



ℓ -generalized prime factors counting functions

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Abstract

To study the number of extra prime factors beyond the first $\ell - 1$ occurrences of a prime p in the factorization of an integer n , this paper introduces the ℓ -generalized prime factors counting arithmetic function $\Omega_{A,\ell}$. Key results include the Dirichlet series, an exact summatory formula, and the asymptotic mean. In addition, a distributional limit result is derived using Kubilius model and Kolmogorov three series theorem.

1 Introduction

Arithmetic functions play a fundamental role in modern computer science and cryptography, particularly in the design and analysis of secure systems and efficient algorithms. In cryptography, the security of widely-used protocols such as RSA encryption and digital signatures relies critically on the computational hardness of integer factorization and the distribution of prime numbers [15]. Arithmetic functions like Ω and ω help analyze the complexity of factorization algorithms and assess the security of cryptographic keys. The study of prime gaps and distributions informs parameter selection in lattice-based cryptography and is crucial for post-quantum cryptographic schemes [1]. In computer science, these functions appear in the analysis of algorithms for integer multiplication, greatest common divisor computation, and modular arithmetic [21].

The quantitative analysis of prime factor multiplicities represents a central theme in analytic number theory, with profound implications across mathematics and computational sciences. Classical arithmetic functions such as

Key Words: Multiplicative functions, Dirichlet series, mean value, Kubilius model, distributional limit.

2010 Mathematics Subject Classification: Primary 11A25, 11N37; Secondary 11M41, .

Received: 03.10.2025

Accepted: 20.12.2025

$\omega(n)$ and $\Omega(n)$ —counting distinct prime divisors and total prime factors with multiplicity of n , respectively—trace their origins to Hardy and Ramanujan’s foundational work [13], which established their characteristic order $\log \log n$. This was elevated by the Erdős-kac theorem [12], which revealed Gaussian fluctuations under suitable normalization, cementing the deep interconnection between probabilistic methods and multiplicative number theory.

Despite the extensive and in-depth advances studying the functions ω and Ω , see [3], [5], [6], [8], [9], [13], [17], [18] for example, the systematic study of *structured prime subsets* with controlled multiplicity thresholds remains underdeveloped. This paper bridges the gap by introducing and analyzing the arithmetic function

$$\Omega_{A,\ell}(n) := \sum_{\substack{p \in A \\ v_p(n) \geq \ell}} (v_p(n) - \ell + 1),$$

where $A \subseteq \mathbb{P}$ (the primes), $\ell \geq 1$ is an integer threshold, and $v_p(n)$ denotes the p -adic valuation. This function quantifies the *excess multiplicity* of primes in A beyond $\ell - 1$ occurrences in the factorization of n , unifying and extending classical cases:

1. $\ell = 1$: $\Omega_{A,1}(n) = \sum_{p \in A} v_p(n)$ (*Total prime factors in A with multiplicity*).
2. $\ell = 1$ and $A = \mathbb{P}$: $\Omega_{\mathbb{P},1}(n) = \sum_{p \in \mathbb{P}} v_p(n) = \Omega(n)$ (*Total prime factors with multiplicity*).
3. $\ell = 2$: $\Omega_{A,2}(n) = \sum_{\substack{p \in A \\ v_p(n) \geq 2}} (v_p(n) - 1)$ (*Factors beyond first occurrence*).

Combinatorially, $\Omega_{A,\ell}$ isolates higher-order multiplicities, providing a refined lens for investigating structured irregularities in prime factor distributions. In this paper, We establish comprehensive analytic and probabilistic properties of $\Omega_{A,\ell}$, revealing a fundamental dichotomy governed by the density of A . In section 2, we prove that $\Omega_{A,\ell}$ is additive and satisfies sharp bounds relative to $\Omega_A(n)$ and $\omega_A(n)$. An explicit formula relates it to truncated valuations is obtained:

$$\Omega_{A,\ell}(n) = \Omega_A(n) - \sum_{p \in A} \min(v_p(n), \ell - 1),$$

where

$$\Omega_A(n) = \sum_{\substack{p \in A \\ v_p(n) \geq 1}} v_p(n).$$

In section 3, we prove that the Dirichlet series admits the closed form:

$$\sum_{n=1}^{\infty} \frac{\Omega_{A,\ell}(n)}{n^s} = \zeta(s) \sum_{p \in A} \frac{p^{-\ell s}}{1 - p^{-s}}, \quad \Re(s) > 1.$$

In section 4, an exact summatory function is derived:

$$\sum_{n \leq x} \Omega_{A,\ell}(n) = \sum_{p \in A} \sum_{j=\ell}^{\infty} \left\lfloor \frac{x}{p^j} \right\rfloor.$$

In section 5, an asymptotic average:

$$\frac{1}{x} \sum_{n \leq x} \Omega_{A,\ell}(n) = \lambda + O(x^{-1/\ell}), \quad \lambda = \sum_{p \in A} \frac{1}{p^{\ell-1}(p-1)},$$

is proved for $\ell \geq 1$ and $A \subseteq \mathbb{P}$ that satisfies $\sum_{p \in A} p^{-(\ell)} < \infty$.

