



A note on the extension of some triangular Diophantine sets

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Abstract

Let n be an integer. We call a set of m distinct positive integers a triangular $D(n)$ - m -tuple if the product of its two distinct elements increased by n is the triangular number. In this paper, we consider the extension of the triangular $D(k)$ -pair $\{k, 2k\}$, for $k = 10$ and $k = 28$, and prove that such a set cannot be extended to a quadruple.

1 Introduction

In this paper, we study one variant of the problem of Diophantus, which involves triangular numbers, i.e. numbers of the form $\Delta_r = \frac{r(r+1)}{2}$, where $r \in \mathbb{N}$ (this is the sequence A000217 in the OEIS [8]). In [4], the following definition is introduced:

Definition 1. Let n be an integer. A set $\{a_1, \dots, a_m\}$ of m distinct positive integers with $a_i a_j + n$ is a triangular number, for $1 \leq i < j \leq m$, is called triangular Diophantine m -tuple with the property $D(n)$, or simply triangular $D(n)$ - m -tuple.

In this variant of the problem, as in the similar problems of Diophantus, one of the question of interest is how large such sets can be. Jukić Bokun and Soldo [6] considered triangular $D(-1)$ -tuples and proved that the only triangular

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$D(-1)$ -triples of the form $\{1, 2, c\}$, $c = 2^n p$, where n is a non-negative integer and p is a prime, are those with $c \in \{11, 46, 352, 11936\}$. They proved that for these c 's no triangular $D(-1)$ -quadruple of the form $\{1, 2, c, d\}$ exists. The first author and Szalay [5] studied the extensibility of the triangular $D(1)$ -pair $\{1, 2\}$, and proved that it cannot be extended to a triangular $D(1)$ -quadruple $\{1, 2, c, d\}$, while the first and second author, together with Szalay [4] studied the extension the triangular $D(3)$ -pair $\{3, 6\}$ and proved the analogous result that such pair cannot be extended to a quadruple.

The main motivation of this paper is to generalize the results from [5] and [4] and consider the extension of the triangular $D(k)$ -pair $\{k, 2k\}$. However, that problem seems very difficult in general and we are not able to solve it for general k for now. The main difficulty that arises is the determination of the fundamental solutions of the observed system of Pell-like equations, mostly because it is hard to obtain suitable estimates on these fundamental solutions. There is some work in progress about the extension of the triangular $D(k)$ -pair $\{k, 2k\}$ for integer k of special form, i.e. $k = 2l^2 \pm l$ where l is an odd positive integer and $4l \pm 1$ is prime. But, for now it looks like that the problem of triangular Diophantine m -tuples is not easy to generalize. Here, we will consider another variant of the triangular $D(k)$ -pair $\{k, 2k\}$, the cases $k = 10$ and $k = 28$, with both k of the form $10 = 2 \cdot 2^2 + 2$ and $28 = 2 \cdot 4^2 - 4$, but $4 \cdot 2 + 1 = 9$ and $4 \cdot 4 - 1 = 15$ are not prime numbers.

Before introducing the problem of triangular Diophantine sets in [5], the authors made a simple computer search for finding the triangular Diophantine quadruples $\{a, b, c, d\}$ in the range $1 \leq a \leq 30$, $a < b \leq 30^2$, $b < c \leq 30^3$ and $c < d \leq 30^4$. They have noticed that triples appear relatively often, but they also found some quadruples, for example the triangular $D(1)$ -quadruple $\{14, 31, 135, 3510\}$. The quadruples seem to have a sparser occurrence than in case of the classical Diophantine quadruples. However, because of the similarity of the methods we use in the proofs, we expect that there should be a finite upper bound for the size of triangular Diophantine sets. Based on preliminary computer searches within a limited range, it appears that this bound is conjectured to be 4.

