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Hamiltonicity of doubly semi-equivelar maps on the torus

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Abstract

There are 22 types of doubly semi-equivelar maps, with curvature 0, on the plane which provide infinitely many doubly semi-equivelar maps of respective types on the torus. In this article, we show that every such doubly semi-equivelar map on the torus contains a Hamiltonian cycle. As a consequence, we establish the Nash-Williams conjecture for the graphs associated with these doubly semi-equivelar maps by showing that these graphs are either 3-connected or 4-connected.

1 Introduction

A cycle C in a graph G is called Hamiltonian cycle if it passes through every vertex of G. A graph G is called Hamiltonian if it contains such a cycle. The existence of a Hamiltonian cycle is one of the important properties of a graph and the problem to find a Hamiltonian cycle is NP-complete.

An embedding of a connected, simple graph G into a (closed) surface F is called a map M on F if the closure of each component of $F \setminus G$ is a p-gonal 2-disk D_p, $(p \geq 3)$, also called face, such that the non-empty intersection of any two faces is either a vertex or an edge. A map M having only triangular faces is also called a triangulation. The face-sequence of a vertex v in a map M is a finite sequence $(p_1^{n_1}, \ldots, p_k^{n_k})$ such that n_i numbers of D_{p_i} incident at v in the given cyclic order, for $1 \leq i \leq k$. The combinatorial curvature of a vertex v with the face sequence $(p_1^{n_1}, \ldots, p_k^{n_k})$, denoted by $\phi(v)$, is defined as $\phi(v) =$

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$1-(\sum_{i=1}^{k} n_i)/2+(\sum_{i=1}^{k} n_i/p_i)$. A map M has the combinatorial curvature k if $\phi(v) = k$ for each vertex v in M. A map M is called a semi-equivelar of type $[p_1^{n_1}.\ldots.p_k^{n_k}]$ if the face-sequence of each vertex is $(p_1^{n_1},\ldots,p_k^{n_k})$, see [19]. Thus, every semi-equivelar map has a constant combinatorial curvature.

Let K_v denote the set of all the faces incident at a vertex v in a map M. The geometric career $|K_v|$ (the union of all faces in K_v) is a 2-disk with the boundary cycle C. The cycle C is called the link of v and is denoted as $C = \mathrm{lk}(v)$. Let $\mathrm{lk}(v) = C_k(v_1, v_2, \ldots, v_k)$, that is, a cycle of length k. We say that the face-sequence of $\mathrm{lk}(v)$ is $(f_1^{n_1}.\ldots.f_l^{n_l})$, if the consecutive n_i numbers vertices in $\mathrm{lk}(v)$ have the face-sequence f_i in the given cyclic order for $1 \leq i \leq l$.

A map M having k distinct face-sequences is called a k-semi-equivelar map, see [20]. In particular, a map having 2 distinct face-sequences is a 2-semi-equivelar map. Let M be a 2-semi-equivelar map with two distinct face-sequences f_1 and f_2. We say that M is a doubly semi-equivelar map (DSEM), if vertices with similar type face-sequence have links with same face-sequence. We denote the map M of type $[f_1^{(f_{11},\ldots,f_{1r_1})} : f_2^{(f_{21},\ldots,f_{2r_2})}]$, where f_{1i} or f_{2j} is f_1 or f_2, for $1 \leq i \leq r_1$ and $1 \leq j \leq r_2$, if vertices with the face-sequence f_1 have links of face-sequence $(f_{11},\ldots,f_{1r_1})$ and vertices with f_2 have links of face-sequence $(f_{21},\ldots,f_{2r_2})$. Observe that f_{ij} is either f_1 or f_2 for each i, j. See the examples of DSEMs on the plane in Example 3.

A graph G is called k-connected if the removal of any $k-1$ vertices leaves a non-trivial connected graph. The connectivity $\kappa(G)$ of G is the maximum non-negative integer k for which G is k-connected. A map is called k-connected (resp. Hamiltonian) if the underlying graph is k-connected (resp. Hamiltonian).

The connectivity of a graph plays an important role to discuss Hamiltonicity. In the case of maps, significant progress has been made using the connectivity of the underlying graphs. It goes back to 1931 when Whitney [22] initiated the theory of Hamiltonian cycle in maps and he used the connectivity of the underlying graphs. He proved that every triangulation of the sphere is Hamiltonian if it is 4-connected. Tutte [21] generalized this result for an arbitrary 4-connected map on the sphere. Grünbaum [6] conjectured that every 4-connected map on the Projective plane is Hamiltonian, which was proved by Thomas and Yu [16]. Brunet et al. [5] showed that every 5-connected triangulation on the Klein bottle is Hamiltonian. For the torus, Grünbaum [6] and Nash-Williams [12] posed the following conjecture.

Conjecture 1. Every 4-connected map on the torus is Hamiltonian.

The above conjecture is yet to prove, however, several interesting results have been established so far. Altshuler [1] established the conjecture for every

6-connected map on the torus. Brunet and Richter [4] established the conjecture for every 5-connected triangulation on the torus, which was further improved by Thomas and Yu [17] for any 5-connected map. Kawarabayashi and Ozeki [7] proved the conjecture for any 4-connected triangulation on the torus. The current progress on connectivity and Hamiltonicity of the underlying graphs on the torus can be found in [8, 11, 13, 18].

There are exactly 11 types of semi-equivelar maps on the torus, namely $[3^6], [4^4], [6^3], [3.12^2], [4.8^2], [4.6.12], [3^4.6], [3^3.4^2], [3^2.4.3.4], [3.6.3.6]$ and $[3^4.6]$. Altshuler [1] showed that all the semi-equivelar maps on the torus of types $[4^4]$ and $[3^6]$ are Hamiltonian. Bouwer and Chernoff [3] proved that every semi-equivelar map on the torus of type $[6^3]$ is Hamiltonian. Recently, Maity and Upadhyay [10] showed that all the semi-equivelar map on the torus of the remaining eight types are also Hamiltonian. In [14], authors have shown that:

Proposition 1. *[14] There are exactly* 22 *types of DSEMs on the plane* $\mathbb{R}^2$ *with combinatorial curvature* 0*.*

The above types T_i's, $1 \leq i \leq 22$, are given in Example 3. Further, the authors have shown that for each type T_i, $1 \leq i \leq 22$, there is a doubly semi-equivelar map on the torus. Recently, the authors [15] have shown that:

Proposition 2. *Let* M *be a DSEM of type* T *with combinatorial curvature 0 on the Klein bottle. Then* $T \in \{T_1, T_2, \ldots, T_{22}\} \setminus \{T_{11}, T_{13}, T_{20}\}$*.*

Since the plane is the universal cover of the torus and Klein bottle, it is natural to explore the Hamiltonicity of these types of DSEMs on the torus and Klein bottle also. In the present work, we determine the Hamiltonicity of these maps on the torus and show:

Theorem 1. *Every DSEM on the torus of the type* T_i*,* $1 \leq i \leq 22$*, is Hamiltonian.*

Theorem 2. *Let* M *be a DSEM of type* T *on the torus.*

1. *If* $T \in \{T_i\} \setminus \{T_{16}, T_{17}\}$*, then* M *is* 4*-connected.*
2. *If* T *is* T_{16} *or* T_{17}*, then* M *is* 3*-connected.*

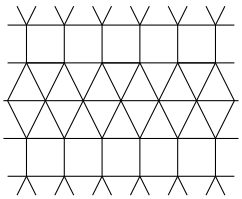
Corollary 1. *All the* 4*-connected DSEMs on the torus of type* T_i *are Hamiltonian.*

2 Definitions and notations

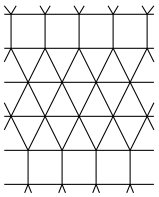
Let G be the underlying graph of a map M with the vertex set $V(G)$ and the edge set $E(G)$. Let u-v denote the edge joining $u, v \in V(G)$. The notation

$P = P(u_1, \ldots, u_n)$ denotes a path u_1-u_2- $\cdots$ -u_n. A path with a single vertex u is referred to as a path of length zero, and is denoted as $P = P(u)$. In P, the vertices u_1 and u_2 are called the boundary vertices, and u_i, for $2 \leq i \leq n-1$, are called the inner vertices. A path P_1 is an extension of another path P_2 if P_2 is a proper sub graph of P_1. A path $P = P(u_1, \ldots, u_n)$ is called a cycle, if it is closed, that is, $u_1 = u_n$. We denote a cycle by $C = C_n(u_1, \ldots, u_n)$. A cycle C is contractible if it bounds a 2-disk D_p, otherwise called non-contractible. The notations $G_1 \cup G_2$ and $G_1 \cap G_2$ denote the usual union and intersection of two graphs G_1 and G_2. For $U \subset V(G)$, $G - U$ is the graph obtained from G by deleting the vertices of U. For the graph theory related terms, we refer to Bondy and Murty [2].

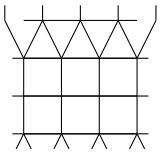
3 Examples: DSEMs on the plane $\mathbb{R}^2$



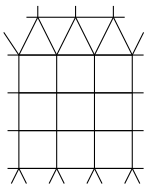
N_1 **of Type** T_1**:**
$[f_1^{({f_1}^2 \cdot f_2 \cdot {f_1}^2 \cdot f_2)} : f_2^{({f_1}^5 \cdot {f_2}^2)}]$
$(f_1, f_2) = ((3^6), (3^3.4^2))$



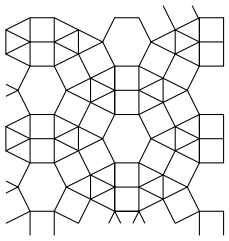
N_2 **of Type** T_2**:**
$[f_1^{({f_1}^2 \cdot {f_2}^4)} : f_2^{({f_1}^5 \cdot {f_2}^2)}]$
$(f_1, f_2) = ((3^6), (3^3.4^2))$



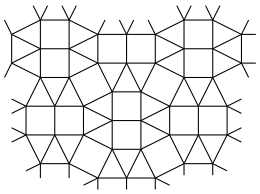
N_3 **of Type** T_3**:**
$[f_1^{({f_1}^4 \cdot {f_2}^3)} : f_2^{({f_1}^3 \cdot f_2 \cdot {f_1}^3 \cdot f_2)}]$
$(f_1, f_2) = ((3^3.4^2), (4^4))$



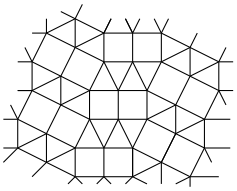
N_4 **of Type** T_4**:**
$[f_1^{({f_1}^4 \cdot {f_2}^3)} : f_2^{({f_1}^3 \cdot {f_2}^5)}]$
$(f_1, f_2) = ((3^3.4^2), (4^4))$



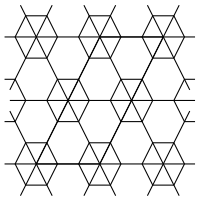
N_5 **of Type** T_5**:**
$[f_1^{({f_2}^2 \cdot f_1 \cdot {f_2}^2 \cdot {f_1}^2)} : f_2^{({f_2}^5 \cdot {f_1}^4)}]$
$(f_1, f_2) = ((3^3.4^2), (3.4.6.4))$



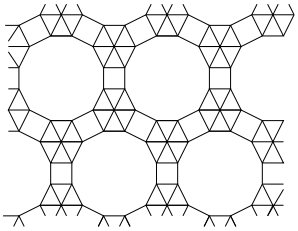
N_6 **of Type** T_6**:**
$[f_1^{(f_1 \cdot {f_2}^6)} : f_2^{({f_1}^2 \cdot {f_2}^3 \cdot f_1 \cdot f_2)}]$
$(f_1, f_2) = ((3^3.4^2), (3^2.4.3.4))$



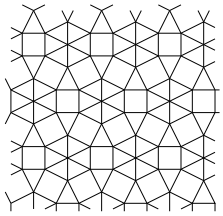
N_7 **of Type** T_7**:**
$[{f_1}^{({f_1}\cdot{f_2}^3\cdot{f_1}\cdot{f_2}^2)} : {f_2}^{({f_1}^3\cdot{f_2}\cdot{f_1}^2\cdot{f_2})}]$
$(f_1, f_2) = ((3^3.4^2), (3^2.4.3.4))$



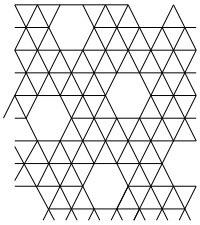
N_8 **of Type** T_8**:**
$[{f_1}^{({f_2}^6)} : {f_2}^{({f_1}\cdot{f_2}^9)}]$
$(f_1, f_2) = ((3^6), (3^2.6^2))$



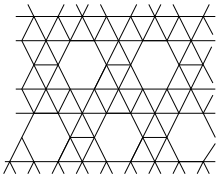
N_9 **of Type** T_9**:**
$[{f_1}^{({f_2}^6)} : {f_2}^{({f_1}\cdot{f_2}^{13})}]$
$(f_1, f_2) = ((3^6), (3^2.4.12))$



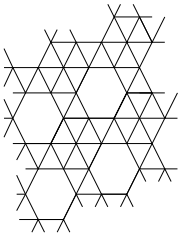
N_{10} **of Type** T_{10}**:**
$[{f_1}^{({f_2}^6)} : {f_2}^{({f_1}\cdot{f_2}^6)}]$
$(f_1, f_2) = ((3^6), (3^2.4.3.4))$



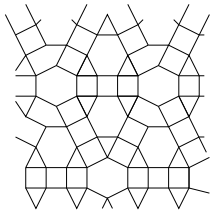
N_{11} **of Type** T_{11}**:**
$[{f_1}^{({f_1}^2\cdot{f_2}^2\cdot{f_1}\cdot{f_2})} : {f_2}^{({f_2}^5\cdot{f_1}^3)}]$
$(f_1, f_2) = ((3^6), (3^4.6))$



