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Bi- Starlike function of complex order involving Rabotnov function subordinated to Lucas Balancing polynomial

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Abstract

Considering the interesting results obtained recently by studying Rabotnov function, in this paper using the normalized Rabotnov function and the concept of subordination related with the Lucas Balancing polynomial, we defined two new subclasses of the bi-starlike and bi-convex function of complex order in the open unit disc and obtained bounds of the initial Taylor-Maclaurin coefficients for functions in these classes. Furthermore, we have determined the Fekete-Szegö inequalities for function in the above mentioned classes. Several related corollaries are also presented.

1 Introduction, Preliminaries and Definitions

Let $\mathfrak{A}$ represent the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, let $\mathcal{S}$ denote the class of all functions in $\mathfrak{A}$ which are univalent in Δ.

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An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on Δ with $\varpi(0) = 0$ and $|\varpi(z)| < 1$ sustaining $f(z) = g(\varpi(z))$.
The convolution or Hadamard product of two functions $f, h \in \mathfrak{A}$ is denoted by $f * h$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n h_n z^n, \tag{1.2}$$

where $f(z)$ is given by (1.1) and $h(z) = z + \sum\limits_{n=2}^{\infty} h_n z^n$.
Some of the significant and well-investigated subclasses of the univalent function class $\mathcal{S}$ comprise (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in Δ and the class $\mathcal{C}(\alpha)$ of convex functions of order α $(0 \leq \alpha < 1)$ in Δ.

Based on the Koebe's one-quarter theorem [11], every $f \in \mathcal{S}$ has the compositional inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \ (z \in \Delta) \text{ and } f(f^{-1}(w)) = w, \quad (w \in \Delta_\rho),$$

where $\rho \geq \frac{1}{4}$ is the radius of the image $f(\Delta)$. It is well known that $f^{-1}(w)$ has the normalized Taylor-Maclaurin's series

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n, \quad (w \in \Delta_\rho), \tag{1.3}$$

where

$$b_n = \frac{(-1)^{n+1}}{n!} |A_{ij}|,$$

and $|A_{ij}|$ is the $(n-1)^{th}$ order determinant whose entries are denoted by

$$|A_{ij}| = \begin{cases} [(i-j+1)n + j - 1] a_{i-j+2}, & \text{if } \ i+1 \geq j, \\ 0, & \text{if } \ i+1 < j. \end{cases}$$

Then

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots . \tag{1.4}$$

A function $f(z) \in \mathfrak{A}$ is said to be bi-univalent in Δ if both $f(z)$ and $f^{-1}(z)$ are univalent in Δ. Let Σ signify the class of bi-univalent functions in Δ given by (1.1). We notice that the class Σ is not empty. For instance, the functions

$$f_1(z) = \frac{z}{1-z}, \quad f_2(z) = \frac{1}{2} \log \frac{1+z}{1-z} \quad \text{and} \quad f_3(z) = -\log(1-z)$$

with their corresponding inverses

$$f_1^{-1}(w) = \frac{w}{1+w}, \quad f_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1} \text{ and } f_3^{-1}(w) = \frac{e^w-1}{e^w},$$

are elements of Σ. However, the Koebe function is not a member of Σ. Formerly, Brannan and Taha [5] presented certain subclasses of bi-univalent function class Σ, namely bi-starlike functions of order $\alpha(0 < \alpha \leq 1)$ denoted by $\mathcal{S}^*_\Sigma(\alpha)$ and bi-convex function of order α denoted by $\mathcal{C}_\Sigma(\alpha)$. For each of the function classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{C}_\Sigma(\alpha)$, non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were established in [5, 42], but the coefficient problem for each of the succeeding Taylor-Maclaurin coefficients:

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1,2\}; \ \ \mathbb{N} := \{1,2,3,\cdots\})$$

is still an open problem (see [4, 5, 22, 27, 42]). Lately, Srivastava et al. [35] have essentially revived the study of analytic and bi-univalent functions, which was followed by such works as those of [5, 13, 23, 41, 33]. Several authors have introduced and examined subclasses of bi-univalent functions and obtained bounds for the initial coefficients [36, 37, 38] and bi-prestarlike functions by Jahangiri and Hamidi [18](see [26]) have familiarized and inspected several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

1.1 Rabotnov function

In 1948, Yu. N. Rabotnov, who worked in solid mechanics included plasticity, creep theory, hereditary mechanics, failure mechanics, nonelastic stability, composites and shell theory, introduced a special function applied in viscoelasticity . This function, known today as the Rabotnov fractional exponential function or briefly Rabotnov function, is defined as follows [32]

$$\mathfrak{Y}_{\wp,\ell}(z) = z^{\wp} \sum_{n=0}^{\infty} \frac{\ell^n}{\Gamma((n+1)(1+\wp))} z^{n(1+\wp)} \qquad (\wp, \ell, z \in \mathbb{C}). \tag{1.5}$$

Rabotnov function is the particular case of the familiar Mittag-Leffler function widely employed to solving fractional order differential equations or integral equations.The relation between the Rabotnov function and Mittag-Leffer function [25]can be written as follows

$$\mathfrak{Y}_{\wp,\ell}(z) = z^{\wp}\mathfrak{M}_{1+\wp,1+\wp}(\ell z^{1+\wp})$$

where $\mathfrak{M}$ is Mittag-Leffler function and $\wp, \ell, z \in \mathbb{C}$.Several characteristics of Mittag-Leffler and generalized Mittag-Lefflerer can be found in [1, 2, 6]and references cited therein.It is clear that the Rabotnov function $\mathfrak{Y}(\wp, \ell)(z)$ does not belong to $\mathcal{A}$. Thus, it is natural to consider the normalization of the Rabotnov function for $\wp \geq 0$ and $\ell > 0$ defined by

$$\mathfrak{T}(\wp, \ell; z) = z^{\frac{1}{1+\wp}}\Gamma(1+\wp)\mathfrak{Y}_{\wp,\ell}(z^{\frac{1}{1+\wp}}) = z+\sum_{n=2}^{\infty}\frac{\ell^{n-1}\Gamma(\wp+1)}{\Gamma(n(1+\wp))}z^n \quad (\wp > -1, \ell \in \mathbb{C}).$$

Geometric properties, including convexity, close-to-convexity and starlikeness, for the normalized Rabotnov function were recently investigated by Eker and Ece in [12]. The convergence of this series at any values of the argument is evident. Noting that for $\wp = 0$ it reduces to the standard exponential $exp(\ell z)$. Using the following normalized, form of Rabotnov function

$$\begin{aligned} \mathfrak{T}(\wp, \ell; z) &= z+\sum_{n=2}^{\infty}\frac{\ell^{n-1}\Gamma(\wp+1)}{\Gamma(n(1+\wp))}z^n \qquad (\wp > -1, \ell \in \mathbb{C}), \qquad (1.6) \\ &= z+\sum_{n=2}^{\infty}\mathfrak{Y}_n(\ell)z^n, \qquad (1.7) \end{aligned}$$

where

$$\mathfrak{Y}_n(\ell) = \frac{\ell^{n-1}\Gamma(\wp+1)}{\Gamma(n(1+\wp))} \qquad (1.8)$$

we define now a new linear operator $\mathfrak{E}_\ell^\wp : \mathcal{A} \to \mathcal{A}$ given by

$$\mathfrak{E}_\ell^\wp f(z) := \mathfrak{T}(\wp, \ell; z) * f(z),\ z \in \Delta,$$

where the symbol "$*$" stands for the Hadamard product. Thus, if $f \in \mathcal{A}$ has the form (1.1), then

$$\mathfrak{E}_\ell^\wp f(z) = z+\sum_{n=2}^{\infty}\mathfrak{Y}_n(\wp, \ell)a_n z^n,\ z \in \Delta. \qquad (1.9)$$

Numerous intriguing and productive applications of a broad range of special functions, q-calculus, and special polynomials, such as the Fibonacci, Faber, Lucas, Pell, Pell-Lucas, and second-kind Chebyshev polynomials, have been found in Geometric Function Theory. Numerous fields in the mathematical, physical, statistical, and engineering sciences may find use for the Horadam polynomials. Behera and Panda were the first to propose the idea of balancing numbers $(B_n), n \geq 0$ [3]. With initial values set at $B_0 = 0$ and $B_1 = 1$, these numbers are defined by the recurrence relation $B_{n=1} = 6B_n - B_{n-1}$ for

$n \geq 1$,.An associated series known as the Lucas-Balancing numbers, which are represented as

$$\mathfrak{B}_n = \sqrt{8B_n^2 + 1},$$

has received a lot of attention. They contain starting terms $\mathfrak{B}_0 = 1$ and $\mathfrak{B}_1 = 3$, and they also fulfill the recurrence relation $\mathfrak{B}_{n+1} = 6\mathfrak{B}_n - \mathfrak{B}_{n-1}$ for $n \geq 1$, like B_n.