Our main result is the following theorem:

Theorem 1.1. *There does not exist a triangular $D(k)$ -quadruple of the form $\{k, 2k, c, d\}$ for $k = 10$ and $k = 28$.*

Our strategy of the proof is similar to the one used in [5] and [4]. We start with the triangular $D(k)$ -pair $\{k, 2k\}$ and first find the sequence (c_ν) of possible values of c that extend our pair to a triple. Then, we transform the problem of extending such a triple to a quadruple by solving a system of simultaneous Pell-like equations. It is furthermore transformed to finding intersections of binary

recurrence sequences. That problem is solved by observing some congruences and applying Baker's theory of linear forms in logarithms of algebraic numbers. At the end of the proof, we must make some reduction, and we use a variant of Baker-Davenport reduction. During the proof, we will do everything we can for general k , to avoid unnecessary repetition. Also, as the methods are similar to previous work on this topic, we won't give all the details here.

2 System of Pell-like equations

Let k be a positive integer and let $\{k, 2k, c\}$ be a triangular $D(k)$ -triple. Let us note that if k is a triangular number, such as $k = 10$ and $k = 28$, then we can extend any triangular $D(k)$ -pair $\{a, b\}$ with $c = 0$. Moreover, similar as in classical problem of Diophantus, we can introduce the following definition:

Definition 2. A triangular $D(n)$ - m -tuple in which elements are not necessarily distinct and may include zero is called improper triangular $D(n)$ - m -tuple.

We assume that $c \neq 0$ and we have

$$\begin{aligned} kc + k &= \frac{x(x+1)}{2}, \\ 2kc + k &= \frac{y(y+1)}{2} \end{aligned}$$

for some integers x and y . That implies

$$8kc + (8k + 1) = X_c^2, \quad (2.1)$$

$$16kc + (8k + 1) = Y_c^2, \quad (2.2)$$

where $X_c = 2x + 1$ and $Y_c = 2y + 1$. Eliminating c , we get a Pell-like equation

$$Y_c^2 - 2X_c^2 = -(8k + 1). \quad (2.3)$$

Hence, the first problem is to solve the equation (2.3). It can be easily solved, for fixed k , but, when we had no idea to solve it for general k or for at least some infinite family of parameter k , because of the difficulties in the proofs, we have focused on the case when $8k + 1 = K^2$ and when $(\pm K, K)$ are the only fundamental solutions of the equation (2.3). Both values $k = 10$ and $k = 28$ satisfy such conditions.

Now, we have that all positive solutions of the equation (2.3) are given by

$$(Y_\nu, X_\nu) = ((Y_c)_\nu, (X_c)_\nu) = (\pm K + K\sqrt{2})(3 + 2\sqrt{2})^\nu, \nu \geq 0.$$

Next, we get all possible values of $c = c_\nu$ from $X_c^2 = 8kc + K^2$. We can see that not all X_c give the c to be an integer, but all integers c that extend

our pair are generated in that way and included in this sequence. So, now we form the sequence $(c_q)_{q \geq 0}$, which includes only integral values of c . Also, when we get c to be an integer, in our cases it can be easily seen that then c is divisible by K^2 , and then we define $C = C_q = \frac{c_q}{K^2}$. In the case $k = 10$, we have $C \geq 357$ and in the case $k = 28$, we get $C \geq 5768133405$. We will use these estimates through the paper. We exclude the case $C = 0$ that leads to an improper triangular $D(k)$ -triple.

Assume now that c is the smallest integer such that $\{k, 2k, c, d\}$ is a triangular $D(k)$ -quadruple. Then, the equations

$$8kd + K^2 = X^2, \quad (2.4)$$

$$16kd + K^2 = Y^2, \quad (2.5)$$

$$8Cd + 1 = Z^2 \quad (2.6)$$

hold, for some integers X, Y , and Z . Eliminating d from the equations (2.4)-(2.6), we get the system of Pell-like equations

$$kZ^2 - CX^2 = k - K^2C, \quad (2.7)$$

$$2kZ^2 - CY^2 = 2k - K^2C. \quad (2.8)$$

Moreover, from the procedure how we got C , we know that there exist positive integers S and T such that

$$8kC + 1 = S^2, \quad (2.9)$$

$$16kC + 1 = T^2. \quad (2.10)$$

From (2.9) and (2.10), we see that S and T are odd. Also, since a square of an odd number has the form $24l + 1$ or $72l + 9$, $l \in \mathbb{Z}$, from (2.9) we get

