



On the Hilbert depth of the quotient ring of the edge ideal of a star graph

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Abstract

Let $S_n = K[x_1, \dots, x_n, y]$ and $I_n = (x_1y, x_2y, \dots, x_ny) \subset S_n$ be the edge ideal of star graph. We prove that $\text{hdepth}(S_n/I_n) \geq \lceil \frac{n}{2} \rceil + \lfloor \sqrt{n} \rfloor - 2$. Also, we show that for any $\varepsilon > 0$, there exists some integer $A = A(\varepsilon) \geq 0$ such that $\text{hdepth}(S_n/I_n) \leq \lceil \frac{n}{2} \rceil + \lfloor \varepsilon n \rfloor + A - 2$. We deduce that $\lim_{n \rightarrow \infty} \frac{1}{n} \text{hdepth}(S_n/I_n) = \frac{1}{2}$.

1 Introduction

Let K be a field and let S be a polynomial ring over K . Let M be a finitely generated graded S -module. The Hilbert depth of M , denoted by $\text{hdepth}(M)$, is the maximal depth of a finitely generated graded S -module N with the same Hilbert series as M ; see [3, 6] for further details. One would expect that it is easy to compute the Hilbert depth of a module, once its Hilbert function is known. But it turns out that even for the powers of the maximal ideal, the computation of the Hilbert depth leads to difficult combinatorial computations; see [4]. In our opinion, the Hilbert depth invariant is unjustly underrated in the literature and deserves a closer attention and more research.

In [1, Theorem 2.4] it was proved that if $0 \subset I \subsetneq J \subset S$ are two squarefree monomial ideals, then $\text{hdepth}(J/I)$ is equal to

$$\max\{q : \beta_k^q(J/I) = \sum_{j=0}^k (-1)^{k-j} \binom{q-j}{k-j} \alpha_j(J/I) \geq 0 \text{ for all } 0 \leq k \leq q\},$$

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where $\alpha_j(J/I)$ is the number of squarefree monomials of degree j in $J \setminus I$, for $0 \leq j \leq n$.

Using this new combinatorial characterization of the Hilbert depth, in [2], the Hilbert depth invariant was studied for several classes of squarefree monomial ideals. In particular, it was showed that if $I_n = (x_1y, \dots, x_ny) \subset S_n := K[x_1, \dots, x_n, y]$, then $\text{hdepth}(I_n) = \lfloor \frac{n+3}{2} \rfloor$. Note that, I_n is the edge ideal of a star graph with n rays, or, equivalently, the edge ideal of the complete bipartite graph $K_{n,1}$. Although, the formula for $\text{hdepth}(I_n)$ is easy to obtain, the problem of computing $\text{hdepth}(S_n/I_n)$ is very difficult and the aim of our paper is to tackle this problem.

We were not able to obtain an exact formula, but we managed to find some sharp bounds for $\text{hdepth}(S_n/I_n)$. In Theorem 2.4 we prove that

$$\binom{k - \lfloor \sqrt{2k+1} \rfloor + 2s + 1}{2s} \geq \binom{k + \lfloor \sqrt{2k+1} \rfloor - 2}{2s - 1},$$

for all $k \geq 1$ and $1 \leq s \leq \lfloor \sqrt{2k+1} \rfloor - 1$. Using this combinatorial inequality, we show in Theorem 2.5 that

$$\text{hdepth}(S_n/I_n) \geq \left\lceil \frac{n}{2} \right\rceil + \lfloor \sqrt{n} \rfloor - 2, \text{ for all } n \geq 1.$$

Using similar techniques, in Theorem 2.6 we prove that that for any $\varepsilon > 0$, there exists some integer $A = A(\varepsilon) \geq 0$ such that

$$\text{hdepth}(S_n/I_n) \leq \left\lceil \frac{n}{2} \right\rceil + \lfloor \varepsilon n \rfloor + A - 2.$$

In particular, from Theorem 2.5 and Theorem 2.6 we deduce in Corollary 2.7 that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{hdepth}(S_n/I_n) = \frac{1}{2}.$$

In the last section, we provide some examples and computer experiments.

2 Main results

The star graph $\mathcal{S}_n$ is the graph on the vertex set $V(\mathcal{S}_n) = \{x_1, \dots, x_n, y\}$ with the edge set $E(\mathcal{S}_n) = \{\{x_i, y\} : 1 \leq i \leq n\}$. Let $S_n = K[x_1, \dots, x_n, y]$. We consider the ideal

$$I_n = (x_1y, x_2y, \dots, x_ny) \subset S_n,$$

which can be interpreted as the edge ideal of the star graph $\mathcal{S}_n$.

