemannian metric on $T_p M$. If $\langle \cdot, \cdot \rangle_p$ is a Riemannian metric for every $p \in M$, then the smooth mapping $\langle \cdot, \cdot \rangle : p \rightarrow \langle \cdot, \cdot \rangle_p$ is called a Riemannian metric on M . Every manifold M is a Riemannian manifold if and only if it is endowed with a Riemannian metric $\langle \cdot, \cdot \rangle_p$ with the corresponding norm denoted by $\| \cdot \|_p$. If there is no ambiguity, the subscript p is omitted. The length of a piecewise smooth curve

$\gamma : [a, b] \rightarrow M$ joining two points p and q is given by $L(\gamma) = \int_a^b \|\gamma'(t)\| dt$,

where $\gamma'(t)$ is the tangent vector at $\gamma(t)$ in the tangent space $T_{\gamma(t)} M$. A Riemannian distance $d(p, q)$ is defined as the minimal length of all curves joining p and q . A single-valued vector field on a smooth manifold is a smooth mapping that assigns to every $p \in M$ a tangent vector $v \in T_p M$.

Let ∇ be the Levi-Civita connection associated with the Riemannian manifold M . A smooth vector field V along a smooth curve γ is said to be parallel $\iff \nabla_{\gamma'} V = \mathbf{0}$. If γ' is parallel to itself, we call it a geodesic, and in this case $\|\gamma'\|$ is constant. The curve γ is considered to be normalized when $\|\gamma'\| = 1$.

If the length of a geodesic joining two points p and q in M is equal to the distance between those points, then we refer to that geodesic as a minimal geodesic.

A Riemannian manifold M is said to be complete if all geodesics can be extended for all values of $t \in \mathbb{R}$. The Hopf-Rinow theorem [23] states that if M is connected and complete, then any two points in M can be joined by a minimizing geodesic. Moreover, in this case, every closed and bounded subset of M is compact.

Let M be a complete Riemannian manifold. The exponential map $\exp_p : T_p M \rightarrow M$ is defined by $\exp_p u = \gamma_u(1, p)$, where $\gamma(\cdot) = \gamma_u(\cdot, p)$ is the geodesic starting at the point p with velocity u , that is, $\gamma_u(0, p) = p$ and $\gamma'_u(0, p) = u$. Then, for each real number t , we have $\exp_p tu = \gamma_u(t, p)$, and it follows that $\exp_p \mathbf{0} = \gamma_u(0, p) = p$.

The exponential mapping $\exp_p : T_p M \rightarrow M$ is a diffeomorphism for $p \in M$. Any two points $p, q \in M$ have a unique minimizing normalized geodesic connecting them, and the exponential map has an inverse $\exp_p^{-1} : M \rightarrow T_p M$ defined by $\exp_p^{-1} q = u$, such that $\gamma_u(0, p) = p$, $\gamma_u(1, p) = q$, and $\gamma'_u(0, p) = u$. We also have $d(p, q) = \|\exp_p^{-1} q\|$ for all $p, q \in M$. A complete, simply connected Riemannian manifold of non-positive sectional curvature is called a Hadamard manifold.

Proposition 1. [23] *Let M be a Hadamard manifold. Then, for any $p \in M$, the exponential map $\exp_p : T_p M \rightarrow M$ is a diffeomorphism, and for any $p, q \in M$, there exists a unique minimal geodesic $\gamma : [0, 1] \rightarrow M$ with $\gamma(0) = p$ and $\gamma(1) = q$, defined as*

$$\gamma(t) = \exp_p t \exp_p^{-1} q,$$

for all $t \in [0, 1]$.

Several concepts and results from convex analysis have been extended to the setting of manifolds. We present some of these that will be used throughout the rest of the paper.

Let M be a Hadamard manifold. A set $C \subset M$ is called geodesically convex if, for any two points p and q in C , the geodesic joining p and q is contained in C . More precisely, if $\gamma : [a, b] \rightarrow M$ is a geodesic such that $p = \gamma(a)$ and $q = \gamma(b)$, then $\gamma(ta + (1 - t)b) \in C$ for all $t \in [0, 1]$.

Definition 1. *A real-valued function $f : M \rightarrow \mathbb{R}$ is called geodesically convex if, for any geodesic $\gamma : [a, b] \rightarrow M$, the composition $f \circ \gamma : [a, b] \rightarrow \mathbb{R}$ is convex, i.e.,*

$$(f \circ \gamma)(ta + (1 - t)b) \leq t(f \circ \gamma)(a) + (1 - t)(f \circ \gamma)(b),$$

for all $t \in [0, 1]$.

Proposition 2. [23] *Let the distance function be $d : M \times M \rightarrow \mathbb{R}$. Then, $d(\cdot, \cdot)$ is a geodesically convex function with respect to the Riemannian metric product. For any pair of geodesics $\gamma_1, \gamma_2 : [0, 1] \rightarrow M$, the following inequality holds:*

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t)d(\gamma_1(0), \gamma_2(0)) + t d(\gamma_1(1), \gamma_2(1)),$$

for all $t \in [0, 1]$.

In particular, for each $q \in M$, the function $d(\cdot, q) : M \rightarrow \mathbb{R}$ is geodesically convex.

We present some geometric properties of finite-dimensional Hadamard manifolds that are analogous to those in Euclidean spaces. A geodesic triangle $\triangle(p_1, p_2, p_3)$ on a Riemannian manifold M is the set consisting of three points p_1, p_2, p_3 and the three minimizing geodesics joining these points.

Proposition 3. [23] *Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle. Then,*

$$d^2(p_1, p_2) + d^2(p_2, p_3) - 2 \langle \exp_{p_2}^{-1} p_1, \exp_{p_2}^{-1} p_3 \rangle \leq d^2(p_3, p_1),$$

and

$$d^2(p_1, p_2) \leq \langle \exp_{p_1}^{-1} p_3, \exp_{p_1}^{-1} p_2 \rangle + \langle \exp_{p_2}^{-1} p_3, \exp_{p_2}^{-1} p_1 \rangle.$$

Moreover, if θ is the angle at p_1 , then we have

$$\langle \exp_{p_1}^{-1} p_2, \exp_{p_1}^{-1} p_3 \rangle = d(p_2, p_1) d(p_1, p_3) \cos \theta.$$

Readers can refer to [4] for further information on the relationship between geodesic triangles on Riemannian manifolds and triangles in $\mathbb{R}^2$.