$$kC \equiv 0 \pmod{3} \text{ or } kC \equiv 1 \pmod{9},$$

and from (2.10) we get

$$kC \equiv 0 \pmod{3} \text{ or } 2kC \equiv 1 \pmod{9}.$$

It is easy to see that the only possibility is $3|kC$. If $k = 10$ or $k = 28$, it means that $3|C$. Moreover, since $c = 81C$ for $k = 10$ and $c = 225C$ for $k = 28$, then $3^5|c$ for $k = 10$ and $3^3|c$ for $k = 28$. For example, $\{10, 20, 28917\}$ is a triangular $D(10)$ -triple and $\{28, 56, 1297830016000\}$ is a triangular $D(28)$ -triple.

From the theory of Pell-like equations [7, Theorem 108.a], using (2.9) and (2.10), we can generate all solutions in positive integers of (2.7) and (2.8) by

$$Z\sqrt{k} + X\sqrt{C} = (Z_0\sqrt{k} + X_0\sqrt{C})(T + 4\sqrt{kC})^m, \quad m \geq 0, \quad (2.11)$$

$$Z\sqrt{2k} + Y\sqrt{C} = (Z_1\sqrt{2k} + Y_1\sqrt{C})(S + 2\sqrt{2kC})^n, \quad n \geq 0, \quad (2.12)$$

where (Z_0, X_0) and (Z_1, Y_1) are fundamental solutions of (2.7) and (2.8), respectively, and, analogously as in [2, Lemma 1], they satisfy

$$\begin{aligned} 0 < X_0^2 &\leq \frac{8k(K^2C - k)}{T - 1}, \\ Z_0^2 &\leq \frac{(T - 1)(K^2C - k)}{2k}, \end{aligned} \quad (2.13)$$

$$\begin{aligned} 0 < Y_1^2 &\leq \frac{4k(K^2C - 2k)}{S - 1}, \\ Z_1^2 &\leq \frac{(S - 1)(K^2C - 2k)}{4k}. \end{aligned} \quad (2.14)$$

We have now transformed our problem to solving the equation

$$Z = V_m = W_n, \quad (2.15)$$

for some non-negative integers m and n , where the sequences $(V_m)_{m \geq 0}$ and $(W_n)_{n \geq 0}$ are given by

$$V_0 = Z_0, \quad V_1 = TZ_0 + 4CX_0, \quad V_{m+2} = 2TV_{m+1} - V_m, \quad (2.16)$$

$$W_0 = Z_1, \quad W_1 = SZ_1 + 2CY_1, \quad W_{n+2} = 2SW_{n+1} - W_n. \quad (2.17)$$

We will use some congruences, obtained analogously as in [5], which will help us in solving the equation (2.15). By induction we easily obtain

$$V_{2m} \equiv V_0 \pmod{8C}, \quad (2.18)$$

$$V_{2m+1} \equiv V_1 \pmod{8C}, \quad (2.19)$$

$$W_{2n} \equiv W_0 \pmod{4C}, \quad (2.20)$$

$$W_{2n+1} \equiv W_1 \pmod{4C}. \quad (2.21)$$

Analogously as in [4], from (2.11) it follows that

$$Z_0^2 \equiv 1 \pmod{8C} \quad (2.22)$$

and, from (2.12), it follows that

$$Z_1^2 \equiv 1 \pmod{4C}. \quad (2.23)$$

3 Finding fundamental solutions of Pell-like equations

In this section, we want to find all possibilities for the fundamental solutions (Z_0, X_0) and (Z_1, Y_1) of the equations (2.7) and (2.8), respectively, that lead to

the extension of a triangular $D(k)$ -triple $\{k, 2k, c\}$ to a quadruple, for $k = 10$ and $k = 28$. We consider the cases depending on the parity of indices m and n in the equation (2.15). For that, we need a suitable estimate on the fundamental solutions. For $C \geq 357$, in the case $k = 10$, and $C \geq 5768133405$, for $k = 28$, from (2.10), we get that