Since then, these numbers have been the subject of numerous generalizations and investigations using a variety of methodologies, such as generating functions, investigations into sum and ratio formulas for balancing numbers, hybrid convolutions, the representation of sums using binomial coefficients, various methods for summing balancing numbers, reciprocals of sequences related to these numbers, incomplete balancing and Lucas-Balancing numbers, and matrix-based methods for studying series. A variety of perspectives and methods are presented in these references, which expand the idea to generalized balancing sequences(see[14, 15, 16, 29, 31]) As was first shown in [30], the study of sequences of Lucas-Balancing polynomials is a logical next step in this area of investigation. These polynomials can be recursively defined as follows

$$\begin{aligned}
\mathfrak{B}_0(x) &= 1 \\
\mathfrak{B}_1(x) &= 3x && (1.10)\\
\mathfrak{B}_2(x) &= 18x^2 - 1, and && (1.11)\\
\mathfrak{B}_n(x) &= 6x\mathfrak{B}_{n-1}(x) - \mathfrak{B}_{n-2}(x), n \geq 2.
\end{aligned}$$

In [12], the generating function of the Lucas-Balancing polynomials is denoted as and given by

$$\Omega(x, z) = \sum_{n=0}^{\infty} \mathfrak{B}_n(x) z^n = \frac{1 - 3xz}{1 - 6xz - z^2}$$

where $x \in [-1, 1]$and $z \in \Delta$

Inspired by the work of Silverman and Silvia [39] (also see [40]) and recent study by Srivastava et al. [35], and by the earlier work in [8, 9, 10, 17, 20, 21, 34, 41] in the present paper first time we introduce two new subclasses of the function class Σ of complex order $\varrho \in \mathbb{C}\backslash\{0\}$, involving the linear operator $\mathfrak{E}_{\ell}^{\wp}$ subordinating with balancing Lucas polynomials .We find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of the function class Σ. Several related classes are also considered and their connection to earlier known results are explained.

Definition 1.1. *A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{S}^{\ell,\wp}_{\Sigma,\Xi}(\varrho,\varpi)$ if the following conditions are satisfied:*

$$1+\frac{1}{\varrho}\left(\frac{z(\mathfrak{E}^{\wp}_{\ell}f(z))'}{\mathfrak{E}^{\wp}_{\ell}f(z)}+\left(\frac{1+e^{i\varpi}}{2}\right)\frac{z^2(\mathfrak{E}^{\wp}_{\ell}f(z))''}{\mathfrak{E}^{\wp}_{\ell}f(z)}-1\right)\prec\Omega(x,z) \tag{1.12}$$

and

$$1+\frac{1}{\varrho}\left(\frac{w(\mathfrak{E}^{\wp}_{\ell}g(w))'}{\mathfrak{E}^{\wp}_{\ell}g(w)}+\left(\frac{1+e^{i\varpi}}{2}\right)\frac{w^2(\mathfrak{E}^{\wp}_{\ell}g(w))''}{\mathfrak{E}^{\wp}_{\ell}g(w)}-1\right)\prec\Omega(x,w) \tag{1.13}$$

where $\varrho\in\mathbb{C}\backslash\{0\}$ $\varpi\in(-\pi,\pi]$, $z,w\in\Delta$ and the function g is given by (1.4).

Definition 1.2. *A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{C}^{\ell,\wp}_{\Sigma,\Xi}(\varrho,\varpi)$ if the following conditions are satisfied:*

$$1+\frac{1}{\varrho}\left(\frac{[z(\mathfrak{E}^{\wp}_{\ell}f(z))'+\left(\frac{1+e^{i\varpi}}{2}\right)z^2(\mathfrak{E}^{\wp}_{\ell}f(z))'']'}{(\mathfrak{E}^{\wp}_{\ell}f(z))'}-1\right)\prec\Omega(x,z) \tag{1.14}$$

and

$$1+\frac{1}{\varrho}\left(\frac{[w(\mathfrak{E}^{\wp}_{\ell}g(w))'+\left(\frac{1+e^{i\varpi}}{2}\right)w^2(\mathfrak{E}^{\wp}_{\ell}g(w))'']'}{(\mathfrak{E}^{\wp}_{\ell}g(w))'}-1\right)\prec\Omega(x,w), \tag{1.15}$$

where $\varrho\in\mathbb{C}\backslash\{0\},\varpi\in(-\pi,\pi]$, $z,w\in\Delta$ and the function g is given by (1.4).

Remark 1.1. *A function $f\in\Sigma$ given by (1.1) and for $\varpi=\pi$, we note that $\mathcal{S}^{\ell,\wp}_{\Sigma,\Xi}(\varrho,\pi)\equiv\mathcal{S}^{\ell,\wp}_{\Sigma,\Xi}(\varrho)$ and $\mathcal{C}^{\ell,\wp}_{\Sigma,\Xi}(\varrho,\pi)\equiv\mathcal{C}^{\ell,\wp}_{\Sigma,\Xi}(\varrho)$ satisfies the following conditions respectively:*

$$\left[1+\frac{1}{\varrho}\left(\frac{z(\mathfrak{E}^{\wp}_{\ell}f(z))'}{\mathfrak{E}^{\wp}_{\ell}f(z)}-1\right)\right]\prec\Omega(x,z)\ \text{ and }\ \left[1+\frac{1}{\varrho}\left(\frac{w(\mathfrak{E}^{\wp}_{\ell}g(w))'}{\mathfrak{E}^{\wp}_{\ell}g(w)}-1\right)\right]\prec\Omega(x,w)$$

and

$$\left[1+\frac{1}{\varrho}\left(\frac{z(\mathfrak{E}^{\wp}_{\ell}f(z))''}{(\mathfrak{E}^{\wp}_{\ell}f(z))'}\right)\right]\prec\Omega(x,z)\ \text{ and }\ \left[1+\frac{1}{\varrho}\left(\frac{w(\mathfrak{E}^{\wp}_{\ell}g(w))''}{(\mathfrak{E}^{\wp}_{\ell}g(w))'}\right)\right]\prec\Omega(x,w),$$

where $\varrho\in\mathbb{C}\backslash\{0\}$ $z,w\in\Delta$ and the function g is given by (1.4).

A function $f\in\Sigma$ given by (1.1) and for $\varpi=0$, we note that $\mathcal{S}^{\ell,\wp}_{\Sigma,\Xi}(\varrho,0)\equiv\mathfrak{M}^{\ell,\wp}_{\Sigma,\Xi}(\varrho)$ and $\mathcal{C}^{\ell,\wp}_{\Sigma,\Xi}(\varrho,0)\equiv\mathfrak{W}^{\ell,\wp}_{\Sigma,\Xi}(\varrho)$ as in the following remarks:

Remark 1.2. *A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathfrak{M}^{\ell,\wp}_{\Sigma,\Xi}(\varrho)$ if the following conditions are satisfied:*

$$1+\frac{1}{\varrho}\left(\frac{z(\mathfrak{E}^{\wp}_{\ell}f(z))'}{\mathfrak{E}^{\wp}_{\ell}f(z)}+\frac{z^2(\mathfrak{E}^{\wp}_{\ell}f(z))''}{\mathfrak{E}^{\wp}_{\ell}f(z)}-1\right)\prec\Omega(x,z) \tag{1.16}$$

and

$$1+\frac{1}{\varrho}\left(\frac{w(\mathfrak{E}^{\wp}_{\ell}g(w))'}{\mathfrak{E}^{\wp}_{\ell}g(w)}+\frac{w^2(\mathfrak{E}^{\wp}_{\ell}g(w))''}{\mathfrak{E}^{\wp}_{\ell}g(w)}-1\right)\prec\Omega(x,w) \tag{1.17}$$

where $\varrho \in \mathbb{C}\backslash\{0\}, z, w \in \Delta$ and the function g is given by (1.4).

Remark 1.3. *A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathfrak{W}^{\ell,\wp}_{\Sigma,\Xi}(\varrho)$ if the following conditions are satisfied:*

$$1+\frac{1}{\varrho}\left(\frac{[z(\mathfrak{E}^{\wp}_{\ell}f(z))'+z^2(\mathfrak{E}^{\wp}_{\ell}f(z))'']'}{(\mathfrak{E}^{\wp}_{\ell}f(z))'}-1\right)\prec\Omega(x,z) \tag{1.18}$$

and

$$1+\frac{1}{\varrho}\left(\frac{[w(\mathfrak{E}^{\wp}_{\ell}g(w))'+w^2(\mathfrak{E}^{\wp}_{\ell}g(w))'']'}{(\mathfrak{E}^{\wp}_{\ell}g(w))'}-1\right)\prec\Omega(x,w), \tag{1.19}$$

where $\varrho \in \mathbb{C}\backslash\{0\}, z, w \in \Delta$ and the function g is given by (1.4).