The following result is a particular case ($m = 1$) of [2, Theorem 2.9].

Proposition 2.1. *We have that $\text{hdepth}(I_n) = \lfloor \frac{n+3}{2} \rfloor$.*

However, $\text{hdepth}(S_n/I_n)$ seems very difficult to compute, in general. As a direct consequence of [2, Theorem 2.6], we get:

Proposition 2.2. *We have that $\text{hdepth}(S_n/I_n) \leq n + 2 - \left\lceil \sqrt{2n + \frac{1}{4}} + \frac{1}{2} \right\rceil$.*

Also, as a direct consequence of [2, Theorem 2.7] and of the identity $\binom{-x}{k} = (-1)^k \binom{x+k-1}{k}$, we get the following result.

Proposition 2.3. *We have that*

$$\text{hdepth}(S_n/I_n) = \max\{d : \binom{n-d+2s-1}{2s} \geq \binom{d-1}{2s-1} \text{ for all } 1 \leq s \leq \left\lfloor \frac{d}{2} \right\rfloor\}.$$

Our key technical tool is the inequality below.

Theorem 2.4. *For all $k \geq 2$ and $1 \leq s \leq \lfloor \sqrt{2k+1} \rfloor - 1$, we have that*

$$\binom{k - \lfloor \sqrt{2k+1} \rfloor + 2s + 1}{2s} \geq \binom{k + \lfloor \sqrt{2k+1} \rfloor - 2}{2s-1}.$$

Proof. If $s = 1$, the conclusion is equivalent to

$$(k - \lfloor \sqrt{2k+1} \rfloor + 3)(k - \lfloor \sqrt{2k+1} \rfloor + 2) \geq 2(k + \lfloor \sqrt{2k+1} \rfloor - 2),$$

which can easily be proved for all $k \geq 1$. Hence, we may assume $s \geq 2$ and $k \geq 4$. Moreover, using case by case verification, we can get the conclusion for $1 \leq k \leq 400$. So, in the following, we will assume $k \geq 401$ and thus $\sqrt{2k+1} > 28$.

We denote $p = \lfloor \sqrt{2k+1} \rfloor$ and we consider the following two cases: (i) $p = 2r + 1$ for some $r \geq 14$ and (ii) $p = 2r$ for some $r \geq 14$.

(i) Since $p = 2r + 1$ and $p \leq \sqrt{2k+1} < p + 1$, it follows that

$$4r^2 + 4r + 1 \leq 2k + 1 \leq 4r^2 + 8r + 3.$$

Hence, we can write $k = 2r^2 + 2r + t$ for some $0 \leq t \leq 2r + 1$. Note that $2 \leq s \leq 2r$. The desired inequality is equivalent to

$$\binom{2r^2 + 2s + t}{2s} \geq \binom{2r^2 + 4r + t - 1}{2s-1} \text{ for all } r \geq 14, 0 \leq t \leq 2r+1, 2 \leq s \leq 2r. \quad (1)$$

Put $x = 2r^2 + t$. By (1), we have to prove the inequality $\binom{x+2s}{2s} \geq \binom{x+4r-1}{2s-1}$, which is equivalent to

$$(x+1)(x+2) \cdots (x+2s) \geq 2s(x+4r-2s+1)(x+4r-2s+2) \cdots (x+4r-1)$$

as well as to

$$0 \leq \log \left(1 + \frac{x}{2s} \right) + \sum_{j=1}^{2s-1} \log \left(1 + \frac{2s-4r}{x+4r-2s+j} \right). \quad (2)$$

Note that (2) is obvious for $s = 2r$, so we may assume $s \leq 2r - 1$.

