



Another inequality inspired by Erdős

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Abstract

One less common inequality in positive integers is carefully examined.

In 1845, J. Bertrand ([2], pg. 129) conjectured such a statement: "For every integer n greater than 6 there always exists at least one prime number contained between $n-2$ and $\frac{n}{2}$." Bertrand himself noticed that his proposition (postulate) was true for all n such that $7 \leq n < 6 \cdot 10^6$ and believed it remained true for all $n \geq 7$. It was P. Erdős in 1932 ([3]) who came up with an elegant and simple proof based on prime decompositions of binomial coefficients. The approach of Erdős may be considered the standard one nowadays. Two remarks could turn out to be of some interest, however. Firstly, a formally weaker form of the postulate is shown in [3]. Namely, that for every positive integer n there is at least one prime p with $n < p \leq 2n$. Of course, all odd primes p are of the form $p = 2n - 1$, $n \geq 2$, whence it is not immediately visible why the lenient form should imply the more severe one. On the other hand, the original proof by Erdős can easily be completed to do the work. Secondly, the proof can be definitely viewed as elementary, but no way calculus-free. On the other hand, if some assertion is formulated within elementary arithmetic, it would be nice to have a really elementary proof. Nonetheless, we are convinced that the methods developed by Erdős can be transmogrified to yield demonstration(s) of the Bertrand postulate if not inside then at least close to Elementary Function Arithmetic – one of the weakest fragments of arithmetic. In our effort to find such a proof we came across several unusual

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inequalities in positive integers. One of them (the inequality (1) below) was carefully examined in [1]. Another kindred inequality (the inequality (2)) is treated in the present short note. Purely from methodological and didactical reasons, we include also a solution of these two inequalities using elementary calculus.

1 Preliminaries

In the sequel, n stands for a positive integer. We denote by

- (a) $z(n)$ the greatest integer such that $3z(n) < 2n$;
- (b) $m(n)$ the greatest integer such that $m(n)^2 \leq 2n$;
- (c) $r(n)$ the smallest non-negative integer such that $n \leq 2^{r(n)}$;
- (d) $x(n) = z(n) - (r(n) + 1)m(n)$;
- (e) $y(n) = 2^{2n-2z(n)-m(n)+2} - n^{m(n)-1}$;
- (f) $c(n) = 2n - 2z(n) + 2$;
- (g) $a(n) = 2^{c(n)-m(n)}$;
- (h) $b(n) = n^{m(n)-1}$.

In [1], we examined the inequality

$$z(n) - (r(n) + 1)m(n) < 0. \quad (1)$$

i.e., the inequality $x(n) < 0$) and we proved the following result (and a bit more):

Theorem 1.1. *Let n be a positive integer. Then $x(n) = 0$ if and only if $n \in \{436, 451, 529, 545, 546\}$ and $x(n) < 0$ if and only if either $n \leq 435$ or $n = 450$ or $513 \leq n \leq 528$.*

Now, our aim is to solve (in positive integers) the inequality

$$2^{2n-2z(n)-m(n)+2} - n^{m(n)-1} < 0 \quad (2)$$

and to find all positive integers n for which the equality holds. In other words, we want to determine the sign of $y(n)$. The answer is given in Theorem 2.7.

Exercise 1.2. it is not complicated to check the following handy table (without using any artificial means):

n	1	2	3	4	5	6	7	8	9
$x(n)$	-1	-3	-5	-4	-9	-9	-8	-11	-15
$c(n) - m(n)$	3	2	4	4	3	5	5	7	6
$y(n)$	7	2	13	12	-17	-4	-17	-496	-665

n	10	11	12	13	14	15
$x(n)$	-14	-13	-13	-17	-16	-21
$c(n) - m(n)$	6	6	8	7	7	9
$y(n)$	-936	-1267	-1472	-28433	-38288	-50113

In order to justify the definition of $y(n)$ (see (e)), we need the following:

Lemma 1.3. $c(n) - m(n) \geq 2$ for every positive integer n and $c(n) - m(n) = 2$ only for $n = 2$.