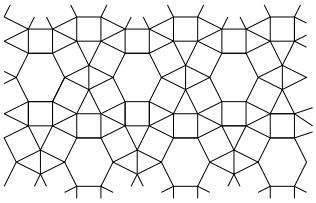
N_{12} **of Type** T_{12}**:**
$[{f_1}^{({f_2}^6)} : {f_2}^{({f_2}^5\cdot{f_1}\cdot{f_2}\cdot{f_1})}]$
$(f_1, f_2) = ((3^6), (3^4.6))$



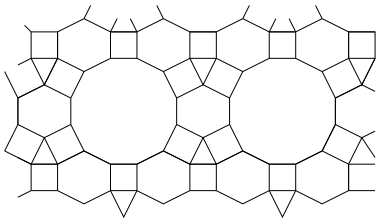
N_{13} **of Type** T_{13}**:**
$[{f_1}^{({f_2}^6)} : {f_2}^{({f_2}^7\cdot{f_1})}]$
$(f_1, f_2) = ((3^6), (3^4.6))$



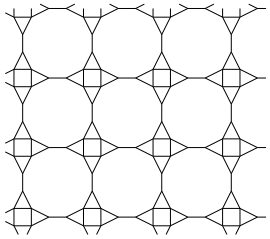
N_{14} **of Type** T_{14}**:**
$[{f_1}^{({f_1}^5\cdot{f_2}^4)} : {f_2}^{({f_1}^2\cdot{f_2}^7)}]$
$(f_1, f_2) = ((3.4.6.4), (3.4^2.6))$



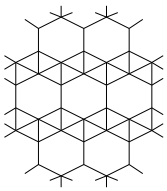
N_{15} **of Type** T_{15}**:**
$[{f_1}^{({f_2}^2\cdot{f_1}\cdot{f_2}^2\cdot{f_1}^2)} : {f_2}^{({f_2}^5\cdot{f_1}^4)}]$
$(f_1, f_2) = ((3^2.4.3.4), (3.4.6.4))$



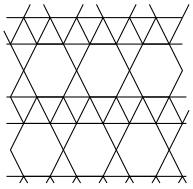
N_{16} **of Type** T_{16}**:**
$[f_1^{(f_2{}^3 \cdot f_1{}^2 \cdot f_2{}^3 \cdot f_1)} : f_2^{(f_2{}^{11} \cdot f_1 \cdot f_2{}^2 \cdot f_1{}^2)}]$
$(f_1, f_2) = ((3.4.6.4), (4.6.12))$



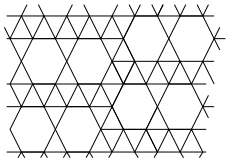
N_{17} **of Type** T_{17}**:**
$[f_{11}^{F_1} : f_{12}^{F_2}]$
$(f_1, f_2) = ((3.4.3.12), (3.12^2))$
$F_1 = (f_2^2 . f_1 . f_2^2 . f_1 . f_2^2 . f_1 . f_2^2 . f_1^3)$
$F_2 = (f_1 . f_2^2 . f_1 . f_2^2 . f_1 . f_2^2 . f_1 . f_2 . f_1 . f_2^2 . f_1 . f_2^2 . f_1 . f_2^2 . f_1)$



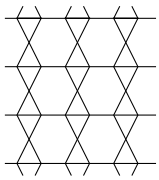
N_{18} **of Type** T_{18}**:**
$[f_1^{(f_2^2 . f_1 . f_2 . f_1 . f_2^2 . f_1)} : f_2^{(f_2^2 . f_1^3 . f_2^2 . f_1 . f_2 . f_1)}]$
$(f_1, f_2) = ((3^4.6), (3^2.6^2))$



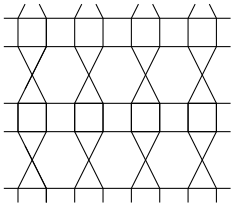
N_{19} **of Type** T_{19}**:**
$[f_1^{(f_1^4 . f_2 . f_1^2 . f_2)} : f_2^{(f_1^4 . f_2 . f_1^4 . f_2)}]$
$(f_1, f_2) = ((3^4.6), (3.6.3.6))$



N_{20} **of Type** T_{20}**:**
$[f_1^{(f_1^7 . f_2)} : f_2^{(f_1^{10})}]$
$(f_1, f_2) = ((3^4.6), (3.6.3.6))$



N_{21} **of Type** T_{21}**:**
$[f_1^{F_1} : f_2^{F_2}]$
$F_1 = (f_2{}^4 . f_1 . f_2{}^4 . f_1)$
$F_2 = (f_2{}^2 . f_1 . f_2 . f_1 . f_2{}^2 . f_1 . f_2 . f_1)$
$(f_1, f_2) = ((3.6.3.6), (3^2.6^2))$



M **of type** T_{22}**:**
$[f_1^{(f_1^5 . f_2 . f_1^2 . f_2)} : f_2^{(f_1^4 . f_2 . f_1^4 . f_2)}]$
$(f_1 = (3.4^2.6),\ f_2 = (3.6.3.6))$

Figure 3.1: Doubly semi-equivelar maps on the plane $\mathbb{R}^2$

4 Proof of Theorem 2

The Nash-Williams conjecture states about the Hamiltonicity of a map with respect to the k-connectivity of the underlying graph. To establish the con-

jecture in our context, here, we determine the k-connectivity of the graphs associated with certain DSEMs.

Lemma 1. *Let M be a DSEM on the torus of type T_i, $1 \leq i \leq 22$. Then removal of any two distinct vertices leaves the map connected.*

Proof. Let $V(G)$ denote the vertex set of the underlying graph G of M. Note that G is connected. Suppose, if possible, there is a vertex $z_1 \in V(G)$ such that $G' = G - z_1$ is disconnected. Then, there exist distinct vertices $u, v \in V(G)$ such that every path between u and v passes through z_1. Choose such a path $P = u\text{-}a_1\text{-}a_2\text{-} \cdots \text{-}a_{n-1}\text{-}z_1\text{-}a_{n+1}\text{-} \cdots \text{-}v$. Let $\mathrm{lk}(z_1) = C_i(w_1, w_2, \ldots, w_i)$, $P_1 = u\text{-}a_1\text{-}a_2\text{-} \cdots \text{-}a_{n-1}$ and $P_2 = a_{n+1}\text{-}a_{n+2}\text{-} \cdots \text{-}v$. Note that $a_{n-1}, a_{n+1} \in \mathrm{lk}(z_1)$, there is a path P_3 between a_{n-1} and a_{n+1} in $\mathrm{lk}(z_1)$. Now, observe that $\hat{P} = P_1 \cup P_2 \cup P_3$ is a connected sub graph of G' such that $u, v \in V(\hat{P})$ and $z_1 \notin V(\hat{P})$. Therefore G' is connected.