Since $\log(1+y) \geq \frac{y}{1+y}$ for all $y > -1$, in order to prove (2) it suffices to show

$$0 \leq \log \left(1 + \frac{x}{2s} \right) - (4r-2s) \sum_{j=1}^{2s-1} \frac{1}{x+j}. \quad (3)$$

We recall the following classical bounds (see [5]) for the harmonic series:

$$\begin{aligned} \sum_{m \leq X} \frac{1}{m} &\leq \log X + \gamma + \frac{1}{2X} + \frac{1}{4X^2}, \\ \sum_{m \leq X} \frac{1}{m} &\geq \log X + \gamma - \frac{1}{2X} - \frac{1}{4X^2}, \end{aligned} \quad (4)$$

where $\gamma \approx 0.57721$ is the Euler–Mascheroni constant. By applying the above inequalities, we deduce that

$$\begin{aligned} \sum_{j=1}^{2s-1} \frac{1}{x+j} &\leq \log(x+2s-1) + \frac{1}{2(x+2s-1)} + \frac{1}{4(x+2s-1)^2} - \log x + \frac{1}{2x} + \frac{1}{4x^2} \\ &\leq \log \left(1 + \frac{2s-1}{x} \right) + \frac{1}{2(x+2s-1)} + \frac{1}{300(x+2s-1)} + \frac{1}{2x} + \frac{1}{288x} \\ &< \log \left(1 + \frac{2s-1}{x} \right) + \frac{A}{x}, \end{aligned}$$

with $A = 1.00128$. Here, we used the fact that $x \geq 2r^2 \geq 392$ and $x+2s-1 \geq 395$.

Now, in order to prove (3), using the above upper bound for $\sum_{j=1}^{2s-1} \frac{1}{x+j}$, it is enough to show that

$$\log \left(1 + \frac{x}{2s} \right) > (4r-2s) \left(\log \left(1 + \frac{2s-1}{x} \right) + \frac{A}{x} \right).$$

As the expression on the left side is increasing w.r.t. x and the expression on the right side is decreasing w.r.t. x , it is enough to consider the case $x = 2r^2$, that is, to prove that for all $r \geq 14$ and $2 \leq s \leq 2r - 1$ we have

$$\log \left(1 + \frac{r^2}{s} \right) > (4r-2s) \left(\log \left(1 + \frac{2s-1}{2r^2} \right) + \frac{A}{2r^2} \right). \quad (5)$$

Using the elementary inequality $\log(1+y) < y$ valid for $y > 0$ and the arithmetic mean-geometric mean inequality, the right-hand side is bounded from above as follows:

$$(4r-2s) \left(\frac{2s-1}{2r^2} + \frac{A}{2r^2} \right) < \frac{(2r+0.5A-0.5)^2}{2r^2} = 2 + \frac{A-1}{r} + \frac{(A-1)^2}{8r^2} < 2.002.$$

The left-hand side is at least

$$\log \left(1 + \frac{r^2}{2r-1} \right) \geq \log(1 + 196/27) > 2.111.$$

Comparison of the last two displayed inequalities ends the proof of part (i).

(ii) Since $p = 2r$ and $p \leq \sqrt{2k+1} < p+1$, it follows that

$$4r^2 \leq 2k+1 \leq 4r^2 + 4r.$$

Hence, we can write $k = 2r^2 + t$ for some $0 \leq t \leq 2r-1$ and the conclusion of the theorem is equivalent to

$$\binom{2r^2 - 2r + t + 2s + 1}{2s} \geq \binom{2r^2 + 2r + t - 2}{2s-1} \quad (6)$$

for all $r \geq 14$, $0 \leq t \leq 2r-1$, and $2 \leq s \leq 2r-1$.

We denote $x = 2r^2 - 2r + t + 1$ and we have to prove that $\binom{x+2s}{2s} \geq \binom{x+4r-3}{2s-1}$. Similarly to the case (i), this is equivalent to

$$0 \leq \log \left(1 + \frac{x}{2s} \right) + \sum_{j=1}^{2s-1} \log \left(1 + \frac{2s+2-4r}{x+4r-2s-2+j} \right). \quad (7)$$

In order to prove (7) it suffices to show that

$$0 \leq \log \left(1 + \frac{x}{2s} \right) - (4r-2s-2) \sum_{j=1}^{2s-1} \frac{1}{x+j}. \quad (8)$$

If $s = 2r-1$, then (8) holds, hence we may assume $s \leq 2r-2$.