Remark 1.4. *A function $f \in \Sigma$ given by (1.1) and for $\varrho = 1$, we note that $\mathcal{S}^{\ell,\wp}_{\Sigma,\Xi}(\varrho,\varpi) \equiv \mathcal{S}^{\ell,\wp}_{\Sigma,\Xi}(\varpi)$ and satisfies the following conditions respectively:*

$$\left(\frac{z(\mathfrak{E}^{\wp}_{\ell}f(z))'}{\mathfrak{E}^{\wp}_{\ell}f(z)}+\left(\frac{1+e^{i\varpi}}{2}\right)\frac{z^2(\mathfrak{E}^{\wp}_{\ell}f(z))''}{\mathfrak{E}^{\wp}_{\ell}f(z)}\right)\prec\Omega(x,z)$$

and

$$\left(\frac{w(\mathfrak{E}^{\wp}_{\ell}g(w))'}{\mathfrak{E}^{\wp}_{\ell}g(w)}+\left(\frac{1+e^{i\varpi}}{2}\right)\frac{w^2(\mathfrak{E}^{\wp}_{\ell}g(w))''}{\mathfrak{E}^{\wp}_{\ell}g(w)}\right)\prec\Omega(x,w).$$

Also $\mathcal{C}^{\ell,\wp}_{\Sigma,\Xi}(\varrho,\varpi) \equiv \mathcal{C}^{\ell,\wp}_{\Sigma,\Xi}(\varpi)$ and satisfies the following conditions

$$\left(\frac{[z(\mathfrak{E}^{\wp}_{\ell}f(z))'+\left(\frac{1+e^{i\varpi}}{2}\right)z^2(\mathfrak{E}^{\wp}_{\ell}f(z))'']'}{(\mathfrak{E}^{\wp}_{\ell}f(z))'}\right)\prec\Omega(x,z)$$

and

$$\left(\frac{[w(\mathfrak{E}^{\wp}_{\ell}g(w))'+\left(\frac{1+e^{i\varpi}}{2}\right)w^2(\mathfrak{E}^{\wp}_{\ell}g(w))'']'}{(\mathfrak{E}^{\wp}_{\ell}g(w))'}\right)\prec\Omega(x,w),$$

where $\varpi \in (-\pi,\pi], z, w \in \Delta$ and the function g is given by (1.4).

2 Results on Initial Coefficient estimates

For notational simplicity, in the sequel we let,

$$\mathfrak{Y}_2(\wp,\ell) = \frac{\ell\Gamma(\wp+1)}{\Gamma(2(1+\wp))}, \tag{2.1}$$

$$\mathfrak{Y}_3(\wp,\ell) = \frac{\ell^2\Gamma(\wp+1)}{\Gamma(3(1+\wp))} \tag{2.2}$$

$$\mathfrak{B}(x,z) := \sum_{n=0}^{\infty} \mathfrak{B}_n z^n = \frac{1-3xz}{1-6xz-z^2}. \tag{2.3}$$

We also let

$$\varrho \in \mathbb{C}\backslash\{0\}\ \varpi \in (-\pi,\pi], z, w \in \Delta, x \in [-1,1]$$

and the function g is given by (1.4) unless otherwise stated.

For deriving our main results, we need the following lemma.

Lemma 2.1. *[28] If $h \in \mathfrak{P}$, then $|c_k| \leq 2$ for each k, where $\mathfrak{P}$ is the family of all functions h analytic in Δ for which $\Re(h(z)) > 0$ and*

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad \textit{for } z \in \Delta.$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \cdots .$$

It follows that

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right]$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \cdots\right].$$

Then $p(z)$ and $q(z)$ are analytic in Δ with $p(0) = 1 = q(0)$.

Since $u, v : \Delta \to \Delta$, the functions $p(z)$ and $q(z)$ have a positive real part in Δ, and $|p_i| \leq 2$ and $|q_i| \leq 2$ for each i.

Theorem 2.1. *Let $f(z)$ given by (1.1) be in the class $\mathcal{S}_{\Sigma,\Xi}^{\ell,\wp}(\varrho,\varpi)$, $\varrho \in \mathbb{C}\backslash\{0\}$ and $\varpi \in (-\pi,\pi]$. Then*

$$|a_2| \leq \frac{3\sqrt{6x}\ |x\varrho|}{\sqrt{|9x^2\varrho[(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)-(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2]-(18x^2-1)(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2|}} \tag{2.4}$$

and

$$|a_3| \leq \frac{36|x\varrho|^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2} + \frac{6|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}. \tag{2.5}$$

Proof. It follows from (1.12) and (1.13) that

$$1+\frac{1}{\varrho}\left(\frac{z(\mathfrak{E}_\ell^\wp f(z))'}{\mathfrak{E}_\ell^\wp f(z)}+\left(\frac{1+e^{i\varpi}}{2}\right)\frac{z^2(\mathfrak{E}_\ell^\wp f(z))''}{\mathfrak{E}_\ell^\wp f(z)}-1\right)=\Xi(u(z)) \tag{2.6}$$

and

$$1+\frac{1}{\varrho}\left(\frac{w(\mathfrak{E}_\ell^\wp g(w))'}{\mathfrak{E}_\ell^\wp g(w)}+\left(\frac{1+e^{i\varpi}}{2}\right)\frac{w^2(\mathfrak{E}_\ell^\wp g(w))''}{\mathfrak{E}_\ell^\wp g(w)}-1\right)=\Xi(v(w)), \tag{2.7}$$

where

$$\begin{aligned}\Xi(u(z)) &= 1+\mathfrak{B}_1(x)p_1z+\Big(\mathfrak{B}_1(x)p_2+\mathfrak{B}_2(x)p_1^2\Big)z^2 \\ &+ \Big(\mathfrak{B}_1(x)p_3+2\mathfrak{B}_2(x)p_1p_2+\mathfrak{B}_3(x)p_1^3\Big)z^3+\cdots . \end{aligned} \tag{2.8}$$

(2.9)

And similarly we get

$$\begin{aligned}\Xi(v(w)) &= 1+\mathfrak{B}_1(x)q_1w+\Big(\mathfrak{B}_1(x)q_2+\mathfrak{B}_2(x)q_1^2\Big)w^2 \\ &+ \Big(\mathfrak{B}_1(x)q_3+2\mathfrak{B}_2(x)q_1q_2+\mathfrak{B}_3(x)q_1^3\Big)w^3+\cdots . \end{aligned} \tag{2.10}$$

(2.11)

Now, equating the coefficients in (2.6) and (2.7), we get

$$\frac{1}{\varrho}(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]a_2=\mathfrak{B}_1(x)p_1, \tag{2.12}$$

$$\frac{1}{\varrho}\left[(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)a_3-(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2a_2^2\right]=\mathfrak{B}_1(x)p_2+\mathfrak{B}_2(x)p_1^2, \tag{2.13}$$

$$-\frac{1}{\varrho}(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]a_2=\mathfrak{B}_1(x)q_1, \tag{2.14}$$

and

$$\frac{1}{\varrho}\left([2(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)-(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2]a_2^2-(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)a_3\right)$$
$$=\mathfrak{B}_1(x)q_2+\mathfrak{B}_2(x)q_1^2. \tag{2.15}$$

From (2.12) and (2.14), we get

$$p_1=-q_1 \tag{2.16}$$

and

$$2(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2a_2^2=[\varrho\mathfrak{B}_1(x)]^2(p_1^2+q_1^2). \tag{2.17}$$

Thus we have

$$\frac{2(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2a_2^2}{[\mathfrak{B}_1(x)\varrho]^2}=p_1^2+q_1^2. \tag{2.18}$$

$$a_2^2=\frac{[\mathfrak{B}_1(x)\varrho]^2(p_1^2+q_1^2)}{2(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2}. \tag{2.19}$$

Now from (2.13), (2.15) and (2.17), we obtain

$$\{2\varrho[\mathfrak{B}_1(x)]^2[(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)-2(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2]-2\mathfrak{B}_2(x)(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2\}a_2^2$$
$$=[\mathfrak{B}_1(x)]^3\varrho^2(p_2+q_2)$$

$$a_2^2=\frac{[\mathfrak{B}_1(x)]^3\varrho^2(p_2+q_2)}{\{2\varrho[\mathfrak{B}_1(x)]^2[(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)-2(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2]-2\mathfrak{B}_2(x)(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2\}} \tag{2.20}$$

Applying Lemma (2.1) to the coefficients p_2 and q_2, from(2.18) and (2.20)we have

$$|a_2|\leq\frac{3\sqrt{6x}\ |x\varrho|}{\sqrt{|9x^2\varrho[(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)-(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2]-(18x^2-1)(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2|}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (2.13) from (2.15) and using (2.16), we get

$$\frac{2}{\varrho}(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)(a_3-a_2^2)=\mathfrak{B}_1(x)(p_2-q_2).$$

Upon substituting the value of a_2^2 from (2.19), we get

$$a_3=\frac{\varrho^2[\mathfrak{B}_1(x)]^2(p_1^2+q_1^2)}{2(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2}+\frac{\varrho\mathfrak{B}_1(x)(p_2-q_2)}{2(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)}. \tag{2.21}$$

Applying Lemma (2.1) once again to the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3| \leq \frac{36|x\varrho|^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2} + \frac{6|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}.$$

□

Theorem 2.2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{C}^{\ell,\wp}_{\Sigma,\Xi}(\varrho,\varpi)$, $\varrho \in \mathbb{C}\backslash\{0\}$ and $\varpi \in (-\pi,\pi]$. Then*

$$|a_2| \leq \frac{6|x\varrho\sqrt{3x}|}{\sqrt{|9x^2\varrho\{3(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell) - 4(2+e^{i\varpi})[\mathfrak{y}_2(\wp,\ell)]^2\} - 4(18x^2-1)(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2|}} \tag{2.22}$$

and

$$|a_3| \leq \frac{9|x\varrho|^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2} + \frac{2|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}. \tag{2.23}$$