Lemma 1. [4] *Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle in M . Then, there exists a triangle $\triangle(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ corresponding to $\triangle(p_1, p_2, p_3)$ such that*

$$d(p_i, p_{i+1}) = \|\bar{p}_i - \bar{p}_{i+1}\|_{\mathbb{R}^2}.$$

This triangle is unique up to an isometry of $\mathbb{R}^2$.

The triangle $\triangle(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ is called a comparison triangle for $\triangle(p_1, p_2, p_3)$, and the points $\bar{p}_1, \bar{p}_2, \bar{p}_3$ are called comparison points corresponding to p_1, p_2, p_3 , respectively.

Lemma 2. [4] *Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle in M and $\triangle(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ be its comparison triangle.*

1. *Let $\theta_1, \theta_2, \theta_3$ (respectively, $\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3$) be the angles of $\triangle(p_1, p_2, p_3)$ (respectively, $\triangle(\bar{p}_1, \bar{p}_2, \bar{p}_3)$) at the vertices p_1, p_2, p_3 (respectively, $\bar{p}_1, \bar{p}_2, \bar{p}_3$). Then*

$$\theta_1 \leq \bar{\theta}_1, \quad \theta_2 \leq \bar{\theta}_2, \quad \text{and} \quad \theta_3 \leq \bar{\theta}_3.$$

2. Let q be a point on the geodesic joining p_1 to p_2 , and let $\bar{q}$ be its comparison point on the interval $[\bar{p}_1, \bar{p}_2]$. If $d(p_1, q) = \|\bar{p}_1 - \bar{q}\|_{\mathbb{R}^2}$ and $d(p_2, q) = \|\bar{p}_2 - \bar{q}\|_{\mathbb{R}^2}$, then

$$d(p_3, q) \leq \|\bar{p}_3 - \bar{q}\|_{\mathbb{R}^2}.$$

We now present the following results, which are essential for proving our main convergence theorems.

Definition 2. [7] Assume that $\{p_n\}$ is a sequence in M and that C is a nonempty subset of M . If $d(p_{n+1}, q) \leq d(p_n, q)$ for all $q \in C$ and $n \in \mathbb{N}$, then $\{p_n\}$ is Fejér monotone with respect to C .

Lemma 3. [7] Let C be a nonempty subset of M , and let $\{p_n\} \subset M$ be a sequence in M such that $\{p_n\}$ is Fejér monotone with respect to C . Then, the following statements hold:

1. For every $q \in C$, the sequence $\{d(p_n, q)\}$ converges.
2. The sequence $\{p_n\}$ is bounded.
3. If every cluster point of $\{p_n\}$ belongs to C , then $\{p_n\}$ converges to a point in C .

Now, we recall the concept of monotonicity for a multivalued vector field on Hadamard manifolds. Let $\Psi(M)$ denote the set of all multivalued vector fields $A : M \rightarrow 2^{TM}$ such that $A(x) \subset T_x M$, and let $D(A) = \{x \in M : A(x) \neq \emptyset\}$ denote the domain of A .

Definition 3. [7] A vector field $A \in \Psi(M)$ is said to be:

1. Monotone if, for all $x, y \in D(A)$,

$$\langle u, \exp_x^{-1} y \rangle \leq \langle v, -\exp_y^{-1} x \rangle,$$

for all $u \in A(x)$ and $v \in A(y)$.

2. Maximal monotone if it is monotone and, for all $x \in M$ and $u \in T_x M$, the condition

$$\langle u, \exp_x^{-1} y \rangle \leq \langle v, -\exp_y^{-1} x \rangle, \text{ for all } y \in D(A), v \in A(y) \text{ implies } u \in A(x)$$

holds.

Li *et al.* [13] provided a definition of the resolvent of a multivalued vector field, as well as of a firmly nonexpansive mapping on Hadamard manifolds.

Definition 4. Given $\lambda > 0$ and a multivalued vector field $A \in \Psi(M)$, the resolvent associated with A of order λ is the multivalued map $J_\lambda^A : M \rightarrow 2^M$ defined by

$$J_\lambda^A(x) := \{y \in M : x \in \exp_y^{-1}(\lambda A(y))\}.$$

Remark 1. [13] Assume that $\lambda > 0$. According to the definition of the resolvent, the range of the resolvent J_λ^A is contained in the domain of A , and $\text{Fix}(J_\lambda^A) = A^{-1}(\mathbf{0})$.

Definition 5. [13] A mapping $T : M \rightarrow M$ is said to be firmly nonexpansive if, for any two points $x, y \in M$, the function $\gamma : [0, 1] \rightarrow [0, +\infty)$ defined by

$$\gamma(t) := d(\exp_x t \exp_x^{-1} T x, \exp_y t \exp_y^{-1} T y),$$

for all $t \in [0, 1]$, is nonincreasing.

It is straightforward to deduce that every firmly nonexpansive mapping is nonexpansive. Moreover, monotonicity and expansiveness are closely related concepts.

Theorem 1. [13] Let $A \in \Psi(M)$ be a vector field. Then, the multivalued vector field A is monotone if and only if J_λ^A is single-valued and firmly nonexpansive for all $\lambda > 0$.

3 Main results

The remainder of the article proceeds under the assumption that M is a finite-dimensional Hadamard manifold. Let $S, T : C \rightarrow C$ be nonexpansive mappings, and let C be a nonempty, closed, and geodesically convex subset of the Hadamard manifold. For an initial point $x_0 \in C$, the new proposed iterative sequence $\{x_n\} \subset C$ is defined by

$$\begin{cases} w_n = T[\exp_{x_n} c_n \exp_{x_n}^{-1} T x_n] \\ z_n = T[\exp_{w_n} b_n \exp_{w_n}^{-1} S w_n] \\ y_n = S[\exp_{z_n} a_n \exp_{z_n}^{-1} T z_n] \\ x_{n+1} = S y_n, \end{cases} \quad (8)$$

for all $n \in \mathbb{N}$, where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $(0, 1)$.

Let $\gamma_1 : [0, 1] \rightarrow M$ be a geodesic connecting the points z_n and $T z_n$, $\gamma_2 : [0, 1] \rightarrow M$ be a geodesic connecting the points w_n and $S w_n$, and $\gamma_3 : [0, 1] \rightarrow M$ be a geodesic connecting the points x_n and $T x_n$. Hence, Algorithm (8) can be written as:

$$\begin{cases} w_n = T[\gamma_3(c_n)] \\ z_n = T[\gamma_2(b_n)] \\ y_n = S[\gamma_1(a_n)] \\ x_{n+1} = S y_n, \end{cases}$$

for all $n \in \mathbb{N}$.