From (4) and the fact that $x \geq 2r^2 - 2r + 1 \geq 1513$ and $x+2s-1 \geq 1516$, it follows that

$$\sum_{j=1}^{2s-1} \frac{1}{x+j} \leq \log \left(1 + \frac{2s-1}{x} \right) + \frac{A}{x},$$

where this time $A = 1.00067$. Therefore, in order to prove (8) it is enough to show that

$$\log \left(1 + \frac{x}{2s} \right) > (4r-2s-2) \left(\log \left(1 + \frac{2s-1}{x} \right) + \frac{A}{x} \right).$$

As in case (i), it is enough to prove the above for $x = 2r^2 - 2r + 1$, that is

$$\log \left(1 + \frac{2r^2 - 2r + 1}{2s} \right) > (4r - 2s - 2) \left(\log \left(1 + \frac{2s - 1}{2r^2 - 2r + 1} \right) + \frac{A}{2r^2 - 2r + 1} \right), \quad (9)$$

for all $r \geq 14$ and $2 \leq s \leq 2r - 2$.

Since $\log(1 + y) < y$ for $y > 0$, in order to prove (9) it suffices to show that

$$\log \left(1 + \frac{2r^2 - 2r + 1}{2s} \right) > \frac{(4r - 2s - 2)(2s + A - 1)}{2r^2 - 2r + 1},$$

for all $r \geq 14$, $2 \leq s \leq 2r - 2$. This is true as seen from the chain of inequalities

$$\begin{aligned} \log \left(1 + \frac{2r^2 - 2r + 1}{2s} \right) &\geq \log \left(1 + \frac{2r^2 - 2r + 1}{4r - 4} \right) \geq \log \left(1 + \frac{365}{52} \right) > 2.081 \\ &> 1.998 > \frac{(2r + 0.5A - 1.5)^2}{2r^2 - 2r + 1} \end{aligned}$$

obtained by arguments similar to those employed in the previous case. Hence, the proof is complete. $\square$

Based on the result just obtained, a short proof for a simple yet sharp lower bound for $\text{hdepth}(S_n/I_n)$ is possible.

Theorem 2.5. *For all $n \geq 1$, we have that $\text{hdepth}(S_n/I_n) \geq \lceil \frac{n}{2} \rceil + \lfloor \sqrt{n} \rfloor - 2$.*

Proof. The conclusion is trivially true for $n \leq 2$, so for the rest of the proof we assume $n \geq 3$. Let $d = \lceil \frac{n}{2} \rceil + \lfloor \sqrt{n} \rfloor - 2$.

If $n = 2k + 1$, then $d = k + \lfloor \sqrt{2k + 1} \rfloor - 1$. Therefore

$$\begin{aligned} \binom{n - d + 2s - 1}{2s} &= \binom{k - \lfloor \sqrt{2k + 1} \rfloor + 2s + 1}{2s} \\ \text{and } \binom{d - 1}{2s - 1} &= \binom{k + \lfloor \sqrt{2k + 1} \rfloor - 2}{2s - 1}. \end{aligned} \quad (10)$$

Also, if $n = 2k$, $d = k + \lfloor \sqrt{2k} \rfloor - 2$. Therefore

$$\begin{aligned} \binom{n - d + 2s - 1}{2s} &= \binom{k - \lfloor \sqrt{2k} \rfloor + 2s + 1}{2s} \\ \text{and } \binom{d - 1}{2s - 1} &= \binom{k + \lfloor \sqrt{2k} \rfloor - 3}{2s - 1}. \end{aligned} \quad (11)$$

Since $\lfloor \sqrt{2k+1} \rfloor \geq \lfloor \sqrt{2k} \rfloor$, in order to prove the inequality

$$\binom{n-d+2s-1}{2s} \geq \binom{d-1}{2s-1} \text{ for } d = \left\lceil \frac{n}{2} \right\rceil + \lfloor \sqrt{n} \rfloor - 2 \text{ and } 1 \leq s \leq \left\lfloor \frac{d}{2} \right\rfloor, \quad (12)$$

it is enough to consider the case $n = 2k + 1$ and to show that

$$\binom{k - \lfloor \sqrt{2k+1} \rfloor + 2s + 1}{2s} \geq \binom{k + \lfloor \sqrt{2k+1} \rfloor - 2}{2s-1}, \quad (13)$$

for all $1 \leq s \leq \left\lfloor \frac{k + \lfloor \sqrt{2k+1} \rfloor - 2}{2} \right\rfloor$.

Obviously, (13) holds for $s \geq \lfloor \sqrt{2k+1} \rfloor$. From Theorem 2.4 and the above discussions, it follows that equation (12) holds. Hence, the conclusion follows from Proposition 2.3. $\square$

We can bound from above $\text{hdepth}(S_n/I_n)$ by an expression of a different nature than that given in the previous theorem.