Proof. From the table in Exercise 1.2, we can assume that $n \geq 10$. Then $2n \geq 20$, $z(n) \geq 6$, $2z(n) \geq 12$, $2z(n)^2 \geq 12z(n) \geq 10z(n) + 12 > 10z(n) + 8$, $(z(n) - 1)^2 = z(n)^2 - 2z(n) + 1 = \frac{2z(n)^2 - 4z(n) + 2}{2} > \frac{10z(n) + 8 - 4z(n) + 2}{2} = 3z(n) + 5 > 3(z(n) + 1) \geq 2n$. Now, $(z(n) - 1)^2 > 2n$ implies $z(n) - 1 > m(n)$ and $z(n) - m(n) - 1 \geq 1$. On the other hand, $2n - 3z(n) \geq 1$ and, consequently, $c(n) - m(n) - 3 = 2n - 2z(n) - m(n) - 1 = (2n - 3z(n)) + (z(n) - m(n) - 1) \geq 2$. Thus $c(n) - m(n) \geq 5$. $\square$

One observes readily that the sequences $z(n)$, $m(n)$, $r(n)$ are non-decreasing, the sequence $b(n)$ is increasing and $y(n) = a(n) - b(n)$. Further, $c(1) = c(2) = 4$, $c(3k) = c(3k+1) = c(3k+2) = 2k+4$ for every $k \geq 2$, $c(n+3k) = c(n) + 2k$ and the sequence $c(n)$ is non-decreasing.

2 A purely arithmetic solution to (2)

Observation 2.1. In order to facilitate the calculation of $x(n)$, $y(n)$, it is useful to distinguish intervals where the values of $m(n)$ and $r(n)$ are constant (and thus the values of $x(n)$ are non-decreasing).

(i) Let a be a positive integer with $r(a) = s$ and $m(a) = t$, p be the largest integer such that $2p < (t+1)^2$ and $f(a) = \min(p, 2^{s+1} - 1)$. Then $f(a)$ is the largest positive integer k such that $r(k) = s$, $m(k) = t$ and if n_1, n_2 are positive integers such that $a \leq n_1 \leq n_2 \leq b(a)$ then $x(a) \leq x(n_1) = z(n_1) - (r(a) + 1)m(a) \leq x(n_2) \leq x(b(a))$ (indeed, if $n > f(a)$ then either $n \geq 2^{s+1}$ and $r(n) \geq s+1$, or $n > p$ and $2p \geq (t+1)^2$, and hence $m(n) \geq t+1$ and the rest follows from the fact that the sequence $z(n)$ is non-decreasing).

(ii) We have $f(2^k) = 2^k$ for every $k \geq 0$.

Construction 2.2. Put $g(1) = 1$ ($= f(1)$) and $g(i+1) = f(g(i)) + 1$ for each $i \geq 1$. Clearly, $g(i) \leq n \leq f(g(i))$, $i = 1, 2, \dots$, are maximal intervals where $r(n), m(n)$ are constant and if $g(i) \leq n \leq f(g(i))$ then $x(n) = z(n) - (r(g(i)) + 1)m(g(i))$.

The first 41 intervals $g(i) \leq n \leq f(g(i))$ are presented in the following list (the necessary calculations, although a bit fatigant, are extremely easy):