Proof. We can write the argument inequalities in (1.14) and (1.15) equivalently as follows:

$$1 + \frac{1}{\varrho}\left(\frac{[z(\mathfrak{E}^{\wp}_{\ell} f(z))' + \left(\frac{1+e^{i\varpi}}{2}\right) z^2(\mathfrak{E}^{\wp}_{\ell} f(z))'']'}{(\mathfrak{E}^{\wp}_{\ell} f(z))'} - 1\right) = \Xi(u(z)) \tag{2.24}$$

and

$$1 + \frac{1}{\varrho}\left(\frac{[w(\mathfrak{E}^{\wp}_{\ell} g(w))' + \left(\frac{1+e^{i\varpi}}{2}\right) w^2(\mathfrak{E}^{\wp}_{\ell} g(w))'']'}{(\mathfrak{E}^{\wp}_{\ell} g(w))'} - 1\right) = \Xi(v(w)). \tag{2.25}$$

Now proceeding as in the proof of Theorem 2.1, from (2.24) and (2.25), we obtain the following relations

$$\frac{2}{\varrho}(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]a_2 = \mathfrak{B}_1(x)p_1, \tag{2.26}$$

$$\frac{1}{\varrho}[3(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)a_3 - 4(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2 a_2^2] = \mathfrak{B}_1(x)p_2 + \mathfrak{B}_2(x)p_1^2, \tag{2.27}$$

and

$$-\frac{2}{\varrho}(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]a_2 = \mathfrak{B}_1(x)q_1, \tag{2.28}$$

$$\frac{1}{\varrho}[3(5+3e^{i\varpi})(2a_2^2 - a_3)\mathfrak{Y}_3(\wp,\ell) - 4(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2 a_2^2] = \mathfrak{B}_1(x)q_2 + \mathfrak{B}_2(x)q_1^2 \tag{2.29}$$

From (2.26) and (2.28), we get

$$p_1 = -q_1 \tag{2.30}$$

and

$$\begin{aligned} 8(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2 a_2^2 &= \varrho^2[\mathfrak{B}_1(x)]^2(p_1^2+q_1^2), && (2.31)\\ a_2^2 &= \frac{\varrho^2[\mathfrak{B}_1(x)]^2 p_1^2}{4(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2}, && (2.32)\end{aligned}$$

Now from (2.27), (2.29) and (2.32), we obtain

$$a_2^2 = \frac{\varrho^2[\mathfrak{B}_1(x)]^3(p_2+q_2)}{2\{[\varrho[\mathfrak{B}_1(x)]^2\{3(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)-4(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2\}-4\mathfrak{B}_2(x)(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2]\}}. \tag{2.33}$$

Applying Lemma (2.1) to the coefficients p_2 , q_2 and (1.10) we have the desired inequality given in (2.22).

Next, in order to find the bound on $|a_3|$, by subtracting (2.27) from (2.29), and using (2.30), we get

$$\frac{6}{\varrho}(5+3e^{i\varpi})(a_3-a_2^2)\mathfrak{Y}_3(\wp,\ell) = \mathfrak{B}_1(x)(p_2-q_2). \tag{2.34}$$

Upon substituting the value of a_2^2 given by (2.31), the above equation leads to

$$a_3 = \frac{\varrho(p_2-q_2)\mathfrak{B}_1(x)}{6(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)} + \frac{\varrho^2[\mathfrak{B}_1(x)]^2 p_1^2}{4(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2}. \tag{2.35}$$

Applying the Lemma (2.1) once again to the coefficients p_1, p_2, q_1 and q_2, and (1.10)we get the desired coefficient given in (2.23). $\square$

3 Bi-univalent function class $\mathcal{G}^{\ell}_{\Sigma}(\varrho,\mu,\Xi)$

In this section we define another new subclass of bi univalent functions based on Rabotnov fractional exponential function subordinating with Lucas Balancing polynomial and obtain the initial Taylor estimates $|a_2|; |a_3|$ and making use of it we derive the Fekete-Szegö inequality for $f \in \mathcal{G}^{\ell}_{\Sigma}(\varrho,\mu,\Xi)$.

Definition 3.1. *A function $f \in \Sigma$ given by (1.1)is said to be in the class $\mathcal{G}^{\ell}_{\Sigma}(\varrho,\mu,\Xi)$ if the following conditions are satisfied:*

$$1+\frac{1}{\varrho}\left(\frac{z(\mathfrak{E}^{\wp}_{\ell} f(z))'}{(1-\mu)\mathfrak{E}^{\wp}_{\ell} f(z)+\mu z(\mathfrak{E}^{\wp}_{\ell} f(z))'}-1\right) \prec \Omega(x,z) \tag{3.1}$$

and

$$1+\frac{1}{\varrho}\left(\frac{w(\mathfrak{E}_{\ell}^{\wp}g(w))'}{(1-\mu)\mathfrak{E}_{\ell}^{\wp}g(w)+\mu z(\mathfrak{E}_{\ell}^{\wp}g(w))'}-1\right)\prec\Omega(x,w) \tag{3.2}$$

where $\varrho\in\mathbb{C}\backslash\{0\};\ 0\leqq\mu<1;\ z,w\in\Delta$ *and* g *is given by* (1.4).

Example 3.1. *For* $\mu=0$ *and* $\varrho\in\mathbb{C}\backslash\{0\}$, *a function* $f\in\Sigma$, *given by* (1.1)*is said to be in the class* $\mathcal{S}_{\Sigma}^{\ell}(\varrho,\Xi)$ *if the following conditions are satisfied:*

$$1+\frac{1}{\varrho}\left(\frac{z(\mathfrak{E}_{\ell}^{\wp}f(z))'}{\mathfrak{E}_{\ell}^{\wp}f(z)}-1\right)\prec\Omega(x,z), \tag{3.3}$$

and

$$1+\frac{1}{\varrho}\left(\frac{w(\mathfrak{E}_{\ell}^{\wp}g(w))'}{\mathfrak{E}_{\ell}^{\wp}g(w)}-1\right)\prec\Omega(x,w), \tag{3.4}$$

where $z,w\in\Delta$ *and the function* g *is given by* (1.4).

Note that $\mathcal{S}_{\Sigma}^{\ell}(\varrho,\Xi)=\mathcal{S}_{\Sigma,\Xi}^{\ell,\wp}(\varrho,\pi)\equiv\mathcal{S}_{\Sigma,\Xi}^{\ell,\wp}(\varrho)$ as given in Remark 1.1.

Theorem 3.1. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{G}_{\Sigma}^{\ell}(\varrho,\mu,\Xi)$. *Then*

$$|a_2|\leq\frac{6x\sqrt{3x}|\varrho|}{\sqrt{|[18x^2\varrho(\mu^2-1)-2(18x^2-1)(1-\mu)^2][\mathfrak{Y}_2(\wp,\ell)]^2+36x^2\varrho(1-\mu)\mathfrak{Y}_3(\wp,\ell)|}} \tag{3.5}$$

and

$$|a_3|\leq\frac{36x^2|\varrho|^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2}+\frac{3x|\varrho|}{(1-\mu)\mathfrak{Y}_3(\wp,\ell)}. \tag{3.6}$$

For $\hbar\in\mathbb{R}$, *we have*

$$|\ a_3-\hbar a_2^2\ |\leqq\begin{cases}\frac{|\varrho|}{2(1-\mu)\mathfrak{Y}_3(\wp,\ell)}, & 0\leq|\ \phi(\hbar)\ |\leq\frac{|\varrho|}{8(1-\mu)\mathfrak{Y}_3(\wp,\ell)}\\ 2|\varrho||\phi(\hbar)|, & |\phi(\hbar)|\geq\frac{|\varrho|}{8(1-\mu)\mathfrak{Y}_3(\wp,\ell)},\end{cases} \tag{3.7}$$

where

$$\phi(\hbar)=\frac{\varrho^2(1-\hbar)}{2\left([\varrho(\mu^2-1)+(1-\mu)^2][\mathfrak{Y}_2(\wp,\ell)]^2+4\varrho(1-\mu)\mathfrak{Y}_3(\wp,\ell)\right)}.$$

Proof. It follows from (3.1) and (3.2) that

$$1+\frac{1}{\varrho}\left(\frac{z(\mathfrak{E}_{\ell}^{\wp}f(z))'}{(1-\mu)\mathfrak{E}_{\ell}^{\wp}f(z)+\mu z(\mathfrak{E}_{\ell}^{\wp}f(z))'}-1\right)=\Xi(u(z)) \tag{3.8}$$

and

$$1+\frac{1}{\varrho}\left(\frac{w(\mathfrak{E}_{\ell}^{\wp}g(w))'}{(1-\mu)\mathfrak{E}_{\ell}^{\wp}g(w)+\mu z(\mathfrak{E}_{\ell}^{\wp}g(w))'}-1\right)=\Xi(v(w)). \tag{3.9}$$