In section 6, a distributional limit result is established. Namely, It is shown that, for n chosen uniformly from $[1, x]$,

$$\Omega_{A,\ell}(n) \xrightarrow{d} Y \quad \text{as } x \rightarrow \infty,$$

where $Y = \sum_{p \in A} Y_p$ and $\{Y_p\}_{p \in A}$ are independent geometric-type variables.

2 The function $\Omega_{A,\ell}$

This section exposes some main properties of the arithmetic function $\Omega_{A,\ell}$.

Lemma 1. *For $\ell \geq 1$, $0 \leq \Omega_{A,\ell}(n) \leq \Omega_A(n)$. In addition, $\Omega_{A,\ell}(n) \leq \Omega_A(n) - \omega_A(n)$ for $\ell \geq 2$.*

Proof. Clearly, $v_p(n) - \ell + 1 \geq 1 > 0$ for $p \in A$ with $v_p(n) \geq \ell$. If no such p exists, the sum is empty and equals 0. Thus, $\Omega_{A,\ell}(n) \geq 0$. Note that $\Omega_{A,\ell}(n)$ sums a subset of the terms in $\Omega_A(n) = \sum_{p \in A} v_p(n)$, and each term $v_p(n) - \ell + 1 \leq v_p(n)$. Thus, $\Omega_{A,\ell}(n) \leq \Omega_A(n)$.

For $\ell \geq 2$, we have $v_p(n) - \ell + 1 \leq v_p(n) - 1$ for $v_p(n) \geq \ell$. Summing over $p \in A$ with $v_p(n) \geq \ell$, we obtain

$$\Omega_{A,\ell}(n) \leq \sum_{\substack{p \in A \\ v_p(n) \geq \ell}} (v_p(n) - 1).$$

Now, observe that for primes $p \in A$ with $1 \leq v_p(n) < \ell$, the term $(v_p(n) - 1)$ is non-negative. Therefore, extending the sum to all primes dividing n (i.e.,

$v_p(n) \geq 1$) preserves the inequality:

$$\sum_{\substack{p \in A \\ v_p(n) \geq \ell}} (v_p(n) - 1) \leq \sum_{\substack{p \in A \\ v_p(n) \geq 1}} (v_p(n) - 1) = \Omega_A(n) - \omega_A(n),$$

where the last equality holds because $\Omega_A(n) = \sum_{p \in A} v_p(n)$ and $\omega_A(n)$ counts the number of distinct primes in A dividing n . $\square$

Lemma 2. *The function $\Omega_{A,\ell}$ is additive.*

Proof. Let m and n be coprime. Then their prime factors are disjoint. Recall that $v_p(mn) = v_p(m) + v_p(n)$ for all p , see [2]. For $p \in A$,

- If $p \mid m$, then $v_p(n) = 0$, so $v_p(mn) = v_p(m)$.
- If $p \mid n$, then $v_p(m) = 0$, so $v_p(mn) = v_p(n)$.

Thus,

$$\Omega_{A,\ell}(mn) = \sum_{\substack{p \in A \\ v_p(mn) \geq \ell}} (v_p(mn) - \ell + 1) + \sum_{\substack{p \in A \\ v_p(n) \geq \ell}} (v_p(n) - \ell + 1) = \Omega_{A,\ell}(m) + \Omega_{A,\ell}(n).$$

$\square$

Remark 1. *The function $\Omega_{A,\ell}$ is not completely additive. For example, taking $m = n = p$, $\ell = 2$, $A = \{p\}$, we get $\Omega_{A,2}(p^2) = 1$ and $\Omega_{A,2}(p) + \Omega_{A,2}(p) = 0 + 0 = 0 \neq 1$.*

An explicit relation between $\Omega_{A,\ell}(n)$ and $\Omega_A(n)$ is provided as follows.

Lemma 3. *For $A \subseteq \mathbb{P}$ and $\ell \geq 1$, the following holds:*

$$\Omega_{A,\ell}(n) = \Omega_A(n) - \sum_{p \in A} \min(v_p(n), \ell - 1).$$

Proof. We prove the identity by analyzing the contribution of each prime $p \in A$ to both sides of the equation. For each prime $p \in A$, we consider two cases based on the value of $v_p(n)$:

Case 1: $v_p(n) < \ell$. The prime p does not contribute to $\Omega_{A,\ell}(n)$ since the condition $v_p(n) \geq \ell$ is not satisfied. For the right-hand side, $\Omega_A(n)$ contributes $v_p(n)$. In addition, $\min(v_p(n), \ell - 1) = v_p(n)$ since $v_p(n) < \ell$. Therefore, the net contribution is $v_p(n) - v_p(n) = 0$.