1. If $n = 1$ then $r(n) = 0$, $m(n) = 1$, $x(n) = -1$.
2. If $n = 2$ then $r(n) = 1$, $m(n) = 2$, $x(n) = -3$.
3. If $3 \leq n \leq 4$ then $r(n) = 2$, $m(n) = 2$, $-5 \leq x(n) \leq -4$.
4. If $5 \leq n \leq 7$ then $r(n) = 3$, $m(n) = 3$, $-9 \leq x(n) \leq -8$.
5. If $n = 8$ then $r(n) = 3$, $m(n) = 4$, $x(n) = -11$.
6. If $9 \leq n \leq 12$ then $r(n) = 4$, $m(n) = 4$, $-15 \leq x(n) \leq -13$.
7. If $13 \leq n \leq 16$ then $r(n) = 4$, $m(n) = 5$, $-17 \leq x(n) \leq -15$.
8. If $n = 17$ then $r(n) = 5$, $m(n) = 5$, $x(n) = -19$.
9. If $18 \leq n \leq 24$ then $r(n) = 5$, $m(n) = 6$, $-25 \leq x(n) \leq -21$.
10. If $25 \leq n \leq 31$ then $r(n) = 5$, $m(n) = 7$, $-26 \leq x(n) \leq -22$.
11. If $n = 32$ then $r(n) = 5$, $m(n) = 8$, $x(n) = -27$.
12. If $33 \leq n \leq 40$ then $r(n) = 6$, $m(n) = 8$, $-35 \leq x(n) \leq -30$.
13. If $41 \leq n \leq 49$ then $r(n) = 6$, $m(n) = 9$, $-36 \leq x(n) \leq -31$.
14. If $50 \leq n \leq 60$ then $r(n) = 6$, $m(n) = 10$, $-37 \leq x(n) \leq -31$.
15. If $61 \leq n \leq 64$ then $r(n) = 6$, $m(n) = 11$, $-37 \leq x(n) \leq -35$.
16. If $65 \leq n \leq 71$ then $r(n) = 7$, $m(n) = 11$, $-45 \leq x(n) \leq -41$.
17. If $72 \leq n \leq 84$ then $r(n) = 7$, $m(n) = 12$, $-49 \leq x(n) \leq -41$.
18. If $85 \leq n \leq 97$ then $r(n) = 7$, $m(n) = 13$, $-48 \leq x(n) \leq -40$.
19. If $98 \leq n \leq 112$ then $r(n) = 7$, $m(n) = 14$, $-47 \leq x(n) \leq -38$.
20. If $113 \leq n \leq 127$ then $r(n) = 7$, $m(n) = 15$, $-45 \leq x(n) \leq -36$.
21. If $n = 128$ then $r(n) = 7$, $m(n) = 16$, $x(n) = -43$.
22. If $129 \leq n \leq 144$ then $r(n) = 8$, $m(n) = 16$, $-59 \leq x(n) \leq -49$.
25. If $181 \leq n \leq 199$ then $r(n) = 8$, $m(n) = 19$, $-51 \leq x(n) \leq -39$.
26. If $200 \leq n \leq 220$ then $r(n) = 8$, $m(n) = 20$, $-47 \leq x(n) \leq -34$.
27. If $221 \leq n \leq 241$ then $r(n) = 8$, $m(n) = 21$, $-42 \leq x(n) \leq -29$.
28. If $242 \leq n \leq 256$ then $r(n) = 8$, $m(n) = 22$, $-37 \leq x(n) \leq -28$.
29. If $257 \leq n \leq 264$ then $r(n) = 9$, $m(n) = 22$, $-49 \leq x(n) \leq -45$.
30. If $265 \leq n \leq 287$ then $r(n) = 9$, $m(n) = 23$, $-54 \leq x(n) \leq -39$.
31. If $288 \leq n \leq 312$ then $r(n) = 9$, $m(n) = 24$, $-49 \leq x(n) \leq -33$.
32. If $313 \leq n \leq 337$ then $r(n) = 9$, $m(n) = 25$, $-42 \leq x(n) \leq -26$.
33. If $338 \leq n \leq 364$ then $r(n) = 9$, $m(n) = 26$, $-35 \leq x(n) \leq -18$.
34. If $365 \leq n \leq 391$ then $r(n) = 9$, $m(n) = 27$, $-27 \leq x(n) \leq -10$.
35. If $392 \leq n \leq 420$ then $r(n) = 9$, $m(n) = 28$, $-19 \leq x(n) \leq -1$.
36. If $421 \leq n \leq 449$ then $r(n) = 9$, $m(n) = 29$, $-10 \leq x(n) \leq 9$.
37. If $450 \leq n \leq 480$ then $r(n) = 9$, $m(n) = 30$, $-1 \leq x(n) \leq 19$.

38. If $481 \leq n \leq 511$ then $r(n) = 9$, $m(n) = 31$, $10 \leq x(n) \leq 30$.
 39. If $n = 512$ then $r(n) = 9$, $m(n) = 32$, $x(512) = 21$.
 40. If $513 \leq n \leq 544$ then $r(n) = 10$, $m(n) = 32$, $-11 \leq x(n) \leq 10$.
 41. If $545 \leq n \leq 577$ then $r(n) = 10$, $m(n) = 32$, $0 \leq x(n) \leq 21$

Taking into account Theorem 1.1, we see that $x(129) = -59$ and $x(n) \geq -58$ for $n \neq 129$.

Lemma 2.3. *Let u, v, n be positive integers such that $u \leq n \leq v$ and $m(u) = m(v) = t$.*

- (i) $a(u) - b(v) \leq y(n) = a(n) - b(n) = 2^{c(n)-t} - n^{t-1} \leq a(v) - b(u)$.
- (ii) *If $b(u) = u^{t-1} > 2^{c(v)-t} = a(v)$ then $y(n) < 0$.*
- (iii) *If $b(v) = v^{t-1} < 2^{c(u)-t} = a(u)$ then $y(n) > 0$.*
- (iv) *If $n+1 \leq v$ and $c(n) = c(n+1)$ then $y(n+1) < y(n)$.*

Proof. It follows from the fact that the sequence $c(n)$ is non-decreasing and the sequence $b(n)$ is increasing. $\square$

Lemma 2.4. *Let n be a positive integer. Put $r(n) = s$ and $m(n) = t$. Then:*

- (i) *If $c(n) \leq s(t-1) + 1$ then $y(n) < 0$.*
- (ii) *If $c(n) > s(t-1) + t$ then $y(n) > 0$.*