$$|T| < 0.67C. \quad (3.1)$$

Then, from (2.13), it follows that

$$|Z_0| < 1.648C. \quad (3.2)$$

Similarly, from (2.9) we get

$$|S| < 0.48C \quad (3.3)$$

and then, from (2.14), it follows that

$$|Z_1| < 0.986C. \quad (3.4)$$

1.) $Z = V_{2m} = W_{2n}$

From the congruences (2.18) and (2.20), we have $Z_0 \equiv Z_1 \pmod{4C}$. Then, from the estimates (3.2) and (3.4), we conclude that $Z_0 = Z_1$. By (2.22), there exists $d_0 \in \mathbb{Z}$ such that $d_0 = \frac{Z_0^2 - 1}{8C}$ and, from the estimate (3.2), we easily get $d_0 < c$. By plugging that into (2.11) and (2.12), we obtain

$$\begin{aligned} 8Cd_0 + 1 &= Z_0^2, \\ 8kd_0 + K^2 &= X_0^2, \\ 16kd_0 + K^2 &= Y_1^2. \end{aligned}$$

This implies that we have the triangular $D(k)$ -quadruple $\{k, 2k, d_0, c\}$ with $d_0 < c$. By assumption on the minimality of c , we conclude that $d_0 = 0$, i.e. this $D(k)$ -quadruple is improper. Therefore, we obtain

$$Z_0 = Z_1 = \pm 1, \quad X_0 = K, \quad Y_1 = K.$$

2.) $Z = V_{2m+1} = W_{2n}$

The relations (2.19), (2.20), (2.16), and (2.17) imply

$$TZ_0 \equiv Z_1 \pmod{4C}. \quad (3.5)$$

Firstly, if $Z_0 = \pm 1$, then $\pm T \equiv Z_1 \pmod{4C}$. By (3.1) and (3.4), we conclude $Z_1 = \pm T$. But now (2.8) implies

$$Y_1^2 = 32k^2 + K^2 = 32k^2 + 8k + 1$$

or

$$2Y_1^2 = (8k + 1)^2 + 1$$

and finally we have

$$K^4 - 2Y_1^2 = -1,$$

which is well known to not have solutions except $(K, Y) = (\pm 1, \pm 1)$.

Hence, we have proved that $|Z_0| > 1$. By (2.22) and using (2.7), we know that there exists $d_0 \in \mathbb{Z}$ such that

$$Z_0^2 = 8Cd_0 + 1, \tag{3.6}$$

$$X_0^2 = 8kd_0 + K^2. \tag{3.7}$$

Now, we use a simple computer program in Pari/GP to check what is happening for $1 \leq d_0 \leq 100$ and for our two values of k . First, we check if $8kd_0 + K^2$ is a square. If it is, then, for such d_0 , we check if $8Cd_0 + 1$, $8kC + 1$, and $16kC + 1$ can simultaneously be squares. We do this by using Magma, as it was explained in more details in [5] and [4], and by finding integral points of some elliptic curves. At the end, we see that the only possibility is $Z_0 = \pm T$ and $X_0 = 4k + 1$, which we get for $d_0 = 2k$. From (3.5) and (2.10), we get $\pm 1 \equiv Z_1 \pmod{4C}$ and then, by (3.4), $Z_1 = \pm 1$. Using (2.8) to calculate Y_1 , we obtain

$$Z_0 = \pm T, \quad Z_1 = \pm 1, \quad X_0 = 4k + 1, \quad Y_1 = K.$$

Now, we have a lower bound $X_0 \geq \sqrt{8k \cdot 101 + K^2}$, which is good enough to prove the following property:

$$\begin{aligned} 0 < 4CX_0 - T|Z_0| &= \frac{16C^2X_0^2 - T^2Z_0^2}{4CX_0 + T|Z_0|} = \frac{16K^2C^2 - 16kC - 8Cd_0 - 1}{4CX_0 + T|Z_0|} \\ &< \frac{16K^2C^2}{2T|Z_0|} < 3C. \end{aligned} \tag{3.8}$$