Suppose, if possible, there is a vertex $z_2 \in V(G)$ such that $G'' = G' - z_2$ is disconnected. Then, there exist distinct vertices $u', v' \in G''$ such that z_2 is a vertex of any path connecting u' and v', say $P' = u'\text{-}b_1\text{-}b_2\text{-} \cdots \text{-}b_{n-1}\text{-}b_n\text{-}b_{n+1}\text{-} \cdots \text{-}v'$. Then, either $z_2 \in \mathrm{lk}(z_1)$ or $z_2 \notin \mathrm{lk}(z_1)$.

First, suppose $z_2 \notin \mathrm{lk}(z_1)$, then G'' is connected by a similar argument used for G'. Now suppose, $z_2 \in \mathrm{lk}(z_1)$. Then $z_2 = w_k$ and $z_2\text{-}w_{k-1}$, $z_2\text{-}w_{k+1}$ are edges for some $1 \leq k \leq i$. Since G' is connected, we get paths P_1' from u' to w_{k+1} and P_2' from w_{k-1} to v' which are not passing through z_2. Also, we get a path P_3' from w_{k-1} to w_{k+1} in $\mathrm{lk}(z_2)$. Note that P_3' is a path in G not passing through z_1. Thus we get a connected sub graph $\hat{P'} = P_1' \cup P_2' \cup P_3'$ such that $u', v' \in V(\hat{P'})$ and $z_2 \notin V(\hat{P'})$. Hence G'' is connected. □

The proof of the next theorem follows from Lemma 1.

Theorem 3. *Let M be doubly semi-equivelar map of types T_{16} and T_{17} on the torus. Then M is 3-connected.*

Theorem 4. *Let M be a doubly semi-equivelar map of type T on the torus, where $T \in \{T_i\} \setminus \{T_{16}, T_{17}\}$. Then M is 4-connected.*

Proof. Let $V(G)$ denote the vertex set of the underlying graph G of M, which is connected. Note that the degree of every vertex of M is grater or equal to 4. Let $z_1, z_2 \in V(G)$. Then by Lemma 1, $G' = G - \{z_1\}$ and $G'' = G - \{z_1, z_2\}$ are connected. Suppose, if possible, there is a vertex $z_3 \in V(G)$ such that $G''' = G - \{z_1, z_2, z_3\}$ is disconnected. Then we get $u, v \in V(G)$ such that there is no path between u and v in G'''. Let $\mathrm{lk}(z_1) = C_i(w_1, w_2, \cdots, w_i)$, $\mathrm{lk}(z_2) = C_j(x_1, x_2, \cdots, x_j)$ and $\mathrm{lk}(z_3) = C_k(y_1, y_2, \cdots, y_k)$. Now, as per the positions of z_1, z_2, z_3, we have the following cases:

1. $z_a \notin \mathrm{lk}(z_b)$, where $a \neq b$ and $a, b \in \{1, 2, 3\}$.

2. $z_a \in \mathrm{lk}(z_b)$ such that $z_c \notin \mathrm{lk}(z_a)$ and $z_c \notin \mathrm{lk}(z_b)$, where $a \neq b \neq c \neq a$ and $a, b, c \in \{1, 2, 3\}$.

3. $z_2, z_3 \in \mathrm{lk}(z_1)$ and $z_2 \in \mathrm{lk}(z_3)$.

4. $z_a, z_b \in \mathrm{lk}(z_c)$ such that $z_a \notin \mathrm{lk}(z_b)$, where $a \neq b \neq c \neq a$ and $a, b, c \in \{1, 2, 3\}$.

In Case 1 and 2, G''' remains connected by Lemma 1. So we discuss the remaining cases.

Case 3: Since $z_2, z_3 \in \mathrm{lk}(z_1)$, $z_2 = w_h$, $z_3 = w_l$ for some $1 \leq h, l \leq i$. Note that, w_{h-1}, w_{h+1} are adjacent to z_2 and w_{l-1}, w_{l+1} are adjacent to z_3. Hence, $w_{h-1}, w_{h+1} \in \mathrm{lk}(z_2)$, $w_{l-1}, w_{l+1} \in \mathrm{lk}(z_3)$ and $w_{h-1} = x_m, w_{h+1} = x_n$, $w_{l-1} = y_r, w_{l+1} = y_s$ for some $1 \leq m, n \leq j$, $1 \leq r, s \leq k$. Let $R = x_m$-x_{m+1}- $\cdots$ -x_{n-1}-x_n be the path in $\mathrm{lk}(z_2)$ from x_m to x_n such that $z_1 \notin V(R)$ and $S = y_r$-y_{r+1}- $\cdots$ -y_{s-1}-y_s be the path in $\mathrm{lk}(z_3)$ from y_r to y_s such that $z_1 \notin V(S)$.

Subcase 3.1: If z_2, z_3 are not adjacent such that $w_{l-1}, w_{l+1} \notin \{w_{h-1}, w_{h+1}\}$. Then $A_1 = w_{l+1}$-w_{l+2}- $\cdots$ -w_{h-2}-w_{h-1} and $A_2 = w_{h+1}$-w_{h+2}- $\cdots$ -w_{l-2}-w_{l-1} are paths in $\mathrm{lk}(z_1)$ such that $z_2, z_3 \notin V(A_1) \cup V(A_2)$. Since $z_2 \in \mathrm{lk}(z_3)$ and $z_3 \in \mathrm{lk}(z_2)$, we have $z_2 = y_p, z_3 = x_q$ for some $1 \leq p \leq k$, $1 \leq q \leq j$ and z_2-y_{p+1}, z_2-y_{p-1}, z_3-x_{q-1}, z_3-x_{q+1} are edges. Consider the path $R = x_m$-x_{m+1}- $\cdots$ -x_{q-1}-$x_q(= z_3)$-x_{q+1}- $\cdots$ -x_{n-1}-x_n in $\mathrm{lk}(z_2)$, then we get sub paths $A_3 = x_{q+1}$-x_{q+2}- $\cdots$ -$x_n(= w_{h+1})$ and $A_4 = x_{q-1}$-x_{q-2}- $\cdots$ -$x_m(= w_{h-1})$ of R. Similarly, the path $S = y_r$-y_{r+1}- $\cdots$ -y_{p-1}-$y_p(= z_2)$-y_{p+1}- $\cdots$ -y_{s-1}-y_s in $\mathrm{lk}(z_3)$ gives sub paths $A_5 = y_{p+1}$-y_{p+2}- $\cdots$ -$y_s(= w_{l+1})$ and $A_6 = y_{p-1}$-y_{p-2}- $\cdots$ -$y_r(= w_{l-1})$. In this manner, we get connected sub graphs $W_1 = A_3 \cup A_2 \cup A_6$ and $W_2 = A_5 \cup A_1 \cup A_4$ of G'''. If $V(W_1) \cap V(W_2) \neq \phi$, then $W_1 \cup W_2$ is connected, so G''' is connected. If $V(W_1) \cap V(W_2) = \phi$, then the links of all the vertices of G''' remain complete except for the vertices of W_1 and W_2 as they belong to the links of z_1, z_2 or z_3. Note that there exists either a vertex or an edge in W_1 contained in a face F_1, where F_1 is a face in M that does not contain the vertices z_1, z_2 or z_3. Similarly, for a vertex or an edge in W_2, we get a face H_1. Corresponding to the faces F_1, H_1, we get faces F_2, H_2 such that $F_1 \cap F_2 \neq \phi, H_1 \cap H_2 \neq \phi$, otherwise links of some vertices of these faces are not complete. In this manner, we get finite sequences of faces $F_1, F_2, \cdots, F_e$ and $H_1, H_2, \cdots, H_f$ corresponding to W_1 and W_2, respectively. Since M is a map on the finite number of vertices, after some stage, we get $F_u \cap H_v \neq \phi$, for some $1 \leq u \leq e$, $1 \leq v \leq f$. Therefore, every vertex of W_1 is connected to every vertex of W_2. Hence, G''' is connected.