Case 2: $v_p(n) \geq \ell$. The prime p contributes $(v_p(n) - \ell + 1)$ to $\Omega_{A,\ell}(n)$. For the right-hand side, $\Omega_A(n)$ contributes $v_p(n)$ and $\min(v_p(n), \ell - 1) = \ell - 1$ since $v_p(n) \geq \ell$. Thus, the net contribution is $v_p(n) - (\ell - 1) = v_p(n) - \ell + 1$.

In both cases, the contribution of each prime $p \in A$ is identical on both sides of the equation. Since the sums are finite (as n has finitely many prime factors), we conclude that

$$\Omega_{A,\ell}(n) = \Omega_A(n) - \sum_{p \in A} \min(v_p(n), \ell - 1).$$

□

3 The Dirichlet series of $\Omega_{A,\ell}$

Theorem 1. *Let $A \subseteq \mathbb{P}$ and $\ell \geq 1$. Then*

$$\sum_{n=1}^{\infty} \frac{\Omega_{A,\ell}(n)}{n^s} = \zeta(s) \sum_{p \in A} \frac{p^{-\ell s}}{1 - p^{-s}}, \quad \Re(s) > 1,$$

where $\zeta(s)$ is the Riemann zeta function.

Proof. Let

$$h_p(n) := \max(v_p(n) - \ell + 1, 0) = \begin{cases} v_p(n) - \ell + 1 & \text{if } v_p(n) \geq \ell, \\ 0 & \text{otherwise.} \end{cases}$$

As $h_p(n)$ counts the "excess multiplicity" of p in n beyond $\ell - 1$, we get

$$\sum_{n=1}^{\infty} \frac{h_p(n)}{n^s} = \sum_{\substack{n=1 \\ p^\ell | n}}^{\infty} \frac{v_p(n) - \ell + 1}{n^s}.$$

The sum is non-zero only when n is divisible by p^ℓ . That is, $n = p^j \cdot m$, where $p \nmid m$ and $j \geq \ell$. Thus,

$$\sum_{\substack{n=1 \\ p^\ell | n}}^{\infty} \frac{v_p(n) - \ell + 1}{n^s} = \sum_{j=\ell}^{\infty} \sum_{\substack{m=1 \\ p \nmid m}}^{\infty} \frac{j - \ell + 1}{(p^j \cdot m)^s}.$$

The right side equals

$$\sum_{j=\ell}^{\infty} (j - \ell + 1) p^{-js} \sum_{\substack{m=1 \\ p \nmid m}}^{\infty} \frac{1}{m^s}.$$

The inner sum is the Dirichlet series for integers coprime to p . This is given by

$$\sum_{m=1}^{\infty} \frac{1}{m^s} \cdot \mathbf{1}_{p \nmid m} = \left(1 - \frac{1}{p^s}\right) \zeta(s),$$

where $(1 - p^{-s})$ removes terms divisible by p from $\zeta(s)$. Let $r = p^{-s}$. Then

$$\sum_{j=\ell}^{\infty} (j - \ell + 1) r^j = r^{\ell} \sum_{m=0}^{\infty} (m + 1) r^m = r^{\ell} \cdot \frac{1}{(1 - r)^2},$$

where the last equality uses $\sum_{m=0}^{\infty} (m + 1) r^m = (1 - r)^{-2}$ for $|r| < 1$. Substituting $r = p^{-s}$ gives

$$\sum_{j=\ell}^{\infty} (j - \ell + 1) p^{-js} = \frac{p^{-\ell s}}{(1 - p^{-s})^2}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{\Omega_{A,\ell}(n)}{n^s} = \zeta(s) \sum_{p \in A} \frac{p^{-\ell s}}{1 - p^{-s}}, \quad \Re(s) > 1.$$

□

Remark 2. Taking $\ell = 1$ and $A = \mathbb{P}$, the Dirichlet series reduces to the classical formula

$$\sum_{n=1}^{\infty} \frac{\Omega(n)}{n^s} = \zeta(s) \sum_{p \in \mathbb{P}} \frac{1}{p^s - 1}, \quad \Re(s) > 1.$$

Example 1. Let $A = \{2, 3\}$, $\ell = 3$, and $s = 2$. We compute the Dirichlet series for $\Omega_{A,3}(n)$ using both the closed-form formula and direct computation. The Dirichlet series is:

$$\sum_{n=1}^{\infty} \frac{\Omega_{A,3}(n)}{n^2} = \zeta(2) \sum_{p \in A} \frac{p^{-6}}{1 - p^{-2}}, \quad \Re(s) > 1.$$