Proof. Since $r(n) = s$, we have $2^{s-1} < n \leq 2^s$. If $c(n) \leq st - s + 1$ then $c(n) - t \leq (s-1)(t-1)$, hence $2^{c(n)-m(n)} \leq (2^{s-1})^{t-1} < n^{t-1}$ and $y(n) < 0$. If $c(n) > st - s + t$ then $s(t-1) < c(n) - t$, and therefore $0 \leq (2^s)^{t-1} - n^{t-1} < 2^{c(n)-t} - n^{t-1} = y(n)$. $\square$

Lemma 2.5. *If $n \geq 1$ is such that $y(n) \leq 0$ then $x(n) \leq -r(n) - 3 \leq -6$.*

Proof. In view of 1.2, we have $n \geq 5$. As $3z(n) \leq 2n - 1$, we have $z(n) - m(n) + 3 \leq z(n) - m(n) + 2 + 2n - 3z(n) = c(n) - m(n)$. However $y(n) \leq 0$, hence $2^{c(n)-m(n)} \leq n^{m(n)-1} \leq 2^{r(n)(m(n)-1)}$ and $z(n) - m(n) + 3 \leq c(n) - m(n) \leq r(n)m(n) - r(n)$. Then $x(n) = z(n) - r(n)m(n) - m(n) \leq -r(n) - 3 \leq -6$. $\square$

Proposition 2.6. *$y(n) > 0$ for every $n \geq 404$.*

Proof. Assume $n \geq 404$. Then $r(n) \geq 9$, $-r(n) - 3 \leq -12 < -11$. On the other hand, taking into account Theorem 1.1, Construction 2.2 and the fact that $x(404) = -11$, we have $-11 \leq x(n)$. Thus $-r(n) - 3 < x(n)$ and $y(n) > 0$ by Lemma 2.5. $\square$

Theorem 2.7. (i) *$y(n) < 0$ if and only if either $5 \leq n \leq 335$ or $338 \leq n \leq 350$ or $365 \leq n \leq 368$.*

(ii) *$y(n) \neq 0$ for every positive integer n .*

(iii) *$y(n) > 0$ if and only if either $1 \leq n \leq 4$ or $336 \leq n \leq 337$ or $351 \leq n \leq 364$ or $n \geq 369$.*

Proof. With respect to Exercise 1.2 and Proposition 2.6, we can restrict ourselves to $17 \leq n \leq 403$. We will make use of Construction 2.2 and compare the powers of 2 with those of n (the necessary calculations are non-demanding in principle, nonetheless they might turn out to be somewhat tiresome occasionally – a calculator and/or a homemade table of powers of 2 will help a lot). The proof will be divided into several parts.

(i) From Lemma 2.4(i) and the table in Construction 2.2, we see that $y(n) < 0$ for $n \leq 220$ (i.e., for $i = 1, \dots, 26$).

(ii) Let $221 \leq n \leq 241$. Then $r(n) = 8$, $m(n) = 21$ and $c(241) = 164$. As $221^3 > 2^{23}$, we have $221^{20} = (221^3)^6 \cdot 221^2 > 221^{138} \cdot 221^2 > 2^{153}$ and $y(n) < 0$ for $221 \leq n \leq 241$ by Lemma 2.3(ii).

(iii) Let $242 \leq n \leq 264$. Then $m(n) = 22$, $c(264) = 180$ and $242^4 > 2^{31}$. Thus $242^{21} = (242^4)^5 \cdot 242 > 2^{31 \cdot 5} \cdot 242 > 2^{158}$ and $y(n) < 0$ by Lemma 2.3(ii).

(iv) Let $265 \leq n \leq 287$. Then $m(n) = 23$, $c(287) = 194$, $265^{22} > 256^{22} = (2^8)^{22} = 2^{176} > 2^{194-23}$ and $y(n) < 0$ by Lemma 2.3(ii).

(v) Let $288 \leq n \leq 312$. Then $r(n) = 9$, $m(n) = 24$ and $c(312) = 212$. As $c(308) = c(306) = 208 \leq 9 \cdot 23 + 1$, $y(n) < 0$ for $288 \leq n \leq 308$ by Lemma 2.4(i). Further, $309 > 2^8$ and $309^4 > 2^{33}$. Hence $309^{23} = (309^4)^5 \cdot 309^3 > 2^{33 \cdot 5} \cdot 2^{8 \cdot 3} > 2^{188}$ and $y(n) < 0$ for $309 \leq n \leq 312$ by Lemma 2.3(ii).