If z_2, z_3 are not adjacent such that either $w_{l-1} = w_{h+1}$ or $w_{l+1} = w_{h-1}$, then proceeding similarly, we get connected sub graphs W_1' and W_2' of G'''. By using the similar argument as above, G''' is connected.

Subcase 3.2: Let z_2, z_3 are adjacent. As $z_2, z_3 \in \mathrm{lk}(z_1)$, $z_2 = w_h, z_3 =$

w_{h+1} for some $1 \leq h \leq i$ and z_2-w_{h-1}, z_2-$z_3(= w_{h+1})$, z_3-w_{h+2} are edges. Clearly, there is a path $B_1 = w_{h+2}$-w_{h+3}- $\cdots$ -w_{h-2}-w_{h-1} between w_{h+2} and w_{h-1} such that $z_1, z_2, z_3 \notin V(B_1)$. Since $z_3 \in \mathrm{lk}(z_1)$ and z_3-w_{h+2} is an edge, so $w_{h+2} \in \mathrm{lk}(z_3)$. Now $z_2(= w_h)$-$z_3(= w_{h+1})$ is an edge, so $z_2 \in \mathrm{lk}(z_3)$. If $z_2 = y_p$ for some $1 \leq p \leq k$ in $\mathrm{lk}(z_3)$, then $B_2 = y_{p+1}$-y_{p+2}- $\cdots$ -w_{h+2} is a path between y_{p+1}-w_{h+2} such that $z_1, z_2, z_3 \notin V(B_2)$. As w_{h-1}-z_2 and y_{p+1}-z_2 are edges, this gives $w_{h-1}, y_{p+1} \in \mathrm{lk}(z_2)$. Hence, we have a path B_3 between w_{h-1} and y_{p+1}. Thus, from above, we get a connected sub graph $W = B_1 \cup B_3 \cup B_2$ of G'''. Since G'' is connected, every vertices of G''' is connected with some vertices of W, therefore G''' is connected.

Case 4: Without loss of generality, suppose that $z_2, z_3 \in \mathrm{lk}(z_1)$ and $z_2 \notin \mathrm{lk}(z_3)$. As discussed in Case 3, $z_2, z_3 \in \mathrm{lk}(z_1)$ gives $z_2 = w_h$, $z_3 = w_l$ for some $1 \leq h, l \leq i$, $w_{h-1}, w_{h+1} \in \mathrm{lk}(z_2)$, $w_{l-1}, w_{l+1} \in \mathrm{lk}(z_3)$ and $w_{h-1} = x_m, w_{h+1} = x_n$, $w_{l-1} = y_r, w_{l+1} = y_s$ for some $1 \leq m, n \leq j$, $1 \leq r, s \leq k$. Also, we get a path $R = x_m$-x_{m+1}- $\cdots$ -x_{n-1}-x_n in $\mathrm{lk}(z_2)$ from x_m to x_n such that $z_1 \notin V(R)$ and a path $S = y_r$-y_{r+1}- $\cdots$ -y_{s-1}-y_s in $\mathrm{lk}(z_3)$ from y_r to y_s such that $z_1 \notin V(S)$.

If z_2, z_3 are such that $w_{l-1}, w_{l+1} \notin \{w_{h-1}, w_{h+1}\}$. Then $A_1 = w_{l+1}$-w_{l+2}- $\cdots$ -w_{h-2}-w_{h-1} and $A_2 = w_{h+1}$-w_{h+2}- $\cdots$ -w_{l-2}-w_{l-1} are paths in $\mathrm{lk}(z_1)$ such that $z_2, z_3 \notin V(A_1) \cup V(A_2)$. Clearly, $Z = R \cup A_2 \cup S \cup A_3$ is a connected sub graph of G'''. Therefore G''' is connected.

If z_2, z_3 are such that either $w_{l-1} = w_{h+1}$ or $w_{l+1} = w_{h-1}$, then proceeding similarly, we get a connected sub graph Z' of G'''. By using the similar argument as above, G''' is connected.

Hence, by the Cases 1-4, M is 4-connected. □

Proof of Theorem 2: The proof follows from Theorem 3 and Theorem 4. □

5 Proof of Theorem 1 and Corollary 1

To explore a Hamiltonian cycle in a DSEM M, we construct its planar representation (called $M(i, j, k)$ representation) by cutting M along two non-homologous cycles at a vertex. Evidently, finding a Hamiltonian cycle in M is equivalent to find a Hamiltonian cycle in its $M(i, j, k)$. We give a detailed description of $M(i, j, k)$ representations for DSEMs of types T_1 and T_2 and avoid a similar discussion for the remaining types. Throughout this section by a DSEM, we mean a doubly semi-equivelar map on the torus. Let M be a DSEM of type $[f_1 : f_2]$ with the vertex set $V(M)$. For $j = 1, 2$, let V_{f_j} denote the set of vertices with face-sequence f_j. The notations $|V(M)|$, $|V_{f_j}|$ denote the cardinality of the vertex set $V(M)$, V_{f_j} respectively.

5.1 DSEMs of types $T_1 = [f_1^{(f_2^2 \cdot f_1 \cdot f_2^2 \cdot f_1)} : f_2^{(f_1^2 \cdot f_2^5)}]$ and $T_2 = [f_1^{(f_1^4 \cdot f_2^2)} : f_2^{(f_1^2 \cdot f_2^5)}]$, where $f_1 = (3^6)$, $f_2 = (3^3.4^2)$

Let M^r be a DSEM of type T_r with the vertex set $V(M^r)$, where $r \in \{1, 2\}$. For M^1, we see easily that the number of triangular faces is $4|V_{(3^6)}|$ or $2|V_{(3^3.4^2)}|$. Thus, if M^1 exists, then $2|V_{(3^6)}| = |V_{(3^3.4^2)}|$. Similarly, for the existence of M^2, we get $|V_{(3^6)}| = |V_{(3^3.4^2)}|$. Let u be a vertex with face-sequence (3^6) or $(3^3.4^2)$. We denote their respective links by $\mathrm{lk}(u) = C_6(u_1, u_2, u_3, u_4, u_5, u_6)$ or $\mathrm{lk}(u) = C_7(\boldsymbol{u_1}, u_2, \boldsymbol{u_3}, u_4, u_5, u_6, u_7)$ (labeling of vertices in the links is considered anti-clockwise). The bold appearance of some u_i means u is not adjacent with u_i by an edge. Now, we define certain types of paths in M^r as follows.