Theorem 2.6. *For any $\frac{1}{2} > \varepsilon > 0$ define the positive integer $A = A(\varepsilon)$ by*

$$A = \left\lceil \frac{3B+6}{8\varepsilon} - \frac{B+1}{2} \right\rceil,$$

where $B = \log\left(\frac{2\varepsilon+1}{4\varepsilon}\right)$. Then

$$\text{hdepth}(S_n/I_n) \leq \left\lceil \frac{n}{2} \right\rceil + \lfloor \varepsilon n \rfloor + A - 2, \text{ for all } n \geq 2.$$

Proof. Let $\varepsilon > 0$. Having in view Proposition 2.3, the conclusion of the theorem is equivalent to the fact that there exists some nonnegative integer A such that, for any $n \geq 2$ and $d = \left\lceil \frac{n}{2} \right\rceil + \lfloor \varepsilon n \rfloor + A - 1$, there exists some $1 \leq s \leq \left\lfloor \frac{d}{2} \right\rfloor$ for which

$$\binom{n-d+2s-1}{2s} < \binom{d-1}{2s-1}$$

or equivalently

$$\binom{\left\lfloor \frac{n}{2} \right\rfloor - \lfloor \varepsilon n \rfloor - A + 2s}{2s} < \binom{\left\lceil \frac{n}{2} \right\rceil + \lfloor \varepsilon n \rfloor + A - 2}{2s-1}. \quad (14)$$

If $n = 2k$, then $d = k + \lfloor 2k\varepsilon \rfloor + A - 1$ and (14) becomes

$$\binom{k - \lfloor 2k\varepsilon \rfloor - A + 2s}{2s} < \binom{k + \lfloor 2k\varepsilon \rfloor + A - 2}{2s-1}, \quad (15)$$

for some $1 \leq s \leq \left\lfloor \frac{k + \lfloor 2k\varepsilon \rfloor + A - 1}{2} \right\rfloor$.

If $n = 2k + 1$, then $d = k + \lfloor (2k + 1)\varepsilon \rfloor + A$ and (14) is equivalent to

$$\binom{k - \lfloor (2k + 1)\varepsilon \rfloor - A + 2s}{2s} < \binom{k + \lfloor (2k + 1)\varepsilon \rfloor + A - 1}{2s - 1}, \quad (16)$$

for some $1 \leq s \leq \left\lfloor \frac{k + \lfloor (2k + 1)\varepsilon \rfloor + A}{2} \right\rfloor$. Since $\binom{k + \lfloor (2k + 1)\varepsilon \rfloor + A - 1}{2s - 1} > \binom{k + \lfloor 2k\varepsilon \rfloor + A - 2}{2s - 1}$ and $\binom{k - \lfloor (2k + 1)\varepsilon \rfloor - A + 2s}{2s} \leq \binom{k - \lfloor 2k\varepsilon \rfloor - A + 2s}{2s}$, while

$$\left\lfloor \frac{k + \lfloor 2k\varepsilon \rfloor + A - 1}{2} \right\rfloor \leq \left\lfloor \frac{k + \lfloor (2k + 1)\varepsilon \rfloor + A}{2} \right\rfloor,$$

it is enough to consider only the case $n = 2k$.

Let $A \geq 0$ and $x = k - \lfloor 2k\varepsilon \rfloor - A$. Without any loss of generality, we may assume $x \geq 0$, otherwise (15) is trivially true. Note that (15) is equivalent to

$$(x + 1)(x + 2) \cdots (x + 2s) < 2s(x + 2 \lfloor 2k\varepsilon \rfloor + 2A - 2s) \cdots (x + 2 \lfloor 2k\varepsilon \rfloor + 2A - 2),$$

as well as to

$$\log \left(1 + \frac{x}{2s} \right) < \sum_{j=1}^{2s-1} \log \left(1 + \frac{2 \lfloor 2k\varepsilon \rfloor + 2A - 2s - 1}{x + j} \right).$$

We claim that $s = \lfloor 2k\varepsilon \rfloor$ does the job. Explicitly, this means there exists $A \geq 0$ which depends only on ε such that, for all $k \geq 1$ with $k - \lfloor 2k\varepsilon \rfloor \geq A$, we have

$$\log \left(1 + \frac{k - \lfloor 2k\varepsilon \rfloor - A}{2 \lfloor 2k\varepsilon \rfloor} \right) < \sum_{j=1}^{2 \lfloor 2k\varepsilon \rfloor - 1} \log \left(1 + \frac{2A - 1}{k - \lfloor 2k\varepsilon \rfloor - A + j} \right). \quad (17)$$