It is known that $\zeta(2) = \frac{\pi^2}{6} \approx 1.644934$. Thus,

$$\sum_{p \in \{2,3\}} \frac{p^{-6}}{1 - p^{-2}} = \left(\frac{1}{48} + \frac{1}{648}\right) \approx 0.022376.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{\Omega_{A,3}(n)}{n^2} \approx 1.644934 \times 0.022376 \approx 0.03681.$$

We compute the partial sum $\sum_{n \leq 40} \frac{\Omega_{A,3}(n)}{n^2}$ directly. Recall that $\Omega_{A,3}(n)$ counts excess prime factors beyond $\ell - 1 = 2$ for primes in $A = \{2, 3\}$:

$$\Omega_{A,3}(n) = \sum_{\substack{p \in \{2,3\} \\ v_p(n) \geq 3}} (v_p(n) - 2).$$

n	Factorization	$\Omega_{A,3}(n)$	$\frac{\Omega_{A,3}(n)}{n^2}$
8	2^3	$3 - 2 = 1$	$1/64 \approx 0.015625$
16	2^4	$4 - 2 = 2$	$2/256 = 0.0078125$
24	$2^3 \cdot 3$	1	$1/576 \approx 0.001736$
27	3^3	$3 - 2 = 1$	$1/729 \approx 0.001372$
32	2^5	$5 - 2 = 3$	$3/1024 \approx 0.002930$
40	$2^3 \cdot 5$	1	$1/1600 = 0.000625$
Partial Sum			≈ 0.030100

4 Summatory function of $\Omega_{A,\ell}$

Theorem 2. Let $A \subseteq \mathbb{P}$, $\ell \geq 1$, and $x \geq 1$. Then

$$\sum_{n \leq x} \Omega_{A,\ell}(n) = \sum_{p \in A} \sum_{j=\ell}^{\infty} \left\lfloor \frac{x}{p^j} \right\rfloor. \quad (1)$$

Proof. Define:

$$\Omega_{A,\ell}(n) = \sum_{\substack{p \in A \\ v_p(n) \geq \ell}} (v_p(n) - \ell + 1).$$

For a fixed prime $p \in A$ and n with $v_p(n) \geq \ell$, observe that:

$$v_p(n) - \ell + 1 = \sum_{\ell=1}^{v_p(n)-\ell+1} 1 = \sum_{j=\ell}^{v_p(n)} 1,$$

since there are $(v_p(n) - \ell + 1)$ integers j in the interval $[\ell, v_p(n)]$. Thus:

$$\Omega_{A,\ell}(n) = \sum_{p \in A} \underbrace{\sum_{j=\ell}^{v_p(n)} 1}_{\text{defined only if } v_p(n) \geq \ell}.$$

Equivalently, we can extend the sum to all $j \geq \ell$ using an indicator function:

$$\Omega_{A,\ell}(n) = \sum_{p \in A} \sum_{j=\ell}^{\infty} \mathbf{1}_{p^j | n},$$

where $\mathbf{1}_{p^j | n} = 1$ if $p^j \mid n$, and 0 otherwise. Summing over over $n \leq x$ entails that

$$\sum_{n \leq x} \Omega_{A,\ell}(n) = \sum_{n \leq x} \sum_{p \in A} \sum_{j=\ell}^{\infty} \mathbf{1}_{p^j | n}$$

The inner sum $\sum_{n \leq x} \mathbf{1}_{p^j | n}$ counts the number of positive integers $n \leq x$ that are divisible by p^j which are exactly $\left\lfloor \frac{x}{p^j} \right\rfloor$. Interchange the first two sums (justified by Tonelli's theorem) to get

$$\sum_{n \leq x} \Omega_{A,\ell}(n) = \sum_{p \in A} \sum_{j=\ell}^{\infty} \sum_{n \leq x} \mathbf{1}_{p^j | n} = \sum_{p \in A} \sum_{j=\ell}^{\infty} \left\lfloor \frac{x}{p^j} \right\rfloor.$$

□

5 Asymptotic Average of $\Omega_{A,\ell}$

Theorem 3. *Let $\ell \geq 1$ and $A \subseteq \mathbb{P}$ satisfies $\sum_{p \in A} p^{-\ell} < \infty$. Then*

$$\frac{1}{x} \sum_{n \leq x} \Omega_{A,\ell}(n) = \lambda + O\left(x^{-1/\ell}\right) \quad \text{as } x \rightarrow \infty,$$

where $\lambda = \sum_{p \in A} \frac{1}{p^{\ell-1}(p-1)}$.