Moreover, if $Z_0 > 0$, then

$$-3C < TZ_0 - 4CX_0 < 0.$$

From the congruence (3.5) and the estimate (3.4), we conclude that $TZ_0 - 4CX_0 = Z_1$. Then, $Z_1^2 \equiv 1 \pmod{8C}$, which implies that there exists $d_1 \in \mathbb{Z}$ such that

$$Z_1^2 = 8Cd_1 + 1. \tag{3.9}$$

Again, as before we conclude that $d_1 = \frac{Z_1^2 - 1}{8C} < c$. Furthermore, we have

$$\begin{aligned} 8kd_1 + K^2 &= k \frac{Z_1^2 - 1}{C} + K^2 = \frac{k(TZ_0 - 4CX_0)^2 - k + K^2C}{C} \\ &= (4kZ_0 - TX_0)^2, \\ 16kd_1 + K^2 &= 2k \frac{Z_1^2 - 1}{C} + K^2 = \frac{2kZ_1^2 - 2k + K^2C}{C} = Y_1^2. \end{aligned}$$

Subsequently, there exists a triangular $D(k)$ -quadruple $\{k, 2k, d_1, c\}$ with $d_1 < c$. Then, from the minimality of c , we conclude that $d_1 = 0$ and we have an improper $D(k)$ -quadruple. By (3.9), $Z_1 = \pm 1$. Multiplying (3.5) by T , applying (3.1), and using the estimates (3.1) and (3.2), we see that $Z_0 = \pm T$. This leads to a solution, which we have already found.

Finally, if $Z_0 < 0$, we have

$$0 < TZ_0 + 4CX_0 < 3C.$$

As before, from (3.5), we deduce that $TZ_0 + 4CX_0 = Z_1$ and conclude that there exists a nonnegative integer d_1 such that $d_1 = \frac{Z_1^2 - 1}{8C} < c$. Furthermore,

$$\begin{aligned} 8kd_1 + K^2 &= k \frac{Z_1^2 - 1}{C} + K^2 = \frac{k(TZ_0 + 4CX_0)^2 - k + K^2C}{C} \\ &= (4kZ_0 + TX_0)^2, \\ 16kd_1 + K^2 &= 2k \frac{Z_1^2 - 1}{C} + K^2 = \frac{2kZ_1^2 - 2k + K^2C}{C} = Y_1^2. \end{aligned}$$

Thus, it yields that there exists a triangular $D(k)$ -quadruple $\{k, 2k, d_1, c\}$ such that $d_1 < c$. Again, from the minimality of c , we conclude that $d_1 = 0$ and $Z_1 = \pm 1$, and we have arrived at a known solution again.

The other two cases depending on parity of indices m and n are solved in the exactly same way as it was done in [5] and [4], with slightly more time consuming work when searching for small values of fundamental solutions.

In conclusion, we have proved the following lemma:

Lemma 3.1. *The fundamental solutions of the equations (2.7) and (2.8), which lead to the extension of the triangular $D(k)$ -triple $\{k, 2k, c\}$, for $k = 10$ and $k = 28$, are*

- $Z_0 = Z_1 = \pm 1$, $X_0 = Y_1 = K$, for $V_{2m} = W_{2n}$,
- $Z_0 = \pm T$, $Z_1 = \pm 1$, $X_0 = 4k + 1$, $Y_1 = K$, for $V_{2m+1} = W_{2n}$.

4 Linear forms in logarithms

In this section, we consider the equation $Z = V_m = W_n$, for $m, n \neq 0$. We use the standard method to transform this equation to an inequality that includes a linear form in logarithms. We have

$$V_m = \frac{1}{2\sqrt{k}}((Z_0\sqrt{k} + X_0\sqrt{C})(T + 4\sqrt{kC})^m + (Z_0\sqrt{k} - X_0\sqrt{C})(T - 4\sqrt{kC})^m)$$

and

$$W_n = \frac{1}{2\sqrt{2k}}((Z_1\sqrt{2k} + Y_1\sqrt{C})(S + 2\sqrt{2kC})^n + (Z_1\sqrt{2k} - Y_1\sqrt{C})(S - 2\sqrt{2kC})^n).$$