Remark 2. We note that the proposed iterative scheme is asymmetric with respect to the nonexpansive mappings S and T . This asymmetry is intentional and reflects the hierarchical structure of the algorithm. It does not affect the convergence analysis, which relies solely on the nonexpansiveness of both mappings and on the Fejér monotonicity of the generated sequence. Moreover, the asymmetric use of S and T allows greater flexibility in applications and includes, as particular cases, several known symmetric and alternating schemes studied in the literature.

Next, we show that the iterative algorithm defined in (8) converges to common fixed points of the nonexpansive mappings under certain conditions.

Theorem 2. Let C be a nonempty, closed, and geodesically convex subset of a Hadamard manifold M , and let $S, T : C \rightarrow C$ be nonexpansive mappings such that $\text{Fix}(S) \cap \text{Fix}(T) \neq \emptyset$. Suppose that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences satisfying $0 < k_1 \leq a_n \leq \bar{k}_1 < 1$, $0 < k_2 \leq b_n \leq \bar{k}_2 < 1$, and $0 < k_3 \leq c_n \leq \bar{k}_3 < 1$, for all $n \in \mathbb{N}$. Then, the iterative sequence $\{x_n\}$ defined by Algorithm (8) converges to a common fixed point of S and T .

Proof. Let $\eta \in \text{Fix}(S) \cap \text{Fix}(T)$. Using the geodesic convexity of the Riemannian distance and the first step of the iterative algorithm (8), we obtain

$$\begin{aligned} d(w_n, \eta) &= d[T(\gamma_3(c_n)), \eta] \\ &\leq (1 - c_n) d(x_n, \eta) + c_n d(Tx_n, \eta) \\ &\leq (1 - c_n) d(x_n, \eta) + c_n d(x_n, \eta) \\ &= d(x_n, \eta). \end{aligned} \tag{9}$$

Using the second step of the iterative algorithm (8), we have

$$\begin{aligned} d(z_n, \eta) &= d[T(\gamma_2(b_n)), \eta] \\ &\leq d(\gamma_2(b_n), \eta) \\ &\leq (1 - b_n) d(w_n, \eta) + b_n d(Sw_n, \eta) \\ &\leq (1 - b_n) d(w_n, \eta) + b_n d(w_n, \eta) \\ &= d(w_n, \eta). \end{aligned} \tag{10}$$

Using the third step of the iterative algorithm (8), we get

$$\begin{aligned} d(y_n, \eta) &= d[S(\gamma_1(a_n)), \eta] \\ &\leq d(\gamma_1(a_n), \eta) \\ &\leq (1 - a_n) d(z_n, \eta) + a_n d(Tz_n, \eta) \\ &\leq (1 - a_n) d(z_n, \eta) + a_n d(z_n, \eta) \\ &= d(z_n, \eta). \end{aligned} \tag{11}$$

From (9), (10) and (11), we conclude that

$$\begin{aligned} d(x_{n+1}, \eta) &= d(Sy_n, \eta) \\ &\leq d(y_n, \eta) \\ &\leq d(x_n, \eta). \end{aligned} \quad (12)$$

Therefore, the sequence $\{x_n\}$ is Fejér monotone with respect to $\text{Fix}(S) \cap \text{Fix}(T)$. By Lemma 3 (ii), the sequence $\{x_n\}$ is bounded. Hence, the sequence has at least one cluster point $p \in \text{Fix}(S) \cap \text{Fix}(T)$.

Let $\triangle(z_n, Tz_n, \eta) \subseteq M$ be a geodesic triangle with vertices z_n, Tz_n, η , and let $\triangle(\bar{z}_n, \overline{Tz_n}, \bar{\eta}) \subseteq \mathbb{R}^2$ be the corresponding comparison triangle in the plane. Then, we obtain

$$\begin{aligned} d(z_n, \eta) &= \|\bar{z}_n - \bar{\eta}\|_{\mathbb{R}^2}, \\ d(Tz_n, \eta) &= \|\overline{Tz_n} - \bar{\eta}\|_{\mathbb{R}^2}, \end{aligned}$$

and

$$d(z_n, Tz_n) = \|\bar{z}_n - \overline{Tz_n}\|_{\mathbb{R}^2}.$$

Let $\overline{\gamma_1(a_n)} = (1 - a_n)\bar{z}_n + a_n\overline{Tz_n}$ be the comparison point of $\gamma_1(a_n)$. Using (ii) of Lemma 2, we have

$$\begin{aligned} d^2(x_{n+1}, \eta) &= d^2(Sy_n, \eta) \\ &\leq d^2[S(\gamma_1(a_n)), \eta] \\ &\leq d^2(\gamma_1(a_n), \eta) \\ &\leq \|\overline{\gamma_1(a_n)} - \bar{\eta}\|_{\mathbb{R}^2}^2 \\ &= \|(1 - a_n)\bar{z}_n + a_n\overline{Tz_n} - \bar{\eta}\|_{\mathbb{R}^2}^2 \\ &= (1 - a_n)\|\bar{z}_n - \bar{\eta}\|_{\mathbb{R}^2}^2 + a_n\|\overline{Tz_n} - \bar{\eta}\|_{\mathbb{R}^2}^2 \\ &\quad - a_n(1 - a_n)\|\bar{z}_n - \overline{Tz_n}\|_{\mathbb{R}^2}^2 \\ &= (1 - a_n)d^2(z_n, \eta) + a_nd^2(Tz_n, \eta) - a_n(1 - a_n)d^2(z_n, Tz_n) \\ &\leq (1 - a_n)d^2(z_n, \eta) + a_nd^2(z_n, \eta) - a_n(1 - a_n)d^2(z_n, Tz_n) \\ &= d^2(z_n, \eta) - a_n(1 - a_n)d^2(z_n, Tz_n) \\ &\leq d^2(x_n, \eta) - a_n(1 - a_n)d^2(z_n, Tz_n). \end{aligned} \quad (13)$$