Proof. By the summatory function formula, Equation (1), we have

$$\sum_{n \leq x} \Omega_{A,\ell}(n) = \sum_{p \in A} \sum_{j=\ell}^{\infty} \left\lfloor \frac{x}{p^j} \right\rfloor.$$

For $p^j > x$, $\left\lfloor \frac{x}{p^j} \right\rfloor = 0$. Thus

$$\sum_{n \leq x} \Omega_{A,\ell}(n) = \sum_{p \in A} \sum_{j=\ell}^{\lfloor \log_p x \rfloor} \left\lfloor \frac{x}{p^j} \right\rfloor.$$

Decompose the floor function

$$\left\lfloor \frac{x}{p^j} \right\rfloor = \frac{x}{p^j} - \left\{ \frac{x}{p^j} \right\}, \quad 0 \leq \left\{ \frac{x}{p^j} \right\} < 1.$$

Thus,

$$\sum_{n \leq x} \Omega_{A,\ell}(n) = x \sum_{p \in A} \sum_{j=\ell}^{\lfloor \log_p x \rfloor} p^{-j} - \sum_{p \in A} \sum_{j=\ell}^{\lfloor \log_p x \rfloor} \left\{ \frac{x}{p^j} \right\}. \quad (2)$$

The main term is

$$T_1 = x \sum_{p \in A} \sum_{j=\ell}^{\lfloor \log_p x \rfloor} p^{-j}. \quad (3)$$

Extend the inner sum to $j = \infty$ to get

$$\sum_{j=\ell}^{\lfloor \log_p x \rfloor} p^{-j} = \sum_{j=\ell}^{\infty} p^{-j} - \sum_{j=\lfloor \log_p x \rfloor + 1}^{\infty} p^{-j}.$$

The geometric series equals

$$\sum_{j=\ell}^{\infty} p^{-j} = \frac{p^{-\ell}}{1 - p^{-1}} = \frac{1}{p^{\ell-1}(p-1)}.$$

The tail bound is

$$\sum_{j=\lfloor \log_p x \rfloor + 1}^{\infty} p^{-j} \leq \frac{p^{-\lfloor \log_p x \rfloor}}{p-1} \leq \frac{p}{x(p-1)},$$

since

$$p^{-\lfloor \log_p x \rfloor} \leq p \cdot x^{-1}.$$

Thus,

$$\sum_{j=\ell}^{\lfloor \log_p x \rfloor} p^{-j} = \frac{1}{p^{\ell-1}(p-1)} + O\left(\frac{p}{x(p-1)}\right).$$

Sum over $p \in A$ to obtain

$$T_1 = x \sum_{p \in A} \left(\frac{1}{p^{\ell-1}(p-1)} + O\left(\frac{p}{x(p-1)}\right) \right) = x\lambda + O\left(\sum_{\substack{p \in A \\ p^\ell \leq x}} \frac{p}{p-1}\right).$$

Note that

$$\lambda = \sum_{p \in A} \frac{1}{p^{\ell-1}(p-1)} = \sum_{p \in A} \frac{p}{p^\ell(p-1)} \leq 2 \sum_{p \in A} \frac{1}{p^\ell} < \infty.$$

Now, we bound $\sum_{p \in A, p^\ell \leq x} \frac{p}{p-1}$. To this end, let $y = x^{1/\ell}$. Recall that. In addition, we have

$$\pi_A(y) = |\{p \in A : p \leq y\}| \ll y^{\ell-1}.$$

Also, $\frac{p}{p-1} \leq 2$ for $p \geq 2$. Therefore,

$$\sum_{\substack{p \in A \\ p^\ell \leq x}} \frac{p}{p-1} \leq 2\pi_A(y) \ll y^{\ell-1} = x^{(\ell-1)/\ell}.$$

Thus, by Equation (3), we get

$$T_1 = x\lambda + O\left(x^{(\ell-1)/\ell}\right).$$

The error term in Equation (2) is $T_2 = \sum_{p \in A} \sum_{j=\ell}^{\lfloor \log_p x \rfloor} \left\{ \frac{x}{p^j} \right\}$. It satisfies

$$0 \leq T_2 \leq \sum_{p \in A} \sum_{j=\ell}^{\lfloor \log_p x \rfloor} 1 = \sum_{\substack{p \in A \\ p^\ell \leq x}} (\lfloor \log_p x \rfloor - \ell + 1).$$