Now, equating the coefficients in (3.8) and (3.9), we get

$$\frac{(1-\mu)}{\varrho}[\mathfrak{Y}_2(\wp,\ell)]a_2 = \mathfrak{B}_1(x)p_1, \tag{3.10}$$

$$\frac{(\mu^2-1)}{\varrho}[\mathfrak{Y}_2(\wp,\ell)]^2a_2^2 + \frac{2(1-\mu)}{\varrho}\mathfrak{Y}_3(\wp,\ell)a_3 = \mathfrak{B}_1(x)p_2 + \mathfrak{B}_2(x)p_1^2, \tag{3.11}$$

$$-\frac{(1-\mu)}{\varrho}[\mathfrak{Y}_2(\wp,\ell)]a_2 = \mathfrak{B}_1(x)q_1 \tag{3.12}$$

and

$$\frac{(\mu^2-1)}{\varrho}[\mathfrak{Y}_2(\wp,\ell)]^2a_2^2 + \frac{2(1-\mu)}{\varrho}\mathfrak{Y}_3(\wp,\ell)(2a_2^2-a_3) = \mathfrak{B}_1(x)q_2 + \mathfrak{B}_2(x)q_1^2. \tag{3.13}$$

From (3.10) and (3.12), we find that

$$a_2 = \frac{\varrho\mathfrak{B}_1(x)p_1}{(1-\mu)[\mathfrak{Y}_2(\wp,\ell)]} = \frac{-\varrho\mathfrak{B}_1(x)q_1}{(1-\mu)[\mathfrak{Y}_2(\wp,\ell)]}, \tag{3.14}$$

which implies

$$p_1 = -q_1 \tag{3.15}$$

and

$$2(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2a_2^2 = \varrho^2[\mathfrak{B}_1(x)]^2(p_1^2+q_1^2). \tag{3.16}$$

$$a_2^2 = \frac{\varrho^2[\mathfrak{B}_1(x)]^2p_1^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2} \tag{3.17}$$

thus we have,

$$|a_2| \le \frac{6|x\varrho|}{|1-\mu|[\mathfrak{Y}_2(\wp,\ell)]} \tag{3.18}$$

Adding (3.11) and (3.13), by using (3.14) and (3.15), we obtain

$$\begin{aligned}\{[2\varrho(\mu^2-1)[\mathfrak{B}_1(x)]^2-2\mathfrak{B}_2(x)(1-\mu)^2][\mathfrak{Y}_2(\wp,\ell)]^2+4\varrho(1-\mu)[\mathfrak{B}_1(x)]^2\mathfrak{Y}_3(\wp,\ell)\}a_2^2 \\ = [\mathfrak{B}_1(x)]^3\varrho^2(p_2+q_2).\end{aligned} \tag{3.19}$$

Thus,

$$a_2^2 = \frac{[\mathfrak{B}_1(x)]^3\varrho^2(p_2+q_2)}{[2\varrho(\mu^2-1)[\mathfrak{B}_1(x)]^2-2\mathfrak{B}_2(x)(1-\mu)^2][\mathfrak{Y}_2(\wp,\ell)]^2+4\varrho(1-\mu)[\mathfrak{B}_1(x)]^2\mathfrak{Y}_3(\wp,\ell)}. \tag{3.20}$$

Applying Lemma 2.1 for the coefficients p_2, q_2 and using (1.10), we immediately have

$$|a_2|^2 \leqq \frac{54x^3|\varrho|^2}{|[9x^2\varrho(\mu^2-1)-(18x^2-1)(1-\mu)^2][\mathfrak{Y}_2(\wp,\ell)]^2+18x^2\varrho(1-\mu)\mathfrak{Y}_3(\wp,\ell)|}. \tag{3.21}$$

The last inequality gives the desired estimate on $|a_2|$ given in (3.5).

Next, in order to find the bound on $|a_3|$, by subtracting (3.13) from (3.11), we get

$$\begin{aligned}\frac{4(1-\mu)}{\varrho}\mathfrak{Y}_3(\wp,\ell)a_3-\frac{4(1-\mu)}{\varrho}\mathfrak{Y}_3(\wp,\ell)a_2^2 &= \mathfrak{B}_1(x)(p_2-q_2)+\mathfrak{B}_2(x)(p_1^2-q_1^2)\\ a_3 &= a_2^2+\frac{\mathfrak{B}_1(x)\varrho(p_2-q_2)}{4(1-\mu)\mathfrak{Y}_3(\wp,\ell)}. \qquad (3.22)\end{aligned}$$

It follows from (3.14), (3.15) and (3.22) that

$$a_3 = \frac{\varrho^2[\mathfrak{B}_1(x)]^2p_1^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2}+\frac{\varrho\mathfrak{B}_1(x)(p_2-q_2)}{4(1-\mu)\mathfrak{Y}_3(\wp,\ell)}. \tag{3.23}$$

Applying Lemma 2.1 once again for the coefficients p_2 and q_2, we readily get

$$|a_3| \leqq \frac{36x^2|\varrho|^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2}+\frac{3x|\varrho|}{(1-\mu)\mathfrak{Y}_3(\wp,\ell)}.$$

This completes the proof of Theorem 3.1. □

4 Fekete-Szegö inequality Results

In this section, we prove Fekete-Szegö inequalities for functions in the class $\mathcal{S}^{\ell,\wp}_{\Sigma,\Xi}(\varrho,\varpi)$. We shall use the following lemmas, which were introduced by Zaprawa in [43, 44] and by the technique given in[21].

Lemma 4.1. *Let $k \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < R$ and $|z_2| < R$, then*

$$|(k+1)z_1+(k-1)z_2| \leq \begin{cases} 2\,|k|\,R, & |k| \geq 1, \\ 2R, & |k| \leq 1. \end{cases}$$

Lemma 4.2. *Let $k, l \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < R$ and $|z_2| < R$, then*

$$|(k+l)z_1+(k-l)z_2| \leq \begin{cases} 2\,|k|\,R, & |k| \geq |l|\,, \\ 2\,|l|\,R, & |k| \leq |l|\,. \end{cases}$$

Lemma 4.3. *[24]. If $p \in \mathcal{P}$, then there exist some $\varkappa,\ \zeta$ with $|\varkappa| \leq 1,\ |\zeta| \leq 1$, such that*

$$2p_2 = p_1^2 + \varkappa(4 - p_1^2),$$
$$4p_3 = p_1^3 + 2p_1\varkappa(4 - p_1^2) - (4 - p_1^2)p_1\varkappa^2 + 2(4 - p_1^2)(1 - |\varkappa|^2)\zeta.$$

Theorem 4.1. *Let f given by (1.1) be in the class $\mathcal{S}^{\ell,\wp}_{\Sigma,\Xi}(\varrho, \varpi)$ and $\hbar \in \mathbb{R}$. Then*

$$| a_3 - \hbar a_2^2 | \leq \begin{cases} \frac{2|\varrho|}{3\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}, & | 1 - \hbar | \in [0, \frac{16(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{3|\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}) \\ |1 - \hbar|\frac{|\varrho|^2}{4(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}, & |1 - \hbar| \in [\frac{16(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{3|\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}, \infty). \end{cases}$$

Proof. From (2.19) and (2.21) it follows that

$$a_3 - \hbar a_2^2 \quad = \quad \frac{(1 - \hbar)\varrho^2[\mathfrak{B}_1(x)]^2 p_1^2}{(2 + e^{i\varpi})^2[\mathfrak{Y}_2(\wp, \ell)]^2} + \frac{\varrho\mathfrak{B}_1(x)(p_2 - q_2)}{2(5 + 3e^{i\varpi})\mathfrak{Y}_3(\wp, \ell)}.$$

From Lemma 4.3, we have $2p_2 = p_1^2 + \varkappa(4 - p_1^2)$ and $2q_2 = q_1^2 + y(4 - q_1^2)$, hence we get

$$p_2 - q_2 = (\frac{4 - p_1^2}{2})(\varkappa - y),$$

and using triangle inequality, taking $|\varkappa| = \theta, |y| = \kappa$, we can easily obtain that

$$|a_3 - \hbar a_2^2| \leq |1 - \hbar|\frac{|\varrho[\mathfrak{B}_1(x)]|^2 t^2}{(5 + 4cos\varpi)\ [\mathfrak{Y}_2(\wp, \ell)]^2} + \frac{|\mathfrak{B}_1(x)\varrho|}{4\sqrt{34 + 30cos\varpi}\ \mathfrak{Y}_3(\wp, \ell)}(4 - t^2)(\theta + \kappa).$$

For continence let $\mathfrak{M}(t) = |1 - \hbar|\frac{9|x\varrho|^2 t^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2} \geq 0$ and $\mathfrak{N}(t) = \frac{3|x\varrho|}{4\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}(4 - t^2) \geq 0$. Thus ,

$$|a_3 - \hbar a_2^2| \quad \leq \quad \mathfrak{M}(t) + \mathfrak{N}(t)(\theta + \kappa) = \mathfrak{L}(\theta, \kappa)$$

It is clear that the maximum of the function $\mathfrak{L}(\theta, \kappa)$ occurs at $(\theta, \kappa) = (1, 1)$. Therefore,

$$\max \mathfrak{L}(\theta, \kappa) : \theta, \kappa \in [0, 1] = \mathfrak{L}(1, 1) = \mathfrak{M}(t) + 2\mathfrak{N}(t).$$