(vi) Let $313 \leq n \leq 337$. Then $r(n) = 9$, $m(n) = 25$ and $c(337) = 228$. As $c(320) = 216 \leq 9 \cdot 24 + 1$, $y(n) < 0$ for $313 \leq n \leq 320$ by Lemma 2.4(i). Further, $c(329) = 222$ and (see (v)) $321^4 > 2^{33}$. Thus $320^{24} > 2^{33 \cdot 6}$ and $y(n) < 0$ for $320 \leq n \leq 329$ by Lemma 2.3(ii). Moreover, $c(332) = 224$ and $330^3 > 2^{25}$. Hence $330^{24} > 2^{200}$ and $y(n) < 0$ for $330 \leq n \leq 332$ by Lemma 2.3(ii). Further, $333^8 > 1.5 \cdot 10^{20} > 1.48 \cdot 10^{20} > 2^{67}$ (we have $2^{67} = 147, 573, 952, 589, 676, 412, 928$), and hence $333^{24} > 2^{67 \cdot 3} = 2^{201}$. As $c(333) = 226$, $y(333) < 0$ and $y(n) < 0$ for $333 \leq n \leq 335$ by Lemma 2.3(iv). Finally, we have $m(337) = 25$, $c(337) = c(336) = 228$, $337^2 < 2^{17}$ and $337^5 < 2^{42}$. Thus $337^{24} < 2^{42 \cdot 4} \cdot 2^{17 \cdot 2} = 2^{202}$ and $y(337) > 0$. Then $y(336) > 0$ by Lemma 2.3(iv).

(vii) Let $338 \leq n \leq 364$. Then $r(n) = 9$ and $m(n) = 26$. As $c(350) = 236$ and $338^5 > 2^{42}$, we have $338^{25} > 2^{42 \cdot 5}$ and $y(n) > 0$ for $338 \leq n \leq 350$ by Lemma 2.3(ii). As $353^2 < 2^{17}$ and $353^{15} < 1.65 \cdot 10^{38} < 1.7 \cdot 10^{38} < 2^{127}$ (we have $2^{127} = 170, 141, 183, 460, 469, 231, 731, 687, 303, 715, 884, 105, 728$), $353^{25} = 353^{15} \cdot (353^2)^5 > 2^{127} \cdot 2^{17 \cdot 5} = 2^{212}$, hence $y(353) > 0$ and $y(n) > 0$ for $351 \leq n \leq 353$ by Lemma 2.3(iv). Further, $356^2 < 2^{17}$, hence $356^{25} < 2^{17 \cdot 12} \cdot 356 < 2^{213}$. As $c(354) = 240$, $y(n) > 0$ for $354 \leq n \leq 356$ by Lemma 2.3(iii). However, $364^5 < 2^{43}$, and hence $364^{25} < 2^{215}$. As $c(357) = 242$, $y(n) > 0$ for $357 \leq n \leq 364$ by Lemma 2.3(iii).

(viii) Let $365 \leq n \leq 391$. Then $r(n) = 9$ and $m(n) = 27$. We have $c(368) = 248$ and $365^2 > 2^{17}$. Thus $365^{26} > 2^{17 \cdot 13} = 2^{221}$ and $y(n) < 0$

for $365 \leq n \leq 368$ by Lemma 2.3(ii). Further, as $371^3 < 2^{26}$, $371^5 < 2^{43}$ and $371^9 < 1.4 \cdot 10^{23} < 1.5 \cdot 10^{23} < 2^{77} = 151, 115, 727, 451, 828, 646, 838, 272$, we have $371^{26} = (371^9)^2 \cdot 371^5 \cdot 371^3 < 2^{77 \cdot 2 + 43 + 26} = 2^{223}$, hence $y(371) > 0$ and $y(n) > 0$ for $369 \leq n \leq 371$ by Lemma 2.3(iv). Further, $386^5 < 2^{43}$, and hence $386^{26} < 2^{43 \cdot 5} \cdot 386 < 2^{224}$. As $c(372) = 242$, we have $y(n) > 0$ for $372 \leq n \leq 386$ by Lemma 2.3(iii). Finally, we have $c(387) = 262 > 9 \cdot 26 + 27$, $y(n) > 0$ for $387 \leq n \leq 391$ by Lemma 2.4(ii).

(ix) Let $392 \leq n \leq 403$. Then $r(n) = 9$ and $m(n) = 28$. As $c(402) = c(403) = c(404)$ and $y(404) > 0$, $y(n) > 0$ for $n = 402, 403$ by Lemma 2.3(iv). As $c(392) = 264$ and $402^3 < 2^{26}$, we have $402^{27} < 2^{26 \cdot 9} < 2^{264 - 28}$ and $y(n) > 0$ for $392 \leq n \leq 402$ by Lemma 2.3(iii). $\square$

Remark 2.8. (i) $z(368) = 245 = z(369)$, $m(368) = 27 = m(369)$, $r(368) = 9 = r(369)$.

(ii) $x(368) = -25 = x(369)$.