If we define

$$P := \frac{kZ_0 + X_0\sqrt{kC}}{k}(T + 4\sqrt{kC})^m \quad \text{and} \quad Q := \frac{2kZ_1 + Y_1\sqrt{2kC}}{2k}(S + 2\sqrt{2kC})^n,$$

then $Z = V_m = W_n$ yields

$$P - \left(\frac{K^2C}{k} - 1\right)P^{-1} = Q - \left(\frac{K^2C}{2k} - 1\right)Q^{-1}.$$

We easily see that $P, Q > 1$ and

$$\begin{aligned} P - Q &= \left(\frac{K^2C}{k} - 1\right)P^{-1} - \left(\frac{K^2C}{2k} - 1\right)Q^{-1} > \left(\frac{K^2C}{k} - 1\right)(P^{-1} - Q^{-1}) \\ &= \left(\frac{K^2C}{k} - 1\right)(Q - P)P^{-1}Q^{-1}. \end{aligned}$$

Since $\frac{K^2C}{k} - 1 > 0$ and $P^{-1}Q^{-1} > 0$, we have $P > Q$. Furthermore, we conclude

$$P - Q < \left(\frac{K^2C}{k} - 1\right)P^{-1},$$

which gives us

$$\frac{P - Q}{P} < \left(\frac{K^2C}{k} - 1\right)P^{-2}. \quad (4.1)$$

By the lower bound $C \geq 357$ and for the fundamental solutions of the equations (2.7) and (2.8) from Lemma 3.1, for $m \geq 1$, we get

$$P \geq \left(\frac{(4k+1)\sqrt{kC}}{k} - T\right)(T + 4\sqrt{kC}) > 7.997C,$$

hence

$$P^{-1} < \frac{1}{7.997C}.$$

Moreover, we have

$$\left(\frac{K^2C}{k} - 1\right)P^{-2} < 0.000355.$$

As $P > Q$, using (4.1), we conclude that

$$\begin{aligned} 0 < \log \frac{P}{Q} &= -\log \left(1 - \frac{P-Q}{P}\right) \\ &< \left(\frac{K^2C}{k} - 1\right)P^{-2} + \left(\frac{K^2C}{k} - 1\right)^2 P^{-4} \\ &= \left(1 + \left(\frac{K^2C}{k} - 1\right)P^{-2}\right)\left(\frac{K^2C}{k} - 1\right)P^{-2} \\ &< 1.000355 \left(\frac{K^2C}{k} - 1\right) \frac{(T + 4\sqrt{kC})^{-2m}}{\left(\frac{(4k+1)\sqrt{kC}}{k} - T\right)^2} \\ &< (T + 4\sqrt{kC})^{-2m+1}. \end{aligned}$$

In summary, we have just proved the following lemma:

Lemma 4.1. *If $V_m = W_n$, for $m, n \neq 0$, then for $k = 10$ or $k = 28$ we have*

$$\begin{aligned} 0 &< m \log(T + 4\sqrt{kC}) - n \log(S + 2\sqrt{2kC}) + \log \frac{2(kZ_0 + X_0\sqrt{kC})}{2kZ_1 + Y_1\sqrt{2kC}} \\ &< (T + 4\sqrt{kC})^{-2m+1}. \end{aligned}$$

5 Relation between m and n and their lower bounds

In this section, we want to find the relation between indices m and n , if $V_m = W_n$, for $m, n \neq 0$. We have

$$V_m \geq (2T - 1)^{m-1}V_1 \quad \text{and} \quad W_n < (2S)^{n-1}W_1.$$

Remember, there are two possibilities for the fundamental solutions of the equations (2.7) and (2.8), given by Lemma 3.1.

Firstly, if $V_{2m} = W_{2n}$, $Z_0 = Z_1 = \pm 1$, $X_0 = Y_1 = K$, then

$$(2T - 1)^{2m-1}(\pm T + 4KC) < (2S)^{2n-1}(\pm S + 2KC),$$

which easily implies $m < n$.