Since $\lfloor \log_p x \rfloor \leq \log_p x = \frac{\ln x}{\ln p}$, and $-\ell + 1 \leq 1$ for $\ell \geq 2$, we have

$$T_2 \leq \sum_{\substack{p \in A \\ p \leq x^{1/\ell}}} \frac{\ln x}{\ln p}.$$

To bound $\sum_{\substack{p \in A \\ p \leq y}} \frac{1}{\ln p}$, let $y = x^{1/\ell}$. By partial summation,

$$\sum_{\substack{p \in A \\ p \leq y}} \frac{1}{\ln p} \ll \frac{\pi_A(y)}{\ln y} \ll \frac{y^{\ell-1}}{\ln y},$$

Substitute $y = x^{1/\ell}$:

$$\sum_{\substack{p \in A \\ p \leq x^{1/\ell}}} \frac{\ln x}{\ln p} \ll \ln x \cdot \frac{y^{\ell-1}}{\ln y} = \ln x \cdot \frac{x^{(\ell-1)/\ell}}{\ln(x^{1/\ell})} = \ell \cdot x^{(\ell-1)/\ell}.$$

Thus,

$$T_2 \ll x^{(\ell-1)/\ell}.$$

Combine results to get

$$\sum_{n \leq x} \Omega_{A,\ell}(n) = T_1 - T_2 = x\lambda + O\left(x^{(\ell-1)/\ell}\right).$$

Hence,

$$\frac{1}{x} \sum_{n \leq x} \Omega_{A,\ell}(n) = \lambda + O\left(x^{(\ell-1)/\ell-1}\right) = \lambda + O\left(x^{-1/\ell}\right).$$

□

Remark 3. *It is evident that the condition $\sum_{p \in A} p^{-\ell} < \infty$ is satisfied for all $\ell \geq 2$.*

6 Distributional limit of $\Omega_{A,\ell}$

The following theorem is crucial for our investigation. For details, refer to [11].

Theorem 4. (*Kolmogorov Three-Series Theorem*) *For independent random variables $\{X_n\}$, the series $\sum X_n$ converges almost surely if and only if for some $c > 0$ the following hold:*

1. $\sum \mathbb{P}(|X_n| > c) < \infty$,
2. $\sum \mathbb{E}[X_n \cdot \mathbf{1}_{\{|X_n| \leq c\}}] < \infty$,
3. $\sum \text{Var}(X_n \cdot \mathbf{1}_{\{|X_n| \leq c\}}) < \infty$.

Theorem 5. *Let $\ell \geq 1$ and $A \subseteq \mathbb{P}$ satisfies $\sum_{p \in A} p^{-\ell} < \infty$. Then, for n chosen uniformly from $[1, x]$,*

$$\Omega_{A,\ell}(n) \xrightarrow{d} Y \quad \text{as } x \rightarrow \infty,$$

where $Y = \sum_{p \in A} Y_p$ and $\{Y_p\}_{p \in A}$ are independent random variables with distributions:

$$\mathbb{P}(Y_p = 0) = 1 - p^{-\ell}, \quad \mathbb{P}(Y_p = m) = \left(1 - \frac{1}{p}\right) \frac{1}{p^{m+\ell-1}} \quad \text{for } m = 1, 2, 3, \dots$$

Proof. For any finite subset $A_0 \subseteq A$, define:

$$\Omega_{A_0, \ell}(n) = \sum_{p \in A_0} \max(v_p(n) - \ell + 1, 0).$$

The valuations $\{v_p(n)\}_{p \in A_0}$ are asymptotically independent as $x \rightarrow \infty$. Specifically, for any nonnegative integers $\{j_p\}_{p \in A_0}$

$$\lim_{x \rightarrow \infty} \mathbb{P}(v_p(n) = j_p \text{ for all } p \in A_0) = \prod_{p \in A_0} \mathbb{P}(v_p(n) = j_p) = \prod_{p \in A_0} \left(1 - \frac{1}{p}\right) \frac{1}{p^{j_p}}.$$

This follows from the asymptotic independence of prime valuations in the Kubilius model, see [16], [20]. Since

$$\Omega_{A_0, \ell}(n) = \sum_{p \in A_0} g(v_p(n)) \quad \text{where} \quad g(j) = \max(j - \ell + 1, 0),$$

and the g are deterministic, the random variables $\{g(v_p(n))\}_{p \in A_0}$ are asymptotically independent as $x \rightarrow \infty$. The limit distribution of $g(v_p(n))$ is

$$\mathbb{P}(g(v_p(n)) = 0) = \mathbb{P}(v_p(n) < \ell) = 1 - p^{-\ell}.$$

In addition,

$$\mathbb{P}(g(v_p(n)) = m) = \mathbb{P}(v_p(n) = m + \ell - 1) = \left(1 - \frac{1}{p}\right) \frac{1}{p^{m + \ell - 1}} \quad \text{for } m = 1, 2, \dots$$