(iii) $y(368) = 2^{221} - 368^{26} = 2^{104}(2^{117} - 23^{26})$, $167 \cdot 10^{33} < 2^{117}$, $23^{26} < 254 \cdot 10^{33}$, $23^{26} - 2^{117} < 87 \cdot 10^{33}$, $2^{104} < 2 \cdot 10^{31}$, $y(368) < -174 \cdot 10^{64}$.

(iv) $y(369) = 2^{223} - 369^{26} = 2^{223} - 3^{52} \cdot 41^{26}$, $3^{52} < 647 \cdot 10^{22}$, $41^{26} < 856 \cdot 10^{39}$, $3^{52} \cdot 41^{26} < 553832 \cdot 10^{61}$, $2^{223} > 1347998 \cdot 10^{61}$, $y(369) > 794165 \cdot 10^{61} > 794 \cdot 10^{64}$.

(v) The difference between neighbouring values of $y(n)$ can be extremely large: $y(369) - y(368) > 968 \cdot 10^{64}$.

3 A solution of (1) and (2) using elementary calculus

Remark 3.1. Define a function

$$F(x) = \frac{2}{3}x + A - B\sqrt{2x} + C \log_2 x - \sqrt{2x} \log_2 x,$$

where $A, B, C \in \mathbb{R}$.

It is an easy exercise to prove the following assertions.

1. $\lim_{x \rightarrow \infty} F(x) = \infty$.
2. $F'(x) = \frac{2}{3} - \frac{B}{\sqrt{2x}} + \frac{C}{x \ln 2} - \frac{\ln x}{\sqrt{2x} \ln 2} - \frac{\sqrt{2x}}{x \ln 2}$, $\lim_{x \rightarrow \infty} F'(x) = \frac{2}{3}$.
3. $F''(x) = \frac{\sqrt{x} \ln(2^B x) - 2\sqrt{2}C}{2\sqrt{2}x^2 \ln 2}$.
4. Functions F, F', F'' are continuous on $(0, \infty)$.
5. $F''(x) > 0$ for $x \in (d, \infty)$, where $d \geq 2$ and $d \geq \frac{e^{2C}}{2^B}$. Thus F is convex and F' is ascending on $[d, \infty)$, and hence there is at most one $c \in (d, \infty)$ with $F'(c) = 0$.

6. Either F is ascending on $[d, \infty)$ or there is exactly one $c \in (d, \infty)$ such that F is descending on $[d, c]$ and ascending on $[c, \infty)$ (see (2) and (5)).
7. If $F(x_0) < 0$ for some $x_0 \in [d, \infty)$ then there is exactly one $k \in (d, \infty)$ such that $F(x) < 0$ for $x \in [x_0, k]$ and $F(x) > 0$ for $x \in (k, \infty)$ (see (6) and (1)).

Lemma 3.2. *Let n be a positive integer.*

- (i) *If the inequality (1) holds then $n \leq 560$.*
- (ii) *If $n \geq 561$ then $x(n) > 0$.*

Proof. In Remark 3.1, put $A = -1$, $B = 2$ and $C = 0$. Thus we obtain a function

$$f(x) = \frac{2}{3}x - 1 - \sqrt{2x} \log_2 x - 2\sqrt{2x}.$$

We have $f(1) = \frac{2}{3} - 1 - 2\sqrt{2} < 0$ and $f''(x) - \frac{\ln 4x}{2\sqrt{2x}\sqrt{x} \ln 2} > 0$ for $x \in (1, \infty)$ (see Remark 3.1(3) for $A = -1$, $B = 2$, $C = 0$). From Remark 3.1(7) follows that there is exactly one $k \in (1, \infty)$ such that $f(x) < 0$ for $x \in [1, k]$ and $f(x) > 0$ for $x \in (k, \infty)$. As $f(560) < 0$ and $f(561) > 0$, we have $k \in (560, 561)$.

If n is a positive integer then $z(n) \geq \frac{2}{3}n - 1$, $\sqrt{2n} \geq m(n)$ and $\log_2 n + 1 > r(n)$, and hence $f(n) = \frac{2}{3}n - 1 - (\log_2 n + 2)\sqrt{2n} < z(n) - (r(n) + 1)m(n) = x(n)$. If the inequality (1) holds then $f(n) < 0$, and hence $n \leq 560$. On the other hand, if $n \geq 561$ then $0 < f(n) < x(n)$. $\square$

Lemma 3.3. *If n is a positive integer such that $n \leq 384$ then the inequality (1) holds.*

Proof. In Remark 3.1, put $A = B = C = 1$. Thus we have a function

$$g(x) = \frac{2}{3}x - (\log_2 x + 1)(\sqrt{2x} - 1) = \frac{2}{3}x - \sqrt{2x} \log_2 x + \log_2 x - \sqrt{2x} + 1.$$

Then $g(4) = \frac{8}{3} + 3 - 3\sqrt{8} < 0$ and $g''(x) = \frac{\sqrt{x} \ln 2x - 2\sqrt{2}}{2\sqrt{2}x^2 \ln 2} > 0$ for $x \in (4, \infty)$ (see Remark 3.1(3) for $A = B = C = 1$). From Remark 3.1(7) follows that there is exactly one $k \in (4, \infty)$ such that $g(x) < 0$ for $x \in [4, k]$ and $g(x) > 0$ for $x \in (k, \infty)$. However $g(384) < 0$ and $g(385) > 0$, and hence $k \in (384, 385)$.