Secondly, if $V_{2m+1} = W_{2n}$, $X_0 = 4k + 1$, $Y_1 = K$, $Z_0 = \pm T$, $Z_1 = \pm 1$, then we have

$$(2T - 1)^{2m}(\pm 16kC \pm 1 + 4(4k + 1)C) < (2S)^{2n-1}(\pm S + 2KC),$$

which yields

$$(2T - 1)^{2m} < 3 \cdot (2S)^{2n-1}.$$

Because of the parity, $2m \neq 2n - 1$. Furthermore, we see that the last inequality is not possible for $2m \geq 2n - 1$. Therefore, we have $2m + 1 < 2n$. This means that in all cases, for $m, n \neq 0$, the equation $V_m = W_n$ implies $m < n$. Remember that we are interested only in the solutions that lead to the extension of our triple.

Now, we will find the upper bound for n in terms of m . We consider the "worst" possible case of fundamental solutions (Z_0, X_0) and (Z_1, Y_1) concerning the inequality of Lemma 4.1. We obtain

$$n \log(S + 2\sqrt{2kC}) < m \log(T + 4\sqrt{kC}) + \log \frac{2kT + 2(4k+1)\sqrt{kC}}{2k + K\sqrt{2kC}}.$$

Then, we have

$$n < m \frac{\log(T + 4\sqrt{kC})}{\log(S + 2\sqrt{2kC})} + \frac{\log \frac{2kT + 2(4k+1)\sqrt{kC}}{2k + K\sqrt{2kC}}}{\log(S + 2\sqrt{6C})} < 1.06m + 0.4325 < 1.5m.$$

Thus, we have just proved the following lemma:

Lemma 5.1. *If $V_m = W_n$ holds for $m, n \neq 0$, then for $k = 10$ or $k = 28$ we have $m < n < 1.5m$.*

The next congruences can be proved by induction using (2.16) and (2.17):

$$\begin{aligned} V_{2m} &\equiv (32km^2C + 1)Z_0 + 8mTCX_0 \pmod{64kC^2}, \\ V_{2m+1} &\equiv (32km(m+1)C + 1)TZ_0 + 4(2m+1)CX_0 \pmod{64kC^2}, \\ W_{2n} &\equiv (16kn^2C + 1)Z_1 + 4nSCY_1 \pmod{64kC^2}, \\ W_{2n+1} &\equiv (16kn(n+1)C + 1)SZ_1 + 2(2n+1)CY_1 \pmod{64kC^2}. \end{aligned}$$

Firstly, if $V_{2m} = W_{2n}$, by Lemma 3.1 we have $Z_0 = Z_1 = \pm 1, X_0 = Y_1 = K = \sqrt{8k+1}$, which yields

$$\pm(32km^2C + 1) + 8KmTC \equiv \pm(16kn^2C + 1) + 4KnSC \pmod{64kC^2}.$$

Dividing this by $4C$, we get

$$\pm 8km^2 + 2KmT \equiv \pm 4kn^2 + KnS \pmod{16kC}. \quad (5.1)$$

Let us assume that $m \leq \frac{1}{2}\sqrt{C}$. Then, it can be proved that the last congruence becomes equation. Really, we have

$$\begin{aligned} 8km^2 &\leq 2kC, \\ 2KmT &< 2K \cdot \frac{1}{2}\sqrt{C} \cdot \sqrt{16kC+1} < 11.39kC, \\ 4kn^2 &< 4k \cdot (1.5m)^2 = 9km^2 < 2.25C, \\ KnS &< KnT < K \cdot 1.5mT < 8.54kC, \end{aligned}$$

and both sides of congruence (5.1) are positive because of

$$\begin{aligned} 8km^2 &\leq 8km \cdot \frac{1}{2}\sqrt{C} = 4km\sqrt{C} < 2KmT, \\ 4kn^2 &< 4kn \cdot 1.5m \leq 3kn\sqrt{C} < KnS. \end{aligned}$$