Since for $A \geq 1/(2\varepsilon)$ it holds

$$1 + \frac{k - \lfloor 2k\varepsilon \rfloor - A}{2 \lfloor 2k\varepsilon \rfloor} = 1 + \frac{k - A}{2 \lfloor 2k\varepsilon \rfloor} - \frac{1}{2} < \frac{1}{2} + \frac{1}{4\varepsilon} = \frac{2\varepsilon + 1}{4\varepsilon},$$

in order to prove (17) it is enough to find $A \geq 1/(2\varepsilon)$ such that, for all $k \geq 1$ with $k - \lfloor 2k\varepsilon \rfloor \geq A$, we have

$$\log \left(\frac{2\varepsilon + 1}{4\varepsilon} \right) < \sum_{j=1}^{2 \lfloor 2k\varepsilon \rfloor - 1} \log \left(1 + \frac{2A - 1}{k - \lfloor 2k\varepsilon \rfloor + j} \right). \quad (18)$$

Let $B := \log\left(\frac{2\varepsilon+1}{4\varepsilon}\right)$. Since $\log(1+y) > \frac{y}{y+1}$ for $y > 0$, in order to prove (18) it is enough to find $A \geq 1/(2\varepsilon)$ such that, for all $k \geq 1$ with $k - \lfloor 2k\varepsilon \rfloor \geq A$, we have

$$\sum_{j=1}^{2\lfloor 2k\varepsilon \rfloor - 1} \frac{2A - 1}{k - \lfloor 2k\varepsilon \rfloor + j + (2A - 1)} > B. \quad (19)$$

Observe that for the left-hand side we have

$$\begin{aligned} \sum_{j=1}^{2\lfloor 2k\varepsilon \rfloor - 1} \frac{2A - 1}{k - \lfloor 2k\varepsilon \rfloor + j + (2A - 1)} &= (2A - 1) \sum_{\ell=1-\lfloor 2k\varepsilon \rfloor}^{\lfloor 2k\varepsilon \rfloor - 1} \frac{1}{2A + k - 1 - \ell} \\ &> \frac{(2A - 1)(2\lfloor 2k\varepsilon \rfloor - 1)}{2A + k - 1} > \frac{(2A - 1)(4\varepsilon k - 3)}{2A + k - 1}. \end{aligned}$$

Hence, it suffices to find $A \geq 1/(2\varepsilon)$ such that, for all $k \geq 1$ with $k - \lfloor 2k\varepsilon \rfloor \geq A$, we have

$$(2A - 1)(4\varepsilon k - 3) > (2A + k - 1)B,$$

which is equivalent to

$$(2A - 1) \left(4\varepsilon - \frac{B + 3}{k} \right) > B. \quad (20)$$

As $k \geq \lfloor 2k\varepsilon \rfloor + A > A + 2k\varepsilon - 1$, we have

$$(2A - 1) \left(4\varepsilon - \frac{B + 3}{k} \right) > (2A - 1) \left(4\varepsilon - \frac{(B + 3)(1 - 2\varepsilon)}{A - 1} \right).$$

Hence, in order to complete the proof it suffices to find $A \geq 1/(2\varepsilon)$ such that, for all $k \geq 1$ with $k - \lfloor 2k\varepsilon \rfloor \geq A$, we have

$$(2A - 1) \left(4\varepsilon - \frac{(B + 3)(1 - 2\varepsilon)}{A - 1} \right) > B. \quad (21)$$

We denote $y := 2A - 1$ and $N := (B + 3)(1 - 2\varepsilon)$. We have $A - 1 = \frac{y-1}{2}$ and thus (21) is equivalent to

$$4\varepsilon y - \frac{2yN}{y-1} > B,$$

which is also equivalent to

$$f(y) := 4\varepsilon y^2 - (4\varepsilon + 2N + B)y + B > 0. \quad (22)$$

Since the discriminant of $f(y)$ is

$$(4\varepsilon + 2N + B)^2 - 16\varepsilon B = (4\varepsilon - B)^2 + 4N^2 + 16\varepsilon N + 4NB > 0,$$

it follows that $f(y) > 0$ for

$$y > \frac{4\varepsilon + 2N + B + \sqrt{(4\varepsilon + 2N + B)^2 - 16\varepsilon B}}{8\varepsilon}.$$