Definition 1. A path $P_1 = P(\ldots, y_{i-1}, y_i, y_{i+1}, \ldots)$ in M^r is of type A_1 if, either (i) every vertex of P_1 has the face-sequence (3^6), that is, the face-sequence of y_i's is (3^6) or (ii) each vertex of P_1 has the face-sequence $(3^3.4^2)$ such that all the triangles (square) which incident on its inner vertices lie on the one side of the path P_1. See thick black paths in Figures 5.1.1 and 5.1.2.

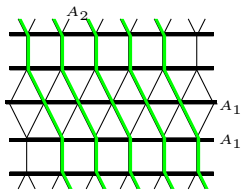


Figure 5.1.1: M^1 **of type** T_1

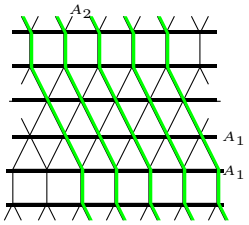


Figure 5.1.2: M^2 **of type** T_2

Definition 2. A path $P_2 = P(\ldots, z_{i-1}, z_i, z_{i+1}, \ldots)$ in M^1, such that z_{i-1}, z_i, z_{i+1} are inner vertices of P_2 or an extended path of P_2, is of type A_2 (shown by green colored paths in Figure 5.1.1), if either of the following three conditions follows for each vertex of the path.

1. If $\mathrm{lk}(z_i) = C_7(\boldsymbol{m}, z_{i-1}, \boldsymbol{n}, o, p, z_{i+1}, q)$ and $\mathrm{lk}(z_{i-1}) = C_7(\boldsymbol{q}, m, r, z_{i-2}, n, \boldsymbol{o}, z_i)$, then $\mathrm{lk}(z_{i+1}) = C_6(s, q, z_i, p, t, z_{i+2})$.
2. If $\mathrm{lk}(z_i) = C_7(\boldsymbol{m}, z_{i+1}, \boldsymbol{n}, o, p, z_{i-1}, q)$ and $\mathrm{lk}(z_{i-1}) = C_6(r, z_{i-2}, s, q, z_i, p)$, then $\mathrm{lk}(z_{i+1}) = C_7(\boldsymbol{o}, z_i, \boldsymbol{q}, m, t, z_{i+2}, n)$.
3. If $\mathrm{lk}(z_i) = C_6(z_{i-1}, m, n, z_{i+1}, o, p)$ and $\mathrm{lk}(z_{i-1}) = C_7(\boldsymbol{r}, z_{i-2}, \boldsymbol{s}, t, m, z_i, p)$, then $\mathrm{lk}(z_{i+1}) = C_7(\boldsymbol{u}, z_{i+2}, \boldsymbol{v}, w, o, z_i, n)$.

Definition 3. A path $P_3 = P(\ldots, z_{i-1}, z_i, z_{i+1}, \ldots)$ in M^2, such that z_{i-1}, z_i, z_{i+1} are inner vertices of P_3 or an extended path of P_3, is of type A_2 (shown by green colored paths in Figure 5.1.2), if either of the following four conditions follows for each vertex of the path.

1. If $\mathrm{lk}(z_i) = C_7(\boldsymbol{m}, z_{i-1}, \boldsymbol{n}, o, p, z_{i+1}, q)$ and $\mathrm{lk}(z_{i-1}) = C_7(r, z_{i-2}, n, \boldsymbol{o}, z_i, \boldsymbol{q}, m)$, then $\mathrm{lk}(z_{i+1}) = C_6(z_i, p, s, z_{i+2}, t, q)$, $\mathrm{lk}(z_{i+2}) = C_6(z_{i+1}, s, u, z_{i+3}, v, t)$.
2. If $\mathrm{lk}(z_i) = C_7(\boldsymbol{m}, z_{i+1}, \boldsymbol{n}, o, p, z_{i-1}, q)$ and $\mathrm{lk}(z_{i-1}) = C_6(z_{i-2}, s, q, z_i, p, r)$, then $\mathrm{lk}(z_{i+1}) = C_7(\boldsymbol{o}, z_i, \boldsymbol{q}, m, t, z_{i+2}, n)$, $\mathrm{lk}(z_{i+2}) = C_6(z_{i+1}, t, u, z_{i+3}, v, n)$.
3. If $\mathrm{lk}(z_i) = C_6(z_{i-1}, m, n, z_{i+1}, o, p)$ and $\mathrm{lk}(z_{i-1}) = C_7(\boldsymbol{q}, z_{i-2}, \boldsymbol{r}, s, m, z_i, p)$, then $\mathrm{lk}(z_{i+1}) = C_6(z_i, n, t, z_{i+2}, u, o)$, $\mathrm{lk}(z_{i+2}) = C_7(\boldsymbol{v}, z_{i+3}, \boldsymbol{w}, x, u, z_{i+1}, t)$.
4. If $\mathrm{lk}(z_i) = C_6(z_{i-1}, m, n, z_{i+1}, o, p)$ and $\mathrm{lk}(z_{i-1}) = C_6(z_{i-2}, r, m, z_i, p, q)$, then $\mathrm{lk}(z_{i+1}) = C_7(\boldsymbol{s}, z_{i+2}, \boldsymbol{t}, u, o, z_i, n)$, $\mathrm{lk}(z_{i+2}) = C_7(\boldsymbol{u}, z_{i+1}, \boldsymbol{n}, s, v, z_{i+3}, t)$.