This is derived from the geometric distribution of $v_p(n)$. Thus, as $x \rightarrow \infty$, we have

$$\Omega_{A_0, \ell}(n) \xrightarrow{d} Y_{A_0} := \sum_{p \in A_0} Y_p.$$

Consider the tail component over $A \setminus A_0$:

$$\Omega_{A \setminus A_0, \ell}(n) = \sum_{p \in A \setminus A_0} \max(v_p(n) - \ell + 1, 0).$$

Its expectation satisfies

$$\mathbb{E}[\Omega_{A \setminus A_0, \ell}(n)] = \sum_{p \in A \setminus A_0} \mathbb{E}[\max(v_p(n) - \ell + 1, 0)].$$

For each p , we compute

$$\mathbb{E}[\max(v_p(n) - \ell + 1, 0)] = \sum_{m=1}^{\infty} m \cdot \mathbb{P}(v_p(n) = m + \ell - 1) = \sum_{m=1}^{\infty} m \left(1 - \frac{1}{p}\right) \frac{1}{p^{m + \ell - 1}}.$$

Using $\sum_{m=1}^{\infty} mr^m = \frac{r}{(1-r)^2}$ for $|r| < 1$ entails

$$\sum_{m=1}^{\infty} m \left(\frac{1}{p} \right)^m = \frac{\frac{1}{p}}{\left(1 - \frac{1}{p} \right)^2} = \frac{p}{(p-1)^2}.$$

So,

$$\mathbb{E}[\max(v_p(n) - \ell + 1, 0)] = \left(1 - \frac{1}{p} \right) \frac{1}{p^{\ell-1}} \cdot \frac{p}{(p-1)^2} = \frac{1}{p^{\ell-1}(p-1)}.$$

Thus,

$$\mathbb{E}[\Omega_{A \setminus A_0, \ell}(n)] = \sum_{p \in A \setminus A_0} \frac{1}{p^{\ell-1}(p-1)} < \infty.$$

Hence,

$$\lim_{A_0 \rightarrow A} \mathbb{E}[\Omega_{A \setminus A_0, \ell}(n)] = 0. \quad (4)$$

To establish that Y is well-defined, we show that $\sum_{p \in A} Y_p$ converges almost surely. To apply Kolmogorov Three-Series Theorem, we use $c = 1$ and verify the conditions. Note that

$$\mathbb{P}(|Y_p| > 1) = \mathbb{P}(|Y_p| \geq 2) = \sum_{m=2}^{\infty} \mathbb{P}(Y_p = m) = \sum_{m=2}^{\infty} \left(1 - \frac{1}{p} \right) p^{-(m+\ell-1)}.$$

This simplifies to

$$\mathbb{P}(Y_p \geq 2) = \frac{1}{p^{\ell+1}}.$$

Thus,

$$\sum_{p \in A} \mathbb{P}(|Y_p| > 1) = \sum_{p \in A} \frac{1}{p^{\ell+1}} < \infty. \quad (5)$$

Hence, the first series converges. For the second series, let $Z_p = Y_p \cdot \mathbf{1}_{\{|Y_p| \leq 1\}}$. Then

$$\mathbb{E}[Z_p] = 0 \cdot \mathbb{P}(Y_p = 0) + 1 \cdot \mathbb{P}(Y_p = 1) = \mathbb{P}(Y_p = 1).$$

In addition,

$$\mathbb{P}(Y_p = 1) = \left(1 - \frac{1}{p} \right) p^{-(1+\ell-1)} = \left(1 - \frac{1}{p} \right) p^{-\ell}.$$

Hence,

$$\sum_{p \in A} \mathbb{E}[Z_p] = \sum_{p \in A} \frac{p-1}{p^{\ell+1}} \leq \sum_{p \in A} \frac{1}{p^{\ell}} < \infty. \quad (6)$$

The truncated variable Z_p is a Bernoulli-like variable:

$$Z_p = \begin{cases} 0 & \text{with probability } \mathbb{P}(Y_p = 0) + \mathbb{P}(Y_p \geq 2), \\ 1 & \text{with probability } \mathbb{P}(Y_p = 1). \end{cases}$$

The variance of Z_p is

$$\text{Var}(Z_p) = \mathbb{P}(Y_p = 1) \cdot [1 - \mathbb{P}(Y_p = 1)] \leq \mathbb{P}(Y_p = 1).$$

This implies that

$$\sum_{p \in A} \text{Var}(Z_p) \leq \sum_{p \in A} \frac{1}{p^\ell} < \infty. \quad (7)$$