Let us define the function $H : [0, 2] \to \mathbb{R}$ as follows

$$H(t) = \mathfrak{M}(t) + 2\mathfrak{N}(t) \tag{4.1}$$

for fixed $\varrho \in \mathbb{C}-\{0\}$. Substituting the value $\mathfrak{M}(t), \mathfrak{N}(t)$ in (4.3), we obtain

$$\begin{aligned} H(t) &= |1-\hbar|\frac{9|x\varrho|^2t^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2} + \frac{3|x\varrho|}{2\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}(4-t^2) \\ &= \left[|1-\hbar|\frac{9|x\varrho|^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2} - \frac{3|x\varrho|}{2\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}\right]t^2 \\ &\quad + \frac{6|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)} \\ &= \frac{9|x\varrho|^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}\left[|1-\hbar| - \frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{6|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}\right]t^2 \\ &\quad + \frac{6|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)} \end{aligned}$$

Now, we must investigate the maximum of the function $H(t)$ in the interval $[0,2]$. By simple computation, we have

$$H'(t) = \frac{18|x\varrho|^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}\left[|1-\hbar| - \frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{6|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}\right]t.$$

It is clear that $H'(t)\leq 0$ if $\frac{18|x\varrho|^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}\left[|1-\hbar| - \frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{6|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}\right] \leq 0$; that is if $|1-\hbar| \in \left(0, \frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{6|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}\right)$. Thus, the function $H(t)$ is a strictly decreasing function if $|1-\hbar| \in \left(0, \frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{6|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}\right)$.

Therefore,

$$\max\{H(t) : t \in [0,2]\} = H(0) = \frac{6|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}.$$

Also, $H'(t) \geq 0$; that is, the function $H(t)$ is an increasing function for $|1-\hbar| \geq \frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{6|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}$. Therefore,

$$\max\{H(t) : t \in [0,2]\} = H(2) = |1-\hbar|\frac{36|x\varrho|^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}.$$

Thus we get

$$|\ a_3-\hbar a_2^2\ | \leq \begin{cases} \frac{6|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}, & |\ 1-\hbar\ | \in [0, \frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{18|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}) \\ |1-\hbar|\frac{36|x\varrho|^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}, & |1-\hbar| \in [\frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{18|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}, \infty). \end{cases}$$

In particular by taking $\hbar = 1$, we get

$$|\ a_3 - a_2^2\ | \leq \frac{6|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}.$$

□

By taking $\varrho \in \mathbb{R}$ and $\varpi = n\pi, (n \in \mathbb{Z})$ in the following theorem we prove the following Fekete-Szegö inequalities .

Theorem 4.2. *Let f given by* (1.1) *be in the class* $\mathcal{S}^{\ell,\wp}_{\Sigma,\Xi}(\varrho,\varpi)$ *and* $\hbar \in \mathbb{R}$. *Then*

$$| a_3 - \hbar a_2^2 |\leq \begin{cases} \frac{6|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}, & 0 \leq| \Psi(\hbar,\varpi) |\leq \frac{|x\varrho|}{2\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)} \\ 4|\Psi(\hbar,\varpi)|, & |\Psi(\hbar,\varpi)| \geq \frac{|\varrho|}{3\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}, \end{cases} \tag{4.2}$$

where

$$\Psi(\hbar,\varpi)=\frac{(1-\hbar)[\mathfrak{B}_1(x)]^3\varrho^2}{\{2\varrho[\mathfrak{B}_1(x)]^2[(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)-2(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2]-2\mathfrak{B}_2(x)(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2\}}.$$

Proof. From (2.20) and (2.21) it follows that

$$\begin{aligned} a_3-\hbar a_2^2 =& \frac{(1-\hbar)[\mathfrak{B}_1(x)]^3\varrho^2(p_2+q_2)}{\{2\varrho[\mathfrak{B}_1(x)]^2[(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)-2(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2]-2\mathfrak{B}_2(x)(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2\}} \\ &+ \frac{\varrho\mathfrak{B}_1(x)(p_2-q_2)}{2(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)} \\ =& \mathfrak{B}_1(x)\left(\left[\Psi(\hbar,\varpi)+\frac{\varrho}{2(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)}\right]p_2+\left[\Psi(\hbar,\varpi)-\frac{\varrho}{2(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)}\right]q_2\right), \end{aligned}$$

where

$$\Psi(\hbar,\varpi) = \frac{(1-\hbar)[\mathfrak{B}_1(x)]^2\varrho^2}{\{2\varrho[\mathfrak{B}_1(x)]^2[(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)-2(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2]-2\mathfrak{B}_2(x)(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2\}}.$$

Thus by applying Lemmas 2.1 4.2 and using (1.10) we get the desired result as given below

$$| a_3 - \hbar a_2^2 |\leq \begin{cases} \frac{6|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}, & 0 \leq| \Psi(\hbar,\varpi) |\leq \frac{|x\varrho|}{2\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)} \\ 4|\Psi(\hbar,\varpi)|, & |\Psi(\hbar,\varpi)| \geq \frac{|\varrho|}{3\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}. \end{cases}$$

In particular by taking $\hbar = 1$, we get

$$| a_3 - a_2^2 |\leq \frac{6|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}.$$

□

Theorem 4.3. *Let f given by* (1.1) *be in the class* $\mathcal{C}^{\ell,\wp}_{\Sigma,\Xi}(\varrho,\varpi)$ *and* $\hbar \in \mathbb{R}$. *Then*

$$| a_3-\hbar a_2^2 |\leq \begin{cases} \frac{2|\varrho|}{3\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}, & | 1-\hbar |\in [0, \frac{16(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{3|\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}) \\ |1-\hbar|\frac{|\varrho|^2}{4(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}, & |1-\hbar| \in [\frac{16(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{3|\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}, \infty). \end{cases}$$

Proof. From (2.32) and (2.35) it follows that

$$a_3 - \hbar a_2^2 \quad = \quad \frac{\varrho(p_2-q_2)\mathfrak{B}_1(x)}{6(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)} + (1-\hbar)\frac{\varrho^2[\mathfrak{B}_1(x)]^2p_1^2}{(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2}$$

From Lemma 4.3, we have $2p_2 = p_1^2 + \varkappa(4-p_1^2)$ and $2q_2 = q_1^2 + y(4-q_1^2)$, hence we get

$$p_2 - q_2 = (\frac{4-p_1^2}{2})(\varkappa - y),$$

and using triangle inequality, taking $|\varkappa| = \theta, |y| = \kappa$, we can easily obtain that

$$|a_3 - \hbar a_2^2| \leq |1-\hbar|\frac{|\varrho[\mathfrak{B}_1(x)]|^2t^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2} + \frac{|\mathfrak{B}_1(x)\varrho|}{12\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}(4-t^2)(\theta+\kappa).$$

For continence let $\mathfrak{M}(t) = |1-\hbar|\frac{9|x\varrho|^2t^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2} \geq 0$ and $\mathfrak{N}(t) = \frac{|x\varrho|}{4\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}(4-t^2) \geq 0$. Thus ,

$$|a_3 - \hbar a_2^2| \quad \leq \quad \mathfrak{M}(t) + \mathfrak{N}(t)(\theta+\kappa) = \mathfrak{L}(\theta,\kappa)$$

It is clear that the maximum of the function $\mathfrak{L}(\theta,\kappa)$ occurs at $(\theta,\kappa) = (1,1)$. Therefore,

$$\max \mathfrak{L}(\theta,\kappa) : \theta,\kappa \in [0,1] = \mathfrak{L}(1,1) = \mathfrak{M}(t) + 2\mathfrak{N}(t).$$

Let us define the function $H : [0,2] \to \mathbb{R}$ as follows

$$H(t) = \mathfrak{M}(t) + 2\mathfrak{N}(t) \tag{4.3}$$

for fixed $\varrho \in \mathbb{C} - \{0\}$. Substituting the value $\mathfrak{M}(t), \mathfrak{N}(t)$ in (4.3), we obtain

$$\begin{aligned} H(t) \quad &= \quad |1-\hbar|\frac{9|x\varrho|^2t^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2} + \frac{|x\varrho|}{2\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}(4-t^2) \\ &= \left[|1-\hbar|\frac{9|x\varrho|^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2} - \frac{|x\varrho|}{2\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}\right]t^2 \\ &\quad + \frac{2|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)} \\ &= \frac{9|x\varrho|^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}\left[|1-\hbar| - \frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{18|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}\right]t^2 \\ &\quad + \frac{2|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)} \end{aligned}$$

Now, we must investigate the maximum of the function $H(t)$ in the interval $[0,2]$. By simple computation, we have

$$H'(t) = \frac{18|x\varrho|^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}\left[|1-\hbar| - \frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{18|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}\right]t.$$

It is clear that $H'(t) \leq 0$ if

$$\frac{18|x\varrho|^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}\left[|1-\hbar| - \frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{18|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}\right] \leq 0;$$

that is if $|1-\hbar| \in \left(0, \frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{18|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}\right)$. Thus, the function $H(t)$ is a strictly decreasing function if$|1-\hbar| \in \left(0, \frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{18|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}\right)$.