This implies that

$$a_n(1 - a_n)d^2(z_n, Tz_n) \leq d^2(x_n, \eta) - d^2(x_{n+1}, \eta).$$

Using the fact that $0 < k_1 \leq a_n \leq \bar{k}_1 < 1$, we have $k_1(1 - \bar{k}_1) \leq a_n(1 - a_n)$. Summing from $i = 0$ to $i = n$, we obtain

$$k_1 (1 - \overline{k_1}) \sum_{i=0}^n d^2(z_i, Tz_i) \leq d^2(x_0, \eta) - d^2(x_{n+1}, \eta).$$

Letting $n \rightarrow +\infty$, we have

$$k_1 (1 - \overline{k_1}) \sum_{i=0}^{\infty} d^2(z_i, Tz_i) \leq d^2(x_0, \eta) < +\infty.$$

This leads us to conclude that

$$\lim_{n \rightarrow +\infty} d(z_n, Tz_n) = 0.$$

Now, let $\triangle(w_n, Sw_n, \eta) \subseteq M$ be a geodesic triangle with vertices w_n, Sw_n, η , and let $\triangle(\overline{w_n}, \overline{Sw_n}, \overline{\eta}) \subseteq \mathbb{R}^2$ be the corresponding comparison triangle in the plane. Then, we obtain

$$d(w_n, \eta) = \|\overline{w_n} - \overline{\eta}\|_{\mathbb{R}^2},$$

$$d(Sw_n, \eta) = \|\overline{Sw_n} - \overline{\eta}\|_{\mathbb{R}^2},$$

and

$$d(w_n, Sw_n) = \|\overline{w_n} - \overline{Sw_n}\|_{\mathbb{R}^2}.$$

Let $\overline{\gamma_2(b_n)} = (1 - b_n)\overline{w_n} + b_n\overline{Sw_n}$ be the comparison point of $\gamma_2(b_n)$. Using (ii) of Lemma 2, we get

$$\begin{aligned} d^2(z_n, \eta) &= d^2[T(\gamma_2(b_n)), \eta] \\ &\leq d^2(\gamma_2(b_n), \eta) \\ &\leq \|\overline{\gamma_2(b_n)} - \overline{\eta}\|_{\mathbb{R}^2}^2 \\ &= \|(1 - b_n)\overline{w_n} + b_n\overline{Sw_n} - \overline{\eta}\|_{\mathbb{R}^2}^2 \\ &= (1 - b_n)\|\overline{w_n} - \overline{\eta}\|_{\mathbb{R}^2}^2 + b_n\|\overline{Sw_n} - \overline{\eta}\|_{\mathbb{R}^2}^2 \\ &\quad - b_n(1 - b_n)\|\overline{w_n} - \overline{Sw_n}\|_{\mathbb{R}^2}^2 \\ &= (1 - b_n)d^2(w_n, \eta) + b_nd^2(Sw_n, \eta) - b_n(1 - b_n)d^2(w_n, Sw_n) \\ &\leq (1 - b_n)d^2(w_n, \eta) + b_nd^2(w_n, \eta) - b_n(1 - b_n)d^2(w_n, Sw_n) \\ &= d^2(w_n, \eta) - b_n(1 - b_n)d^2(w_n, Sw_n). \end{aligned} \tag{14}$$

From (12), (13) and (14), we obtain

$$\begin{aligned} d^2(x_{n+1}, \eta) &\leq d^2(z_n, \eta) - a_n(1 - a_n)d^2(z_n, Tz_n) \\ &\leq d^2(x_n, \eta) - b_n(1 - b_n)d^2(w_n, Sw_n) - a_n(1 - a_n)d^2(z_n, Tz_n). \end{aligned}$$

Using the fact that $k_2(1 - \overline{k_2}) \leq b_n(1 - b_n)$, and summing from $i = 0$ to $i = n$, we obtain

$$k_2(1 - \overline{k_2}) \sum_{i=0}^n d^2(w_i, Sw_i) \leq d^2(x_0, \eta) - d^2(x_{n+1}, \eta).$$

Taking the limit as $n \rightarrow +\infty$ yields

$$k_2(1 - \overline{k_2}) \sum_{i=0}^{+\infty} d^2(w_i, Sw_i) \leq d^2(x_0, \eta) < +\infty.$$

This concludes that

$$\lim_{n \rightarrow +\infty} d(w_n, Sw_n) = 0. \quad (15)$$

Let $\triangle(x_n, Tx_n, \eta) \subseteq M$ represent a geodesic triangle with vertices x_n, Tx_n, η , and let $\triangle(\overline{x_n}, \overline{Tx_n}, \overline{\eta}) \subseteq \mathbb{R}^2$ be the corresponding comparison triangle in the plane. Then, we obtain

$$d(x_n, \eta) = \|\overline{x_n} - \overline{\eta}\|_{\mathbb{R}^2},$$

$$d(Tx_n, \eta) = \|\overline{Tx_n} - \overline{\eta}\|_{\mathbb{R}^2},$$

and

$$d(x_n, Tx_n) = \|\overline{x_n} - \overline{Tx_n}\|_{\mathbb{R}^2}.$$

Let $\overline{\gamma_3(c_n)} = (1 - c_n)\overline{x_n} + c_n\overline{Tx_n}$ be the comparison point of $\gamma_3(c_n)$. Using (ii) of Lemma 2, we obtain

$$\begin{aligned} d^2(w_n, \eta) &= d^2[S(\gamma_3(c_n)), \eta] \\ &\leq \|\overline{\gamma_3(c_n)} - \overline{\eta}\|_{\mathbb{R}^2}^2 \\ &= \|(1 - c_n)\overline{x_n} + c_n\overline{Tx_n} - \overline{\eta}\|_{\mathbb{R}^2}^2 \\ &= (1 - c_n)\|\overline{x_n} - \overline{\eta}\|_{\mathbb{R}^2}^2 + c_n\|\overline{Tx_n} - \overline{\eta}\|_{\mathbb{R}^2}^2 \\ &\quad - c_n(1 - c_n)\|\overline{x_n} - \overline{Tx_n}\|_{\mathbb{R}^2}^2 \\ &= (1 - c_n)d^2(x_n, \eta) + c_nd^2(Tx_n, \eta) - c_n(1 - c_n)d^2(x_n, Tx_n) \\ &\leq (1 - c_n)d^2(x_n, \eta) + c_nd^2(x_n, \eta) - c_n(1 - c_n)d^2(x_n, Tx_n) \\ &= d^2(x_n, \eta) - c_n(1 - c_n)d^2(x_n, Tx_n). \end{aligned} \quad (16)$$