Hence, we get

$$\pm 8km^2 + 2KmT = \pm 4kn^2 + KnS.$$

If we have + sign, then the left-hand side is obviously larger than the right one. On the other hand, if we have the – sign, then

$$2KmT - 8km^2 = m(2KT - 8km) > 18.76km\sqrt{C}$$

and

$$KnS - 4kn^2 < KnS < K \cdot 1.5mS < 12.08km\sqrt{C},$$

which is a contradiction. It implies that in the case when $V_{2m} = W_{2n}$, $m, n \neq 0$, we have $m > \frac{1}{2}\sqrt{C}$, i.e.

$$2m > \sqrt{C}.$$

The case $V_{2m+1} = W_{2n}$ can be similarly studied in the same way as it was done in [5] and [4]. So, we omit details here. Also, remember that we are solving only the cases $k = 10$ and $k = 28$. At the end, we have the following lemma:

Lemma 5.2. *If $V_m = W_n$, with $m, n \neq 0$ and for $k = 10$ or $k = 28$, then $m > \sqrt{C}$.*

6 Proof of the Theorem 1.1

To finish the proof of Theorem 1.1, we apply the following well-known theorem from [1] for the result of our Lemma 4.1:

Theorem 6.1. (see [1, Theorem]) Let $\Lambda \neq 0$ be a linear form in logarithms of ℓ algebraic numbers $\alpha_1, \dots, \alpha_\ell$ with rational integer coefficients $b_1, \dots, b_\ell$. Then

$$\log \Lambda \geq -18(\ell + 1)! \ell^{\ell+1} (32d)^{\ell+2} h'(\alpha_1) \cdots h'(\alpha_\ell) \log(2\ell d) \log B,$$

where $B = \max\{|b_1|, \dots, |b_\ell|\}$, and d is the degree of the number field generated by $\alpha_1, \dots, \alpha_\ell$. Here

$$h'(\alpha) = \max \left\{ h(\alpha), \frac{|\log \alpha|}{d_\alpha}, \frac{1}{d_\alpha} \right\},$$

and $h(\alpha)$ denotes the standard Weil logarithmic height of α , where d_α is the degree of the algebraic number α .

Since this theorem was used many times in similar problems and the details of its application were given in [5] and [4], we will just mention the final result we get, for the linear form that appears in Lemma 4.1. If we define

$$\Lambda = m \log(T + 4\sqrt{kC}) - n \log(S + 2\sqrt{2kC}) + \log \frac{2(kZ_0 + X_0\sqrt{kC})}{2kZ_1 + Y_1\sqrt{2kC}},$$

Theorem 6.1 implies that

$$-\log \Lambda < 2.23 \cdot 10^{15} \cdot \log^3 C \cdot \log n.$$

On the other hand, from Lemma 4.1, we have

$$-\log \Lambda > (2m - 1) \log(T + 4\sqrt{kC}) > 0.5(2m - 1) \log C.$$

Combining the last two inequalities, together with $m > \sqrt{C}$ and $n < 1.5m$ in $V_m = W_n$, for $m, n \neq 0$, we get

$$2m - 1 < 4.46 \cdot 10^{15} \cdot \log^2(m^2) \cdot \log(1.5m),$$

which implies that

$$m < 1.02 \cdot 10^{21}$$

and

$$C < m^2 < 1.0404 \cdot 10^{42},$$

or

$$c = K^2 C < 1.57 \cdot 10^{43}.$$

Hence, we have just proved the following proposition:

Proposition 6.2. *Let $k = 10$ or $k = 28$. Then, the triangular $D(k)$ -triple $\{k, 2k, c\}$ cannot be extended to a quadruple with the larger element, if $c \geq 1.57 \cdot 10^{43}$.*

To finish the proof of Theorem 1.1, as in our previous work, we have applied a version of Baker-Davenport reduction (see [3, Lemma 5]) to the linear form from Lemma 4.1, which is already well known method in proving similar results. We have implemented it in Pari/GP [9], for $k = 10$ and $k = 28$ and $c < 1.57 \cdot 10^{43}$, and in all cases, after only one step of reduction, we have got $m \leq 10$, which is small enough to get a contradiction (and then no extension to a triple, except trivial one with $d = 0$, exists), as we have already proved that $m \geq \sqrt{C} \geq \sqrt{357} > 18.89 > 10$, if $m, n \neq 0$.

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