Therefore,

$$\max\{H(t) : t \in [0,2]\} = H(0) = \frac{2|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}.$$

Also, $H'(t) \geq 0$; that is, the function $H(t)$ is an increasing function for $|1-\hbar| \geq \frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{18|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}$. Therefore,

$$\max\{H(t) : t \in [0,2]\} = H(2) = |1-\hbar|\frac{9|x\varrho|^2 t^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}.$$

Thus we get

$$|\ a_3 - \hbar a_2^2\ | \leq \begin{cases} \frac{2|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}, & |\ 1-\hbar\ | \in [0, \frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{18|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}) \\ |1-\hbar|\frac{9|x\varrho|^2 t^2}{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}, & |1-\hbar| \in [\frac{(5+4cos\varpi)\ [\mathfrak{Y}_2(\wp,\ell)]^2}{18|x\varrho|\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}, \infty). \end{cases}$$

In particular by taking $\hbar = 1$, we get

$$|\ a_3 - a_2^2\ | \leq \frac{2|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}.$$

□

By taking $\varrho \in \mathbb{R}$ and $\varpi = n\pi, (n \in \mathbb{Z})$ in the following theorem we prove the following Fekete-Szegö inequalities .

Theorem 4.4. *Let f given by* (1.1) *be in the class $\mathcal{C}_{\Sigma,\Xi}^{\ell,\wp}(\varrho,\varpi)$ and $\hbar \in \mathbb{R}$. Then*

$$|\ a_3 - \hbar a_2^2\ | \leq \begin{cases} \frac{2|x\varrho|}{\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}, & 0 \leq |\ \Psi(\hbar,\varpi)\ | \leq \frac{|x\varrho|}{2\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)} \\ 4|\Psi(\hbar,\varpi)|, & |\Psi(\hbar,\varpi)| \geq \frac{|\varrho|}{3\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}., \end{cases} \tag{4.4}$$

where

$$\frac{(1-\hbar)[\mathfrak{B}_1(x)]^3\varrho^2}{\{2[\varrho[\mathfrak{B}_1(x)]^2\{3(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)-4(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2\}-4\mathfrak{B}_2(x)(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2]\}}.$$

Proof. From (2.33) and (2.34) it follows that

$$\begin{aligned} a_3-\hbar a_2^2 =& \frac{(1-\hbar)[\mathfrak{B}_1(x)]^3\varrho^2(p_2+q_2)}{\{2[\varrho[\mathfrak{B}_1(x)]^2\{3(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)-4(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2\}-4\mathfrak{B}_2(x)(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2]\}} \\ &+ \frac{\varrho(p_2-q_2)\mathfrak{B}_1(x)}{6(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)} \\ =& \left[\Psi(\hbar,\varpi)+\frac{\varrho\mathfrak{B}_1(x)}{6(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)}\right]p_2+\left[\Psi(\hbar,\varpi)-\frac{\varrho\mathfrak{B}_1(x)}{6(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)}\right]q_2, \end{aligned}$$

where

$$\Psi(\hbar,\varpi)=\frac{(1-\hbar)[\mathfrak{B}_1(x)]^3\varrho^2}{\{2[\varrho[\mathfrak{B}_1(x)]^2\{3(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)-4(2+e^{i\varpi})[\mathfrak{Y}_2(\wp,\ell)]^2\}-4\mathfrak{B}_2(x)(2+e^{i\varpi})^2[\mathfrak{Y}_2(\wp,\ell)]^2]\}}.$$

Now by using (1.10) we get

$$a_3-\hbar a_2^2 \quad = \quad \left[\Psi(\hbar,\varpi)+\frac{3x\varrho}{6(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)}\right]p_2+\left[\Psi(\hbar,\varpi)-\frac{3x\varrho}{6(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)}\right]q_2,$$

Thus by applying Lemmas 2.1 4.2 and using (1.10) we get the desired result as given below

$$|\ a_3-\hbar a_2^2\ |\leq \begin{cases} \frac{2|x\varrho|}{\mathfrak{Y}_3(\wp,\ell)\sqrt{34+30cos\varpi}}, & 0\leq|\ \Psi(\hbar,\varpi)\ |\leq \frac{|x\varrho|}{2\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)} \\ 4|\Psi(\hbar,\varpi)|, & |\Psi(\hbar,\varpi)|\geq \frac{|\varrho|}{3\sqrt{34+30cos\varpi}\ \mathfrak{Y}_3(\wp,\ell)}. \end{cases}$$

In particular by taking $\hbar=1$, we get

$$|\ a_3-a_2^2\ |\leq \frac{2|x\varrho|}{(5+3e^{i\varpi})\mathfrak{Y}_3(\wp,\ell)}.$$

□

Theorem 4.5. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{G}_{\Sigma}^{\ell}(\varrho,\mu,\Xi)$. Then*

$$|\ a_3-\hbar a_2^2\ |\leq \begin{cases} \frac{|\varrho|}{2(1-\mu)\mathfrak{Y}_3(\wp,\ell)}, & |\ 1-\hbar\ |\in[0,\frac{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2}{2|\varrho|(1-\mu)\mathfrak{Y}_3(\wp,\ell)}) \\ |1-\hbar|\frac{|\varrho|^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2}, & |1-\hbar|\in[\frac{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2}{2|\varrho|(1-\mu)\mathfrak{Y}_3(\wp,\ell)},\infty). \end{cases}$$

where $\hbar\in\mathbb{R}$

Proof. From (3.20) and (3.22) it follows that

$$a_3 - \hbar a_2^2 = (1-\hbar)\frac{\varrho^2[\mathfrak{B}_1(x)]^2 p_1^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2} + \frac{\mathfrak{B}_1(x)\varrho(p_2-q_2)}{4(1-\mu)\mathfrak{Y}_3(\wp,\ell)}$$

From Lemma 4.3, we have $2p_2 = p_1^2 + x(4-p_1^2)$ and $2q_2 = q_1^2 + y(4-q_1^2)$, hence we get

$$p_2 - q_2 = (\frac{4-p_1^2}{2})(x-y),$$

and using triangle inequality, taking $|x| = \theta, |y| = \kappa$, we can easily obtain that

$$|a_3 - \hbar a_2^2| \leq |1-\hbar|\frac{|[\mathfrak{B}_1(x)]\varrho|^2 t^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2} + \frac{|\mathfrak{B}_1(x)\varrho|}{8(1-\mu)\mathfrak{Y}_3(\wp,\ell)}(4-t^2)(\theta+\kappa).$$

Using (1.10) and for convenience let

$$\mathfrak{M}_1(t) = |1-\hbar|\frac{9|x\varrho|^2 t^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2} \geq 0$$

and

$$\mathfrak{N}_1(t) = \frac{3|x\varrho|}{8(1-\mu)\mathfrak{Y}_3(\wp,\ell)}(4-t^2)(\theta+\kappa) \geq 0.$$

Thus ,

$$|a_3 - \hbar a_2^2| \leq \mathfrak{M}_1(t) + \mathfrak{N}_1(t)(\theta+\kappa) = \mathfrak{L}_1(\theta,\kappa)$$

It is clear that the maximum of the function $\mathfrak{L}_1(\theta,\kappa)$ occurs at $(\theta,\kappa) = (1,1)$. Therefore,

$$\max \mathfrak{L}_1(\theta,\kappa) : \theta,\kappa \in [0,1] = \mathfrak{L}_1(1,1) = \mathfrak{M}_1(t) + 2\mathfrak{N}_1(t).$$

Let us define the function $H_1 : [0,2] \to \mathbb{R}$ as follows

$$H_1(t) = \mathfrak{M}_1(t) + 2\mathfrak{N}_1(t) \tag{4.5}$$

for fixed $\varrho \in \mathbb{C} - \{0\}$. Substituting the value $\mathfrak{M}_1(t), \mathfrak{N}_1(t)$ in (4.5), we obtain

$$\begin{aligned} H_1(t) &= |1-\hbar|\frac{9|x\varrho|^2 t^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2} + \frac{3|x\varrho|}{4(1-\mu)\mathfrak{Y}_3(\wp,\ell)}(4-t^2) \\ &= \left[|1-\hbar|\frac{9|x\varrho|^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2} - \frac{3|x\varrho|}{4(1-\mu)\mathfrak{Y}_3(\wp,\ell)}\right]t^2 + \frac{3|x\varrho|}{(1-\mu)\mathfrak{Y}_3(\wp,\ell)} \\ &= \frac{9|x\varrho|^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2}\left[|1-\hbar| - \frac{(1-\mu)[\mathfrak{Y}_2(\wp,\ell)]^2}{12|x\varrho|\mathfrak{Y}_3(\wp,\ell)}\right]t^2 + \frac{3|x\varrho|}{(1-\mu)\mathfrak{Y}_3(\wp,\ell)} \end{aligned}$$

Now, we must investigate the maximum of the function $H_1(t)$ in the interval $[0, 2]$. By simple computation, we have

$$H_1'(t) = \frac{18|x\varrho|^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2}\left[|1-\hbar| - \frac{(1-\mu)[\mathfrak{Y}_2(\wp,\ell)]^2}{12|x\varrho|\mathfrak{Y}_3(\wp,\ell)}\right]t.$$

It is clear that $H_1'(t) \leq 0$ if$\frac{18|x\varrho|^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2}\left[|1-\hbar| - \frac{(1-\mu)[\mathfrak{Y}_2(\wp,\ell)]^2}{12|x\varrho|\mathfrak{Y}_3(\wp,\ell)}\right] \leq 0$; that is if $|1-\hbar| \in \left(0, \frac{(1-\mu)[\mathfrak{Y}_2(\wp,\ell)]^2}{12|x\varrho|\mathfrak{Y}_3(\wp,\ell)}\right)$. Thus, the function $H(t)$ is a strictly decreasing function if$|1-\hbar| \in \left(0, \frac{(1-\mu)[\mathfrak{Y}_2(\wp,\ell)]^2}{12|x\varrho|\mathfrak{Y}_3(\wp,\ell)}\right)$.