Combining (13), (14), and (16) yields

$$\begin{aligned} d^2(x_{n+1}, \eta) &\leq d^2(w_n, \eta) - b_n(1 - b_n)d^2(w_n, Sw_n) - a_n(1 - a_n)d^2(z_n, Tz_n) \\ &\leq d^2(x_n, \eta) - c_n(1 - c_n)d^2(x_n, Tx_n) - b_n(1 - b_n)d^2(w_n, Sw_n) \\ &\quad - a_n(1 - a_n)d^2(z_n, Tz_n). \end{aligned}$$

Using the fact that $k_3(1 - \overline{k_3}) \leq c_n(1 - c_n)$, and summing from $j = 0$ to $j = n$, we obtain

$$k_3(1 - \overline{k_3}) \sum_{i=0}^n d^2(x_i, Tx_i) \leq d^2(x_0, \eta) - d^2(x_{n+1}, \eta).$$

By letting $n \rightarrow +\infty$, we have

$$k_3(1 - \overline{k_3}) \sum_{i=0}^{+\infty} d^2(x_i, Tx_i) \leq d^2(x_0, \eta) < +\infty.$$

Hence,

$$\lim_{n \rightarrow +\infty} d(x_n, Tx_n) = 0. \quad (17)$$

Using the geodesic convexity and (17), we obtain

$$\begin{aligned} d(x_n, w_n) &\leq d(x_n, Tx_n) + d(Tx_n, w_n) \\ &\leq 2d(x_n, Tx_n). \end{aligned}$$

This gives

$$\lim_{n \rightarrow \infty} d(x_n, w_n) = 0. \quad (18)$$

Since S is nonexpansive, we get

$$\begin{aligned} d(x_n, Sx_n) &\leq d(x_n, w_n) + d(w_n, Sw_n) + d(Sw_n, Sx_n) \\ &\leq d(x_n, w_n) + d(w_n, Sw_n) + d(w_n, x_n) \\ &= 2d(x_n, w_n) + d(w_n, Sw_n). \end{aligned}$$

By taking the limit as $n \rightarrow \infty$ and using (15) and (18), we conclude that

$$\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0. \quad (19)$$

Finally,

$$\begin{aligned} d(p, Sp) &\leq d(p, x_{n_i}) + d(x_{n_i}, Sx_{n_i}) + d(Sx_{n_i}, Sp) \\ &\leq 2d(p, x_{n_i}) + d(x_{n_i}, Sx_{n_i}). \end{aligned}$$

and

$$\begin{aligned} d(p, Tp) &\leq d(p, x_{n_i}) + d(x_{n_i}, Tx_{n_i}) + d(Tx_{n_i}, Tp) \\ &\leq 2d(p, x_{n_i}) + d(x_{n_i}, Tx_{n_i}). \end{aligned}$$

By letting $i \rightarrow +\infty$ and using (17) and (19), we deduce that $d(p, Sp) = 0$ and $d(p, Tp) = 0$. This shows that $p \in \text{Fix}(S) \cap \text{Fix}(T)$. According to Lemma 3 (iii), the sequence $\{x_n\}$ generated by Algorithm (8) converges to a common fixed point of S and T . $\square$

Remark 3. *The convergence result established in Theorem 2 does not address the uniqueness of the limit point. This is consistent with the general framework of nonexpansive mappings on Hadamard manifolds, where uniqueness typically requires additional assumptions such as strict contractivity or stronger convexity conditions. Investigating conditions that ensure uniqueness of the limit, including stronger geometric assumptions on the manifold or additional properties of the involved mappings, will be considered in future work.*

In what follows, based on an adaptation of an example presented in [9], we utilize the PJ iterative process to develop a numerical simulation that demonstrates the efficiency of the newly introduced iteration.

Example 1. *Let $\mathbb{H}^3 = \{p = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4 : \langle p, p \rangle = -1, p_4 > 0\}$ be the 3-dimensional hyperbolic space endowed with the Lorentz metric $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^4$ defined by*

$$\langle p, q \rangle = \sum_{i=1}^3 p_i q_i - p_4 q_4, \quad p, q \in \mathbb{H}^3.$$

The Riemannian distance $d : \mathbb{H}^3 \times \mathbb{H}^3 \rightarrow \mathbb{R}$ is induced from the Lorentz metric as

$$d(p, q) = \arccos h(-\langle p, q \rangle).$$

Then, $\mathbb{H}^3$ is a Hadamard manifold with sectional curvature -1 . The normalized geodesic $\gamma : \mathbb{R} \rightarrow \mathbb{H}^3$ with starting point $p \in \mathbb{H}^3$ and velocity vector $v \in T_p \mathbb{H}^3$ is defined as

$$\gamma(t) = (\cosh t)p + (\sinh t)v,$$

for all $t \in \mathbb{R}$.

The exponential map $\exp_p : T_p \mathbb{H}^3 \rightarrow \mathbb{H}^3$ is defined as

$$\exp_p tv = (\cosh t)p + (\sinh t)v.$$

Then, the inverse exponential map $\exp_p^{-1} : \mathbb{H}^3 \rightarrow T_p\mathbb{H}^3$ is

$$\exp_p^{-1} q = \arccos h(-\langle p, q \rangle) \frac{q + \langle p, q \rangle p}{\sqrt{\langle p, q \rangle^2 - 1}},$$

for all $p, q \in \mathbb{H}^3$.

Set $r(p, q) = \arccos h(-\langle p, q \rangle)$ and $v(p, q) = \frac{q + \langle p, q \rangle p}{\sqrt{\langle p, q \rangle^2 - 1}}$. Then,

$$\exp_p^{-1} q = r(p, q) \cdot v(p, q).$$

Let $T : \mathbb{H}^3 \rightarrow \mathbb{H}^3$ be a nonexpansive mapping defined by

$$T(p_1, p_2, p_3, p_4) = (-p_1, -p_2, -p_3, p_4),$$

for all $p = (p_1, p_2, p_3, p_4) \in \mathbb{H}^3$. Clearly, the only fixed point of T is $(0, 0, 0, 1)$.