Hence, by (5), (6), and (7), Y is well-defined. Let F be the distribution function of $Y = \sum_{p \in A} Y_p$ and F_{A_0} for $Y_{A_0} = \sum_{p \in A_0} Y_p$. For any continuity point t of F and $\epsilon > 0$, choose a finite subset $A_0 \subseteq A$ such that

$$\mathbb{P}(\Omega_{A \setminus A_0, \ell}(n) > \epsilon) \leq \frac{\mathbb{E}[\Omega_{A \setminus A_0, \ell}(n)]}{\epsilon} < \epsilon,$$

uniformly in x . This is a combination of Markov's inequality [4] and Equation 4. Taking $x \rightarrow \infty$ for fixed A_0 gives

$$\mathbb{P}(\Omega_{A_0, \ell}(n) \leq t \pm \epsilon) \rightarrow F_{A_0}(t \pm \epsilon).$$

By the asymptotic independence and the properties of $\Omega_{A, \ell}$, the event $\mathbb{P}(\Omega_{A, \ell}(n) \leq t)$ satisfies

$$\begin{aligned} \mathbb{P}(\Omega_{A_0, \ell}(n) \leq t - \epsilon) - \mathbb{P}(\Omega_{A \setminus A_0, \ell}(n) > \epsilon) &\leq \mathbb{P}(\Omega_{A, \ell}(n) \leq t) \\ &\leq \mathbb{P}(\Omega_{A_0, \ell}(n) \leq t + \epsilon) + \mathbb{P}(\Omega_{A \setminus A_0, \ell}(n) > \epsilon). \end{aligned}$$

Taking limits as $x \rightarrow \infty$ results in

$$F_{A_0}(t - \epsilon) - \epsilon \leq \liminf_{x \rightarrow \infty} \mathbb{P}(\Omega_{A, \ell}(n) \leq t) \leq \limsup_{x \rightarrow \infty} \mathbb{P}(\Omega_{A, \ell}(n) \leq t) \leq F_{A_0}(t + \epsilon) + \epsilon.$$

As $A_0 \rightarrow A$,

$$Y_{A_0} \xrightarrow{d} Y \quad \text{and} \quad \lim_{A_0 \rightarrow A} F_{A_0}(t \pm \epsilon) = F(t \pm \epsilon).$$

As $\epsilon \rightarrow 0^+$ and since t is a continuity point of F , then

$$F(t) = \lim_{\epsilon \rightarrow 0} F(t - \epsilon) \leq \liminf_{x \rightarrow \infty} \mathbb{P}(\Omega_{A, \ell}(n) \leq t) \leq \limsup_{x \rightarrow \infty} \mathbb{P}(\Omega_{A, \ell}(n) \leq t) \leq \lim_{\epsilon \rightarrow 0} F(t + \epsilon) = F(t).$$

Thus,

$$\lim_{x \rightarrow \infty} \mathbb{P}(\Omega_{A, \ell}(n) \leq t) = F(t).$$

□

Example 2. Let $A = \{2\}$, $\ell = 2$. Then

$$\Omega_{A,2}(n) = \max(v_2(n) - 1, 0), \quad \lambda = \frac{1}{2}.$$

The limit $Y = Y_2$ has distribution:

$$\mathbb{P}(Y_2 = 0) = 1 - 2^{-2} = \frac{3}{4}, \quad \mathbb{P}(Y_2 = m) = \left(1 - \frac{1}{2}\right) \frac{1}{2^{m+1}} = \frac{1}{2^{m+2}} \quad \text{for } m = 1, 2, \dots$$

We have

$$\frac{3}{4} + \sum_{m=1}^{\infty} \frac{1}{2^{m+2}} = \frac{3}{4} + \frac{1}{8} \cdot 2 = 1.$$

For instance, if $n = 72 = 2^3 \cdot 3^2$ then

$$\Omega_{A,2}(72) = \max\{3 - 1, 0\} = 2, \quad \mathbb{P}(Y_2 = 2) = \frac{1}{2^4} = \frac{1}{16}.$$

7 Conclusion

This paper introduces and analyzes the ℓ -generalized prime factor counting function $\Omega_{A,\ell}$. We establish its fundamental properties, including: additivity and its explicit connection to truncated valuations; a closed-form expression for its Dirichlet series; an exact summatory formula; the asymptotic mean for sets A satisfying $\sum_{p \in A} p^{-\ell} < \infty$; and distributional convergence via the Kubilius model.

Several compelling directions emerge from this work. For example, one may investigate Chebyshev-type biases in $\Omega_{A,\ell}(n)$ when A consists of primes in specific residue classes, thereby extending the work of Humphries, Sung, and Tian on factorization biases [14]. Another direction involves analyzing $\Omega_{A,\ell}(n)$ over sumsets and within combinatorial sequences, building on the congruence studies of Rivat, Sárközy, and Stewart [18], as well as the pseudorandomness framework developed by Daboussi and Sárközy [8, 9].

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