Therefore,

$$\max\{H_1(t) : t \in [0, 2]\} = H_1(0) = \frac{3|x\varrho|}{(1-\mu)\mathfrak{Y}_3(\wp,\ell)}.$$

Also, $H'(t) \geq 0$; that is, the function $H(t)$ is an increasing function for $|1-\hbar| \geq \frac{(1-\mu)[\mathfrak{Y}_2(\wp,\ell)]^2}{8\mathfrak{B}_1(x)|\varrho|\mathfrak{Y}_3(\wp,\ell)}$. Therefore,

$$\max\{H_1(t) : t \in [0, 2]\} = H_1(2) = |1-\hbar|\frac{36|x\varrho|^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2}.$$

Thus we get

$$|\ a_3 - \hbar a_2^2\ | \leq \begin{cases} \frac{3|x\varrho|}{(1-\mu)\mathfrak{Y}_3(\wp,\ell)}, & |\ 1-\hbar\ | \in [0, \frac{(1-\mu)[\mathfrak{Y}_2(\wp,\ell)]^2}{12|x\varrho|\mathfrak{Y}_3(\wp,\ell)}) \\ |1-\hbar|\frac{36|x\varrho|^2}{(1-\mu)^2[\mathfrak{Y}_2(\wp,\ell)]^2}, & |1-\hbar| \in [\frac{(1-\mu)[\mathfrak{Y}_2(\wp,\ell)]^2}{12|x\varrho|\mathfrak{Y}_3(\wp,\ell)}, \infty). \end{cases}$$

In particular by taking $\hbar = 1$, we get

$$|\ a_3 - a_2^2\ | \leq \frac{3|x\varrho|}{(1-\mu)\mathfrak{Y}_3(\wp,\ell)}.$$

□

By taking $\varrho \in \mathbb{R}$ and $\varpi = n\pi, (n \in \mathbb{Z})$ in the following theorem we prove the following Fekete-Szegö inequalities .

Theorem 4.6. *Let the function $f(z)$ given by* (1.1) *be in the class $\mathcal{G}_\Sigma^\ell(\varrho, \mu, \Xi)$and $\hbar \in \mathbb{R}$. Then*

$$|\ a_3 - \hbar a_2^2\ | \leq \begin{cases} \frac{3|x\varrho|}{(1-\mu)\mathfrak{Y}_3(\wp,\ell)}, & 0 \leq |\ \Psi(\hbar,\varpi)\ | \leq \frac{3x\varrho}{4(1-\mu)\mathfrak{Y}_3(\wp,\ell)} \\ 4|\Psi(\hbar,\varpi)|, & |\Psi(\hbar,\varpi)| \geq \frac{3x\varrho}{4(1-\mu)\mathfrak{Y}_3(\wp,\ell)}, \end{cases} \tag{4.6}$$

where

$$\frac{(1-\hbar)[\mathfrak{B}_1(x)]^3\varrho^2}{[2\varrho(\mu^2-1)[\mathfrak{B}_1(x)]^2 - 2\mathfrak{B}_2(x)(1-\mu)^2][\mathfrak{Y}_2(\wp,\ell)]^2 + 4\varrho(1-\mu)[\mathfrak{B}_1(x)]^2\mathfrak{Y}_3(\wp,\ell)}.$$

Proof. From (3.20) and (3.22) it follows that

$$
\begin{aligned}
a_3 - \hbar a_2^2 &= \frac{(1-\hbar)[\mathfrak{B}_1(x)]^3 \varrho^2 (p_2+q_2)}{[2\varrho(\mu^2-1)[\mathfrak{B}_1(x)]^2 - 2\mathfrak{B}_2(x)(1-\mu)^2][\mathfrak{Y}_2(\wp,\ell)]^2 + 4\varrho(1-\mu)[\mathfrak{B}_1(x)]^2 \mathfrak{Y}_3(\wp,\ell)} \\
&\quad + \frac{\varrho \mathfrak{B}_1(x)(p_2-q_2)}{4(1-\mu)\mathfrak{Y}_3(\wp,\ell)} \\
&= \left[\Psi(\hbar,\varpi) + \frac{\varrho\mathfrak{B}_1(x)}{4(1-\mu)\mathfrak{Y}_3(\wp,\ell)}\right] p_2 + \left[\Psi(\hbar,\varpi) - \frac{\varrho\mathfrak{B}_1(x)}{4(1-\mu)\mathfrak{Y}_3(\wp,\ell)}\right] q_2,
\end{aligned}
$$

where

$$
\Psi(\hbar,\varpi) = \frac{(1-\hbar)[\mathfrak{B}_1(x)]^3 \varrho^2}{[2\varrho(\mu^2-1)[\mathfrak{B}_1(x)]^2 - 2\mathfrak{B}_2(x)(1-\mu)^2][\mathfrak{Y}_2(\wp,\ell)]^2 + 4\varrho(1-\mu)[\mathfrak{B}_1(x)]^2 \mathfrak{Y}_3(\wp,\ell)}.
$$

Using (1.10)

$$
a_3 - \hbar a_2^2 = \left[\Psi(\hbar,\varpi) + \frac{3x\varrho}{4(1-\mu)\mathfrak{Y}_3(\wp,\ell)}\right] p_2 + \left[\Psi(\hbar,\varpi) - \frac{3x\varrho}{4(1-\mu)\mathfrak{Y}_3(\wp,\ell)}\right] q_2,
$$

and

$$
\Psi(\hbar,\varpi) = \frac{27x^3\varrho^2(1-\hbar)}{[18x^2\varrho(\mu^2-1) - 2(18x^2-1)(1-\mu)^2][\mathfrak{Y}_2(\wp,\ell)]^2 + 36x^2\varrho(1-\mu)\mathfrak{Y}_3(\wp,\ell)}.
$$

Thus by applying Lemmas 2.1 4.2 and we get the desired result as given below

$$
| a_3 - \hbar a_2^2 | \leq \begin{cases} \frac{3|x\varrho|}{(1-\mu)\mathfrak{Y}_3(\wp,\ell)}, & 0 \leq | \Psi(\hbar,\varpi) | \leq \frac{3x\varrho}{4(1-\mu)\mathfrak{Y}_3(\wp,\ell)} \\ 4|\Psi(\hbar,\varpi)|, & |\Psi(\hbar,\varpi)| \geq \frac{3x\varrho}{4(1-\mu)\mathfrak{Y}_3(\wp,\ell)}. \end{cases}
$$

In particular by taking $\hbar = 1$, we get

$$
| a_3 - a_2^2 | \leq \frac{3|x\varrho|}{(1-\mu)\mathfrak{Y}_3(\wp,\ell)}.
$$

□

5 Concluding remarks

We defined three new subclasses of bi-starlike and bi-convex function of complex order concerning Rabotnov function in the open unit disc and found initial coefficients of functions in these classes related with Lucas-Balancing Polynomials. Moreover, we determined the Fekete-Szegö inequalities for function in these classes. Some significance of the results which are new one can derive by fixing $\varpi = \pi$ in Theorems (2.1) and (2.2) , to get the coefficient estimates for $f \in \mathcal{S}_{\Sigma,\Xi}^{\ell,\wp}(\varrho)$ and $f \in \mathcal{C}_{\Sigma,\Xi}^{\ell,\wp}(\varrho)$ defined in Remark 1.1. Also by taking $\varrho = 1$ in Theorems (2.1) and (2.2) , we can stae the coefficient estimates for $f \in \mathcal{S}_{\Sigma,\Xi}^{\ell,\wp}(\varpi)$ and $f \in \mathcal{C}_{\Sigma,\Xi}^{\ell,\wp}(\varpi)$ defined in Remark 1.4 we left this as exercise to interested readers. Also we footnoted that lately various subclasses of starlike functions were familiarized see [7, 19, 34] by fixing some particular functions such as functions linked with Bell numbers, shell-like curve linked with Fibonacci numbers and nephroid domains. Moreover, functions allied with conic domains and rational functions instead of Ξ in (2.3) one can determine new results for the subclasses presented in this paper. The results of the research could be expanded to develop q-fractional calculus, extending the results for bi-univalent functions. Furthermore, we can use the $q-$differential and integral operator and the $q-$fractional differential and integral operator to construct a subclass of univalent and bi-univalent functions. Fekete-Szego inequalities and approximated coefficient constraints can then be obtained for these functions.

Data Availability

No data were used in this paper.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors confirm no competing interests.

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