If n is a positive integer then $z(n) < \frac{2}{3}n$, $m(n) > \sqrt{2n} - 1$ and $r(n) + 1 \geq \log_2 n + 1$. Thus $f(n) = \frac{2}{3}n - (\log_2 n + 1)(\sqrt{2n} - 1) > z(n) - (r(n) + 1)m(n) = x(n)$. If $x(n) \geq 0$ then $n \geq 4$ (see Exercise 1.2) and $g(n) > 0$. Hence $n \geq 385$. $\square$

Remark 3.4. With respect to Lemma 3.2 and Lemma 3.3, it remains to examine $x(n)$ for $385 \leq n \leq 560$. We must investigate intervals where $x(n)$

is non-decreasing, i.e., intervals $g(i) \leq n \leq g(i+1) - 1$. It is clear from Construction 2.2 that we must investigate values $x(n)$ for $34 \leq i \leq 41$. If $i = 34, 35$ then $x(n) < 0$ and if $i = 38, 39$ then $x(n) > 0$.

- (i) For $i = 36$ we have $x(n) = 0$ iff $z(n) = 290$ iff $n = 436$.
- (ii) For $i = 37$ (then $l_i = 480$) we have $x(n) = 0$ iff $z(n) = 300$ iff $n = 451$.
- (iii) For $i = 40$ we have $x(n) = 0$ iff $z(n) = 352$ iff $n = 529$.
- (vi) Finally, for $i = 41$ we have $x(n) = 0$ iff $z(n) = 363$. As this is an odd number, we obtain $n = 545$ or $n = 546$.

Now, the proof of Theorem 1.1 is finished. $\square$

Remark 3.5. For a positive integer n put

- 1. $Y(n) = 2n - 2z(n) + 2 - m(n) + (1 - m(n)) \log_2 n$;
- 2. $d(n) = -m(n) + (1 - m(n)) \log_2 n$.

Clearly, the sequence $d(n)$ is decreasing, $Y(n) = c(n) + d(n)$ and the inequality (2) holds if and only if $Y(n) < 0$. Further, it is easy to see that $Y(n) > 0$ for $n \leq 4$ and $Y(n) < 0$ for $5 \leq n \leq 10$.

Lemma 3.6. *If $Y(n) < 0$ then $5 \leq n \leq 379$.*

Proof. In Remark 3.1 put $A = 2$, $B = C = 1$. We obtain a function

$$h(x) = \frac{2}{3}x + 2 - \sqrt{2n} + \log_2 x - \sqrt{2n} \log_2 x .$$

Using Remark 3.1(3),(7) for $A = 2$, $B = 1$, $C = 1$, we see that $h(4) = \frac{8}{3} + 4 - 3\sqrt{8} < 0$, $h''(x) = \frac{\sqrt{x} \ln 2 - 2\sqrt{2}}{2\sqrt{2}x^2 \ln 2}$, $h''(x) > 0$ for $x \in (4, \infty)$ and there is exactly one $k \in (4, \infty)$ such that $h(x) < 0$ for $x \in [4, k)$ and $h(x) > 0$ for $x \in (k, \infty)$. It is easy to verify that $h(379) < 0$ and $h(380) > 0$, and hence $k \in (379, 380)$.

As $3z(n) < 2n$ and $m(n)^2 \leq 2n$, $Y(n) > \frac{2}{3}n + 2 - \sqrt{2n} + (1 - \sqrt{2n}) \log_2 n = h(n)$ for every positive integer n . However $h(n) < Y(n) < 0$, and hence $n \leq 379$. Finally, $Y(n) > 0$ for $n \leq 4$. $\square$