The Mann iteration (3), the two-step S -iteration (4), the three-step S -iteration (5), Padcharoen's iteration (6), and our proposed PJ iteration (7) are defined as follows:

$$x_{n+1} = (\cosh(1 - a_n) r(x_n, Tx_n)) x_n + (\sinh(1 - a_n) r(x_n, Tx_n)) v(x_n, Tx_n),$$

$$\begin{cases} y_n &= (\cosh(1 - c_n) r(x_n, Tx_n)) x_n \\ &\quad + (\sinh(1 - c_n) r(x_n, Tx_n)) v(x_n, Tx_n) \\ x_{n+1} &= (\cosh(1 - a_n) r(Tx_n, Ty_n)) Tx_n \\ &\quad + (\sinh(1 - a_n) r(Tx_n, Ty_n)) v(Tx_n, Ty_n), \end{cases}$$

$$\begin{cases} z_n &= (\cosh(1 - c_n) r(x_n, Tx_n)) x_n \\ &\quad + (\sinh(1 - c_n) r(x_n, Tx_n)) v(x_n, Tx_n) \\ y_n &= (\cosh(1 - b_n) r(Tx_n, Tz_n)) Tx_n \\ &\quad + (\sinh(1 - b_n) r(Tx_n, Tz_n)) v(Tx_n, Tz_n) \\ x_{n+1} &= (\cosh(1 - a_n) r(Tx_n, Ty_n)) Tx_n \\ &\quad + (\sinh(1 - a_n) r(Tx_n, Ty_n)) v(Tx_n, Ty_n), \end{cases}$$

$$\begin{cases} z_n &= (\cosh(1 - c_n) r(x_n, Tx_n)) x_n \\ &\quad + (\sinh(1 - c_n) r(x_n, Tx_n)) v(x_n, Tx_n) \\ y_n &= (\cosh(1 - b_n) r(z_n, Tz_n)) z_n \\ &\quad + (\sinh(1 - b_n) r(z_n, Tz_n)) v(z_n, Tz_n) \\ x_{n+1} &= (\cosh(1 - a_n) r(Tz_n, Ty_n)) Tz_n \\ &\quad + (\sinh(1 - a_n) r(Tz_n, Ty_n)) v(Tz_n, Ty_n), \end{cases}$$

and

$$\begin{cases} w_n &= T[(\cosh(1 - c_n) r(x_n, Tx_n)) x_n \\ &\quad + (\sinh(1 - c_n) r(x_n, Tx_n)) v(x_n, Tx_n)] \\ z_n &= T[(\cosh(1 - b_n) r(w_n, Tw_n)) w_n \\ &\quad + (\sinh(1 - b_n) r(w_n, Tw_n)) v(w_n, Tw_n)] \\ y_n &= T[(\cosh(1 - a_n) r(z_n, Tz_n)) z_n \\ &\quad + (\sinh(1 - a_n) r(z_n, Tz_n)) v(z_n, Tz_n)] \\ x_{n+1} &= Ty_n. \end{cases}$$

Now, we investigate the convergence behaviour of different iterative algorithms for the initial point

$$x_0 = (0.69445440978475, 1.01382609280137, 0.99360871330745, 1.87012527625153)$$

and the sequences of parameters $a_n = \frac{n}{5n+1}$, $b_n = \frac{n}{(n+11)^{3/2}}$, and $c_n = \frac{n}{7n+2}$. Let $D_n = d(x_n, x^*) < 10^{-6}$ be the stopping criterion.

The following data are presented to justify the efficiency of the newly introduced iterative process, as follows:

- Table 1 displays the fixed point approximation, considering a maximum of 20 iterations. It can be observed that the PJ iteration reaches the exact value as early as the 11th iteration, significantly faster than the other methods;
- Table 2 summarizes the information from Table 1 and additionally provides the execution time of each algorithm. The new iterative process yields the shortest execution time;
- Table 3 and Figure 1 present the error between the exact and numerical solutions, both numerically and graphically. It is evident that the PJ iteration again delivers the best performance, achieving the smallest error.

Iteration	Mean	S-iteration (rank 2)	S-iteration (rank 3)	F-iteration	PJ (New iteration)
1	[0.694454, 1.013826, 0.993608, 1.870125]	[0.694454, 1.013826, 0.993608, 1.870125]	[0.694454, 1.013826, 0.993608, 1.870125]	[0.694454, 1.013826, 0.993608, 1.870125]	[0.694454, 1.013826, 0.993608, 1.870125]
2	[-0.383326, -0.559614, -0.548454, 1.326092]	[0.334271, 0.487998, 0.478267, 1.256431]	[-0.372102, -0.543228, -0.532395, 1.310344]	[-0.245095, -0.357811, -0.350676, 1.145021]	[-0.142647, -0.209349, -0.204096, 1.051396]
3	[0.225234, 0.328947, 0.323388, 1.123793]	[0.167852, 0.245045, 0.240158, 1.079466]	[0.207339, 0.302692, 0.296656, 1.105729]	[0.094196, 0.137515, 0.134773, 1.022715]	[0.034276, 0.050040, 0.049042, 1.000037]
4	[-0.136087, 0.198672, -0.194701, 1.04854]	[0.087749, 0.122265, 0.119828, 1.017998]	[-0.116145, -0.160569, -0.160178, 1.034338]	[-0.055573, -0.053932, -0.050887, 1.032371]	[-0.008124, -0.011860, -0.011623, 1.000171]
5	[0.082947, 0.121093, 0.118678, 1.017658]	[0.040783, 0.050539, 0.058351, 1.004297]	[0.064329, 0.093913, 0.092041, 1.010658]	[0.013004, 0.018984, 0.018606, 1.000438]	[0.001863, 0.002764, 0.002739, 1.000000]
6	[-0.050653, -0.072947, -0.072473, 1.006621]	[0.019344, 0.028240, 0.027677, 1.000068]	[-0.035065, -0.051192, -0.050171, 1.003179]	[-0.004617, -0.006741, -0.006607, 1.000055]	[-0.000436, -0.000626, -0.000624, 1.000000]
7	[0.030012, 0.045126, -0.044228, 1.002371]	[0.008962, 0.013084, 0.012823, 1.000208]	[0.018819, 0.027473, 0.026925, 1.000075]	[0.001601, 0.002337, 0.002296, 1.000007]	[0.000099, 0.000145, 0.000142, 1.000000]
8	[-0.018839, -0.027593, -0.026954, 1.000018]	[0.004071, 0.005943, 0.005825, 1.000043]	[-0.009963, -0.014545, -0.014255, 1.000257]	[-0.000544, -0.000795, -0.000779, 1.000001]	[-0.000023, -0.000033, -0.000032, 1.000000]
9	[0.011465, 0.016778, -0.016484, 1.000040]	[0.001819, 0.002856, 0.002803, 1.000009]	[0.003214, 0.007013, 0.007401, 1.000070]	[0.000152, 0.000236, 0.000201, 1.000000]	[0.000005, 0.000007, 0.000007, 1.000000]
10	[-0.000668, -0.001173, -0.000970, 1.000120]	[0.000092, 0.001170, 0.001147, 1.000002]	[-0.002703, -0.003947, -0.003868, 1.000019]	[-0.000060, -0.000088, -0.000086, 1.000000]	[-0.000001, -0.000002, -0.000002, 1.000000]
11	[0.004231, 0.000176, 0.000653, 1.000046]	[0.000349, 0.000510, 0.000500, 1.000000]	[0.001391, 0.002030, 0.001990, 1.000005]	[0.000020, 0.000029, 0.000028, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]
12	[-0.002566, -0.003746, -0.003672, 1.000017]	[0.000151, 0.000229, 0.000216, 1.000000]	[-0.000711, -0.001037, -0.001017, 1.000001]	[-0.000006, -0.000009, -0.000009, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]
13	[0.001555, 0.002270, 0.002225, 1.000000]	[0.000065, 0.000094, 0.000092, 1.000000]	[0.000361, 0.000527, 0.000517, 1.000000]	[0.000002, 0.000003, 0.000003, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]
14	[-0.000942, -0.001375, -0.001348, 1.000002]	[0.000028, 0.000040, 0.000039, 1.000000]	[-0.000183, -0.000267, -0.000262, 1.000000]	[-0.000001, -0.000001, -0.000001, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]
15	[0.000370, 0.000852, 0.000816, 1.000001]	[0.000012, 0.000017, 0.000017, 1.000000]	[0.000002, 0.000015, 0.000012, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]
16	[-0.000345, -0.000563, -0.000493, 1.000000]	[0.000005, 0.000007, 0.000007, 1.000000]	[-0.000046, -0.000068, -0.000066, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]
17	[0.000299, 0.000394, 0.000298, 1.000000]	[0.000002, 0.000003, 0.000003, 1.000000]	[0.000023, 0.000034, 0.000033, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]
18	[-0.000126, -0.000184, -0.000180, 1.000000]	[0.000001, 0.000001, 0.000001, 1.000000]	[-0.000012, -0.000017, -0.000017, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]
19	[0.000076, 0.000111, 0.000109, 1.000000]	[0.000000, 0.000000, 0.000001, 1.000000]	[0.000006, 0.000009, 0.000008, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]
20	[-0.000046, -0.000067, -0.000066, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]	[-0.000003, -0.000004, -0.000004, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]	[0.000000, 0.000000, 0.000000, 1.000000]