Henceforth, we have $f(y) > 0$ for

$$y > \frac{4\varepsilon + 2N + B}{4\varepsilon}.$$

Since $A = \frac{y+1}{2}$, it follows that $f(y) > 0$ for

$$A > \frac{8\varepsilon + 2N + B}{8\varepsilon} = \frac{3B + 6 - 4B\varepsilon - 4\varepsilon}{8\varepsilon} = \frac{3B + 6}{8\varepsilon} - \frac{B + 1}{2}.$$

Since this lower bound on A is greater than $1/(2\varepsilon)$, the proof is complete. $\square$

Although the upper bound given in Theorem 2.6 is weaker than the lower bound provided in Theorem 2.5, it is sufficiently strong to allow us to conclude that on long run $\text{hdepth}(S_n/I_n)/n$ approaches $1/2$. Note that a similar result

$$\lim_{n \rightarrow \infty} \frac{\text{hdepth}(I_n)}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} \left\lfloor \frac{n+3}{2} \right\rfloor = \frac{1}{2}$$

can be inferred from Proposition 2.1.

Corollary 2.7. *We have that*

$$\lim_{n \rightarrow \infty} \frac{\text{hdepth}(S_n/I_n)}{n} = \frac{1}{2}.$$

Proof. From Theorem 2.5 it follows that

$$\frac{\text{hdepth}(S_n/I_n)}{n} \geq \frac{1}{2} + \frac{\sqrt{n}-3}{n} \text{ for all } n \geq 1. \quad (23)$$

On the other hand, according to Theorem 2.6 it follows that for any $\varepsilon > 0$ there exists some $A(\varepsilon) \geq 0$ such that

$$\frac{\text{hdepth}(S_n/I_n)}{n} \leq \frac{1}{2} + \varepsilon + \frac{A(\varepsilon)-1}{n} \text{ for all } n \geq 1. \quad (24)$$

Now, let δ be positive and less than 1 , and put $\varepsilon := \frac{\delta}{2}$. Then, there exists n_δ such that for all $n \geq n_\delta$ we have $\frac{\sqrt{n}-3}{n} < \delta$ and $\frac{A(\varepsilon)-1}{n} < \varepsilon$. From (23) and (24) it follows that

$$\left| \frac{\text{hdepth}(S_n/I_n)}{n} - \frac{1}{2} \right| < \delta, \text{ for all } n \geq n_\delta.$$

Thus, we get the required conclusion. $\square$

3 Computer experiments

For all $n \geq 1$ we consider:

$$\alpha(n) = \text{hdepth}(S_n/I_n) \text{ and } \beta(n) = \left\lceil \frac{n}{2} \right\rceil + \lfloor \sqrt{n} \rfloor - 2.$$

According to our computer experiments, for $1 \leq n \leq 400$, we have:

$$\begin{aligned} \alpha(n) - \beta(n) &\in \{0, 1\}, \text{ for all } 1 \leq n \leq 160, \\ \alpha(n) - \beta(n) &\in \{0, 1, 2\}, \text{ for all } 161 \leq n \leq 310, \\ \alpha(n) - \beta(n) &\in \{1, 2, 3\}, \text{ for all } 311 \leq n \leq 400. \end{aligned}$$

This shows that the lower bound given by Theorem 2.5 is quite sharp.

Example 3.1. Let $\varepsilon = \frac{1}{6}$. With formula given in Theorem 2.6, we get

$$A = \left\lceil \frac{7 \log 2 + 16}{4} \right\rceil = 6,$$

so we have

$$\text{hdepth}(S_n/I_n) \leq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{6} \right\rfloor + 4, \text{ for all } n \geq 2.$$

Note that, if we consider Eq. (20), we need to find $A \geq 3$ such that

$$(2A - 1) \left(1 - \frac{3 + \log 2}{A - 1} \right) > \frac{3}{2} \log(2).$$

The smallest value of A which satisfies this inequality is indeed 6.

However, this value of the constant A is not optimal. For instance, according to our computer experiments, we believe that

$$\text{hdepth}(S_n/I_n) \leq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{6} \right\rfloor, \text{ for all } n \geq 2.$$

So, it seems, the smallest possible value for A is 2.

Establishing such an upper bound or improving formula for A given in Theorem 2.6 for arbitrary ε would be an interesting subject for future work.

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