Lemma 3.7. *If $Y(n) \geq 0$ then either $1 \leq n \leq 4$ or $n \geq 325$.*

Proof. Suppose, on the contrary, that $Y(n) \geq 0$. In Remark 3.1 put $A = 5$, $B = 1$, $C = 2$. We obtain a function

$$l(x) = \frac{2}{3}x + 5 - \sqrt{2x} + 2 \log_2 x - \sqrt{2x} \log_2 x .$$

By Remark 3.1(3),(7) for $A = 5$, $B = 1$ and $C = 2$, we have $l(9) = 11 - \sqrt{18} + 2 \log_2 9 - \sqrt{18} \log_2 9 < 0$, $l''(x) = \frac{\sqrt{x} \ln 2x - 4\sqrt{2}}{2\sqrt{2}x^2 \ln 2}$, $l''(x) > 0$ for $x \in (9, \infty)$ and

there is exactly one $k \in (9, \infty)$ such that $l(x) < 0$ for $x \in [9, k)$ and $l(x) > 0$ for $x \in (k, \infty)$. However $l(324) < 0$, $l(325) > 0$, and hence $k \in (324, 325)$. As $3(z(n)+1) \geq 2n$ and $(m(n)+1)^2 > 2n$, we have $z(n) \geq \frac{2}{3}n-1$, $m(n) > \sqrt{2n}-1$ and $Y(n) = 2(n-z(n)) + 2 - m(n) + (1-m(n))\log_2 n < \frac{2}{3}n + 5 - \sqrt{2n} + 2\log_2 n - \sqrt{2n}\log_2 n = l(n)$ for every positive integer n . Taking into account Remark 3.5, we have $l(n) > Y(n) \geq 0$, and hence either $n \leq 4$ or $n \geq 325$. $\square$

Lemma 3.8. $Y(n) \neq 0$ for every positive integer n .

Proof. Suppose, on the contrary, that n is such that $Y(n) = 0$. Taking into account Remark 3.1 and Remark 3.7, we have $n \geq 325$ and $k(n) < 0$, hence $n \leq 379$. As $Y(n) = 2(n-z(n)) + 2 - m(n) + (1-m(n))\log_2 n = 0$, $q = \log_2 n = \frac{2(n-z(n))+2-m(n)}{m(n)-1}$ is a rational number. However $n = 2^q$ is an integer, hence q is also an integer. As $2^8 < 325 \leq n \leq 379 < 2^9$, we have a contradiction. $\square$

Remark 3.9. If $5 \leq n \leq 324$ then $Y(n) < 0$ by Remark 3.7. With respect to Remark 3.6, it remains to determine the sign of $Y(n)$ for $325 \leq n \leq 379$. We will distinguish several cases.

- (i) If $325 \leq n \leq 332$ then $c(n) \leq c(325) + 4$, and hence $Y(n) < Y(325) + 4$. However $Y(325) (\approx -5.26) < -5$ and $y(n) < 0$.
- (ii) If $333 \leq n \leq 335$ then $c(n) = c(333)$, and hence $Y(n) \geq Y(337) (\approx 1.48) > 0$.
- (iii) If $338 \leq n \leq 350$ then $c(n) \leq c(338) + 8$ and $Y(n) \leq Y(338) (\approx -8.02) + 8 < 0$.
- (iv) If $351 \leq n \leq 353$ then $c(n) = c(353)$ and $Y(n) \geq Y(353) (\approx 0.41) > 0$.
- (v) If $354 \leq n \leq 364$ then $c(n) \geq c(354) = 240$ and $d(n) \geq d(364) (\approx -238.7) > -239$, and hence $Y(n) > 1$.
- (vi) If $365 \leq n \leq 368$ then $c(n) \leq (365) + 2$ and $Y(n) \leq Y(365) (\approx -2.4) + 2 < 0$.
- (vii) If $369 \leq n \leq 371$ then $c(n) = c(369)$ and $Y(n) \geq Y(371) (\approx 1.08) > 0$.
- (viii) Finally, let $371 \leq n \leq 379$. Then $m(n) = 27$ (by the table in Construction 2.2, this is true till $n = 391$) and $d(n) = -27 - 26\log_2 n$. Define $\Delta Y(n) = Y(n+3) - Y(n)$ and $\Delta d(n) = d(n+3) - d(n)$. As $c(n+3) = c(n) + 2$, we have $\Delta Y(n) = 2 + \Delta d(n)$. If $n+3 \leq 391$ then the sequence $\Delta d(n) = 26\log_2 \frac{n}{n+3}$ is increasing and $\Delta d(371) = 26\log_2 \frac{371}{374} (\approx -0.3) > -0.4$. Thus $\Delta Y(n) \geq 2 - 0.4 > 0$. If i is non-negative and $372 + 3i + 2 \leq 391$ then (as $(c(372+3i) = c(372+3i+1) = c(372+3i+2))$ $Y(372+3i) > Y(372+3i+2) = Y(371+3(i+1)) \geq Y(371) + (i+1)\Delta Y(371) > Y(371) > 0$.

Now, the proof of Theorem 2.7 is finished. $\square$

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