Table 1: Comparison of iterative values for various algorithms.

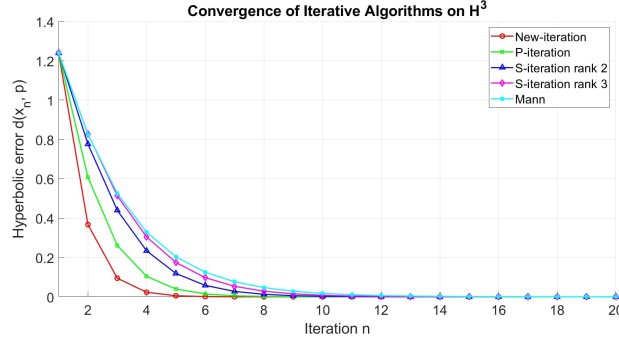


Figure 1: Graphical comparison of errors versus iterations for iterative algorithms.

Algorithm	Iterations	Time (s)
Mann	30	0.00038
S-iteration (rank 2)	20	0.00030
S-iteration (rank 3)	24	0.00024
P-iteration	15	0.00016
PJ (New iteration)	11	0.00014

Table 2: CPU time for iterative algorithms convergence

Iteration number	PJ (New iteration)	P-iteration	S-iteration (rank 2)	S-iteration (rank 3)	Mann
1	1.238499367949	1.238499367949	1.238499367949	1.238499857511	1.238499367949
2	0.366962775688	0.606873847900	0.776010138113	0.827398157377	0.825666245299
3	0.094566665711	0.260806890548	0.439641741275	0.513585546155	0.525423974281
4	0.023087564871	0.104246139624	0.233861189074	0.304549895042	0.328389983926
5	0.005477095348	0.039622691297	0.118701221443	0.174576660201	0.203289037668
6	0.001276410022	0.014498115734	0.058066583086	0.097450796807	0.125100946257
7	0.000293877222	0.005148663886	0.027572185636	0.053252035298	0.076674773513
8	0.000067069714	0.001785045098	0.01277873397	0.028600595029	0.046856806035
9	0.000015205304	0.000606907642	0.005804475991	0.015144999793	0.028571223192
10	0.000003429275	0.000203069691	0.002593577872	0.007927156503	0.017391179334
11	0.000000770550	0.000067057093	0.001143163230	0.004109740336	0.010571109007
12	0.000000000000	0.000021903906	0.000498214374	0.002113957725	0.006418173326
13	0.000000000000	0.000007090886	0.000215118900	0.001080372623	0.003892990378
14	0.000000000000	0.000002278562	0.000092174772	0.000549231618	0.002359388108
15	0.000000000000	0.000000727567	0.000039248278	0.000278015784	0.001428925192
16	0.000000000000	0.000000000000	0.000016627039	0.000140241138	0.000864875774
17	0.000000000000	0.000000000000	0.000007015013	0.000070546929	0.000523196456
18	0.000000000000	0.000000000000	0.000002950054	0.000035410834	0.000316351346
19	0.000000000000	0.000000000000	0.000001237425	0.000017744749	0.000191201364
20	0.000000000000	0.000000000000	0.000000517909	0.000008881092	0.000115517492

Table 3: Numerical comparison of errors versus iterations for iterative algorithms.

On the other hand, in what follows, we conduct a brief study on the behavior

of the PJ iteration under various arbitrary choices of the sequences used in its definition. Figure 2 and Table 4 illustrate the convergence behavior for different selections of control parameters.

It is worth noting that when larger values are chosen (though still strictly less than 1), both the error and the number of iterations required tend to decrease.