Remark 1. *Recall that M^r is a map with a finite vertex set. Hence, for a given maximal path $P = (v_1\text{-}v_2\text{-}\cdots\text{-}v_{r-1}\text{-}v_r)$ of type A_α, $\alpha \in \{1, 2\}$, there is an edge $e = v_1\text{-}v_r$ in M^r such that $P \cup e$ is a cycle Q. Depending on A_α, the corresponding cycle is called a cycle of type A_α. Then we see that such cycles are non-contractible.*

Lemma 2. *The cycle Q of type A_α, for $\alpha \in \{1, 2\}$, is non-contractible.*

Proof. On the contrary, suppose Q is a contractible cycle of type A_1. Then Q is the boundary of a 2-disk, say D. Let v, e, and f denote the number of vertices, edges, and faces of D respectively. Let m and n represent the inner and boundary vertices of D, respectively. Here, $m = m_1 + m_2$, in which m_1 and m_2 represent the number of vertices with face-sequences of (3^6) and $(3^3.4^2)$, respectively. Then we have two cases depending on Q. In the first case, if the incident faces on Q are quadrangles, then $v = n + m_1 + m_2$, $e = 3n/2 + 6m_1/2 + 5m_2/2$, and $f = 2n/4 + 6m_1/3 + 3m_2/3 + 2m_2/4$. In the second case, if the incident faces on Q are triangles, then $v = n + m_1 + m_2$, $e = 4n/2 + 6m_1/2 + 5m_2/2$, and $f = 3n/3 + 6m_1/3 + 2m_2/4 + 3m_2/3$. For both cases, we obtain the Euler characteristic $\chi(D) = v - e + f = 0$. We get a contradiction, as the Euler characteristic of a 2-disk is 1. Hence, Q is not contractible. Similarly, if Q is of type A_2, then Q is non-contractible. □

Let Q be a cycle of type A_1 with the vertex set $V(Q)$. Let S_Q denote the set of all the faces incident at v for all $v \in V(Q)$. Then the geometric carrier $|S_Q|$ is a cylinder, as Q is non-contractible. Let $\partial|S_Q|$ denote the boundary of $|S_Q|$.

Let Q_1 and Q_2 be cycles of the same type in M^r. We say that cycles Q_1 and Q_2 are homologous if there is a cylinder whose boundary is $\{Q_1, Q_2\}$.

Lemma 3. *Let Q be a cycle of type A_1 such that $\partial|S_Q| = \{Q_1, Q_2\}$. Then Q, Q_1, and Q_2 are of the same type with equal length.*

Proof. Let $Q_n \in \{Q_1, Q_2\}$. Consider the faces that are incident with both Q and Q_n. Without loss of generality, let $Q_n = Q_1$. Now, depending on Q, we see that if the faces incident with both Q and Q_1 are quadrangles (respectively triangles), then the faces on the other side of Q_1 must be triangles (respectively triangles or quadrangles). This shows that Q_1 is of type A_1. Similarly, Q_2 is also of type A_1.

Let $Q = C_l(v_1, \ldots, v_l)$, $Q_1 = C_m(u_1, \ldots, u_m)$, and $Q_2 = C_n(w_1, \ldots, w_n)$. Now we show $l = m = n$ by contradiction. For this, it is enough to deal with the case, when $l < m < n$. Following the definition of type A_1, the face-sequences of $v_1, v_2, \ldots, v_{l-1}, v_l$ are same throughout the cycle Q. Since $l < m$, $\mathrm{lk}(v_l)$ contains the vertices u_l, u_{l+i}, and w_l for some $i > 0$. Then, the face-sequences of v_l and v_{l-1} are not same. This is not possible. Therefore, $l = m = n$. □

$M(i, j, k)$-representation of M^r: Let $u \in V(M^r)$ and Q_α be cycles of type A_α through u, where $\alpha \in \{1, 2\}$. Let $Q_1 = C_i(u_1, u_2, \ldots, u_i)$, where $u_k = u$ for one k $(1 \leq k \leq i)$. We cut M^r first along the cycle Q_1. This gives a cylinder, say R_1, bounded by identical cycle Q_1. We say, a cycle is horizontal (resp. vertical) if it is Q_1 or homologous to Q_1 (resp. Q_2 or homologous to Q_2). In R_1, starting from the vertex u, make another cut along the path $P \subset Q_2$, until it reaches Q_1 again for the first time. As a result, we get the unfold torus (a planer representation), say R_2. The idea of $M(i, j, k)$-representation is taken from [9] and a detailed description of such representation for the given type is available in [14].

In M, without loss of generality, suppose that the quadrangular faces are incident on $Q_1 = C_i(a_{11}, a_{12}, \ldots, a_{i1})$. In R_2, let there are j cycles (as shown in Figures 5.1.3 and 5.1.4) which are homologous to Q_1 along P. Since $\mathrm{length}(Q_1) = i$ and the number of horizontal cycles along P is j. So, we denote R_2 by (i, j) representation. To reconstruct the map M^r from its (i, j) representation, one gets a natural way of identification of the vertical cycle of R_2, but identification of the horizontal cycle generally needs some shifting so that the lower horizontal cycle is identified with the upper horizontal cycle. Suppose k is such shifting, *i.e.*, $k + 1$ is the starting vertex of the upper horizontal cycle. This gives another representation say $M^r(i, j, k)$ representation of the (i, j) representation of M^r. The admissible relations among i, j, k of the $M^r(i, j, k)$ is given in Lemma 4.

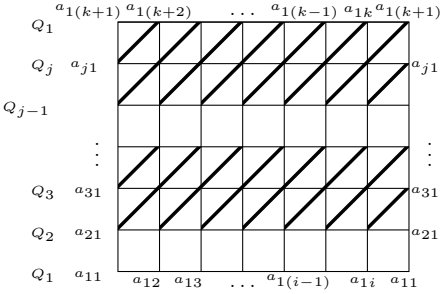


Figure 5.1.3: $M(i,j,k)$ **of DSEM of type** T_1

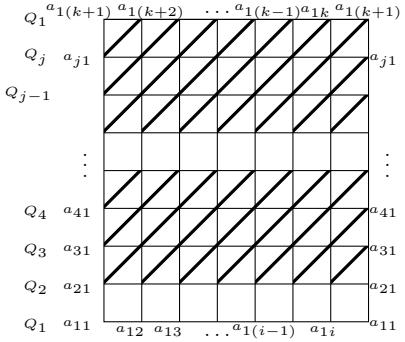


Figure 5.1.4: $M(i,j,k)$ **of DSEM of type** T_2

Lemma 4. *Let M^r be a DSEM of type T_r. Then M^r admits an $M^r(i,j,k)$ representation if and only if the following holds: (i) $i \geq 3$ and $j = (r+2)m$, where $m \in \mathbb{N}$, (ii) $ij \geq 3(r+2)$, (iii) $0 \leq k \leq i-1$.*

Proof. Note that an $M^r(i,j,k)$ of M^r contains j number of A_1 type disjoint horizontal cycles of length i. Since these cycles cover all the vertices of M^r, the number of vertices in M^r is $n = ij$. Clearly if $i \leq 2$, M^r is not a map. So $i \geq 3$. If $j = 1$ then M^r is not a map and if $j = 2$ then M^r has no vertices of face-sequence (3^6). If $j = (r+2)m+1$ or $(r+2)m+2$, then $2|V_{(3^6)}| \neq |V_{(3^3.4^2)}|$ for M^1 and $|V_{(3^6)}| \neq |V_{(3^3.4^2)}|$ for M^2. So $j = (r+2)m$. Thus $n = ij \geq 3(r+2)$. Since the length of the horizontal cycle is i, we get $k \in \{0, 1, \ldots, i-1\}$. The converse part follows directly by constructing $M^r(i,j,k)$ representation for the given values of i, j, and k as in Figures 5.1.3 and 5.1.4. □

From the above lemma, it is clear that every map M^r with the vertex set $V(M^r)$ of type T_r can be obtained from its $M^r(i,j,k)$ representation (shown in Figure 5.1.3 or 5.1.4) for suitable values of i, j, k. In the upcoming subsections, we proceed in a similar way for the remaining types DSEMs. We determine the conditions among i, j, k to construct its $M(i,j,k)$ representation. For this, we consider fixed types of paths on maps, as in Subsection 5.1. Since these DSEMs are on the finite vertex set, given every maximal path P of such types there is an edge e such that $P \cup \{e\}$ is a non-contractible, as in Lemma 2.