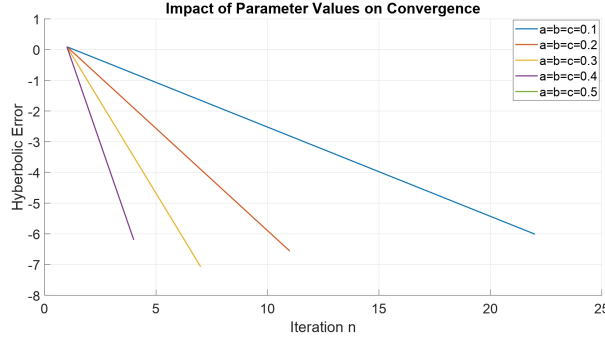


Figure 2: Influence of parameters on the new iterative algorithm

Parameters	Number of iterations
0.1	22
0.2	11
0.3	7
0.4	4
0.5	2

Table 4: Iterations required for different parameter sets

Next, we apply Algorithm (8) to find a common singularity of two multi-valued monotone vector fields. A point $x^* \in M$ is called a singularity of A if $0 \in A(x^*)$. The set of all singularities of A is denoted by $A^{-1}(\mathbf{0}) = \{x \in C : 0 \in A(x)\}$. The inclusion problem in the setting of a Hadamard manifold is defined as: find $x^* \in M$ such that

$$x^* \in A^{-1}(\mathbf{0}) \cap B^{-1}(\mathbf{0}), \quad (20)$$

where $A, B \in \Psi(M)$. Let $\mathfrak{S}$ denote the solution set of the inclusion problem (20). By Remark 1, the problem of finding the singularities of A can be transformed into the problem of finding the fixed points of the mapping J_λ^A .

Theorem 3. *Let $A, B \in \Psi(M)$ be multivalued monotone vector fields such that $\mathfrak{S} \neq \emptyset$. Assume that $\{a_n\}, \{b_n\}$, and $\{c_n\}$ are positive real sequences satisfying the conditions $0 < k_1 \leq a_n \leq \hat{k}_1 < 1$, $0 < k_2 \leq b_n \leq \hat{k}_2 < 1$, and $0 < k_3 \leq c_n \leq \hat{k}_3 < 1$, for all $n \in \mathbb{N}$. Let $x_0 \in M$, and define the iterative sequence $\{x_n\}$ by*

$$\begin{cases} w_n = J_\lambda^A [\exp_{x_n} c_n \exp_{x_n}^{-1} J_\lambda^A(x_n)] \\ z_n = J_\lambda^A [\exp_{w_n} b_n \exp_{w_n}^{-1} J_\mu^B(w_n)] \\ y_n = J_\mu^B [\exp_{z_n} a_n \exp_{z_n}^{-1} J_\lambda^A(z_n)] \\ x_{n+1} = J_\mu^B(y_n), \end{cases}$$

for all $n \in \mathbb{N}$, where $\lambda, \mu > 0$. Then, $\{x_n\}$ converges to an element in $A^{-1}(\mathbf{0}) \cap B^{-1}(\mathbf{0})$.

Proof. Set $T = J_\lambda^A$ and $S = J_\mu^B$. From Theorem 1, T and S are single-valued and firmly nonexpansive mappings. Hence, they are nonexpansive mappings with $\text{Fix}(T) = A^{-1}(\mathbf{0})$ and $\text{Fix}(S) = B^{-1}(\mathbf{0})$. From the hypothesis, $\text{Fix}(T) \cap \text{Fix}(S) = A^{-1}(\mathbf{0}) \cap B^{-1}(\mathbf{0}) \neq \emptyset$. Therefore, according to Theorem 2, we get the desired result. $\square$

Suppose that C is a nonempty, closed, and geodesic convex subset of M . The projection operator $P_C : M \rightarrow C$ is defined by

$$P_C(x) := \{z \in C : d(x, z) \leq d(x, y), \text{ for all } y \in C\}.$$

It is known that the projection operator P_C is single-valued and firmly nonexpansive [13].

Let C_1 and C_2 be nonempty, closed, and geodesic convex subsets of M such that $C_1 \cap C_2 \neq \emptyset$. Then, finding an element in $\text{Fix}(P_{C_1}) \cap \text{Fix}(P_{C_2})$ is equivalent to finding a point in the intersection of two nonempty, closed, and geodesic convex subsets of a Hadamard manifold.

Now, we establish a result to find a point in the intersection of two nonempty, closed, and geodesic convex subsets of Hadamard manifolds using Algorithm (8).

Theorem 4. *Let C_1 and C_2 be nonempty, closed, and geodesic convex subsets of M such that $C_1 \cap C_2 \neq \emptyset$. Assume that $\{a_n\}, \{b_n\}$, and $\{c_n\}$ are positive real sequences satisfying the conditions $0 < k_1 \leq a_n \leq \hat{k}_1 < 1$, $0 < k_2 \leq b_n \leq \hat{k}_2 < 1$, and $0 < k_3 \leq c_n \leq \hat{k}_3 < 1$, for all $n \in \mathbb{N}$. Let $x_0 \in M$, and define the iterative sequence $\{x_n\}$ by*

$$\begin{cases} w_n = P_{C_1} [\exp_{x_n} c_n \exp_{x_n}^{-1} P_{C_1}(x_n)] \\ z_n = P_{C_1} [\exp_{w_n} b_n \exp_{w_n}^{-1} P_{C_2}(w_n)] \\ y_n = P_{C_2} [\exp_{z_n} a_n \exp_{z_n}^{-1} P_{C_1}(z_n)] \\ x_{n+1} = P_{C_2}(y_n), \end{cases}$$

for all $n \in \mathbb{N}$.

Then, $\{x_n\}$ converges to an element in $C_1 \cap C_2$.

Proof. Set $T = P_{C_1}$ and $S = P_{C_2}$. Then T and S are nonexpansive mappings with $\text{Fix}(T) = C_1$ and $\text{Fix}(S) = C_2$. From the hypothesis, $\text{Fix}(T) \cap \text{Fix}(S) = C_1 \cap C_2 \neq \emptyset$.

Therefore, according to Theorem 2, we get the desired result. $\square$

4 Conclusions

In this paper, building upon a recently introduced iterative algorithm in the framework of Hadamard manifolds, we proposed a new four-step iterative scheme aimed at finding the common fixed points of two nonexpansive mappings. We established rigorous convergence results under suitable conditions on the control parameters, thereby extending and generalizing existing methods in the literature. Furthermore, numerical examples were provided to illustrate the efficiency and rapid convergence of the proposed algorithm, demonstrating its advantages over several classical iterative techniques. These results highlight the practical potential of the method for solving fixed point and inclusion problems in the non-Euclidean geometry of Hadamard manifolds.

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