5.2 DSEMs of types $T_3 = [f_1^{(f_1^4.f_2^3)} : f_2^{(f_1^3.f_2.f_1^3.f_2)}]$ and $T_4 = [f_1^{(f_1^4.f_2^3)} : f_2^{(f_1^3.f_2^5)}]$, where $f_1 = (3^3.4^2), f_2 = (4^4)$

Let M^r be a DSEM of type T_r, where $r \in \{3, 4\}$. Then for the existence of M^1 and M^2 we have $|V_{(3^3.4^2)}| = 2|V_{(4^4)}|$ and $|V_{(3^3.4^2)}| = |V_{(4^4)}|$ respectively. Now,

consider the following paths P_1, P_2, and P_3 through a vertex in M^r. One can define such a path using the definition of link as in Subsection 5.1.

• A path $P_1 = P(\ldots, y_{i-1}, y_i, y_{i+1}, \ldots)$ in M^r, say of type B_1, indicated by thick black paths in Figures 5.2.1 and 5.2.2. Here the face-sequence of y_i's is either (4^4) or $(3^3.4^2)$.

• A path $P_2 = P(\ldots, y_{i-2}, z_{i-2}, z_{i-1}, y_i, z_i, z_{i+1}, y_{i+1} \ldots)$ in M^1, say of type B_2, indicated by green paths in Figure 5.2.1. Here the face-sequences of y_i's and z_i's are (4^4) and $(3^3.4^2)$ respectively.

• A path $P_3 = P(\ldots, y_{i-1}, z_{i-2}, z_{i-1}, y_i, y_{i+1}, z_i, z_{i+1}, y_{i+2} \ldots)$ in M^2, say of type B_2, indicated by green paths in Figure 5.2.2. Here the face-sequences of y_i's and z_i's are (4^4) and $(3^3.4^2)$ respectively.

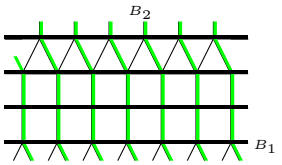


Figure 5.2.1: M^1 **of type** T_3

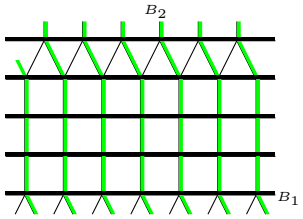


Figure 5.2.2: M^2 **of Type** T_4

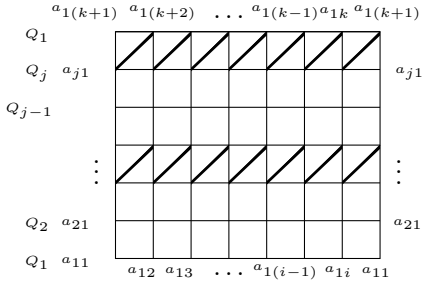


Figure 5.2.3: $M(i, j, k)$ of DSEMs of type T_3

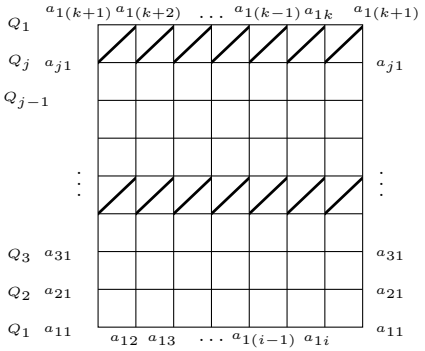


Figure 5.2.4: $M(i, j, k)$ of DSEMs of type T_4

Now, first cutting M^r along a cycle of type B_1 and then along a cycle of type B_2, we get $M^r(i, j, k)$-representation of M^r, shown in Figures 5.2.3 and 5.2.4. To get the relation among i, j, k, we proceed similarly as in Lemma 4. This gives the following result.

Lemma 5. *The DSEM M^r of type T_r admits an $M^r(i, j, k)$-representation if and only if the following holds: (i) $i \geq 3$ and $j = rm$, where $m \in \mathbb{N}$, (ii) $ij \geq 3r$, (iii) $0 \leq k \leq i - 1$.*

5.3 DSEMs of type $T_5 = [f_1^{(f_1^2 \cdot f_2^2 \cdot f_1 \cdot f_2^2)} : f_2^{(f_2^5 \cdot f_1^3)}]$, where $f_1 = (3^3.4^2)$, $f_2 = (3.4.6.4)$

Let M be a DSEM of type T_5. In M, we consider the following path
• A path $P_1 = P(\ldots, y_i, z_i, z_{i+1}, z_{i+2}, y_{i+1}, \ldots)$, say of type S_1, indicated by thick black or green paths (shown in Figure 5.3.1). Here either the vertices y_i's and z_j's have the face-sequences $(3^3.4^2)$ and (3.4.6.4) respectively or the face-sequence of y_i's and z_j's is $(3^3.4^2)$.

An $M(i, j, k)$ representation of M follows by cutting M one by one along a black and green colored cycles of type S_1 through a vertex v with face-sequence (3.4.6.4). Without loss of generality, let the beginning adjacent face to the base horizontal cycle Q_1 is a 4-gon.

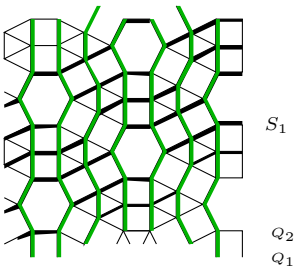


Figure 5.3.1: M of type T_5

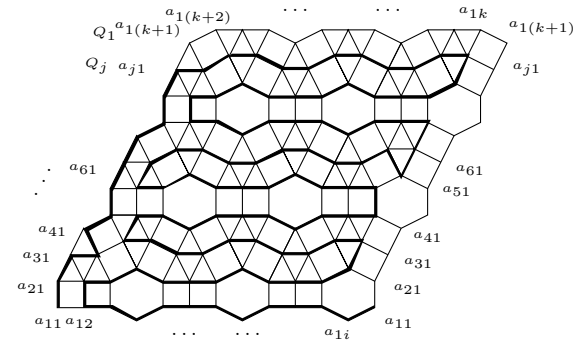


Figure 5.3.2: $M(i, j = 6m + 3, k)$

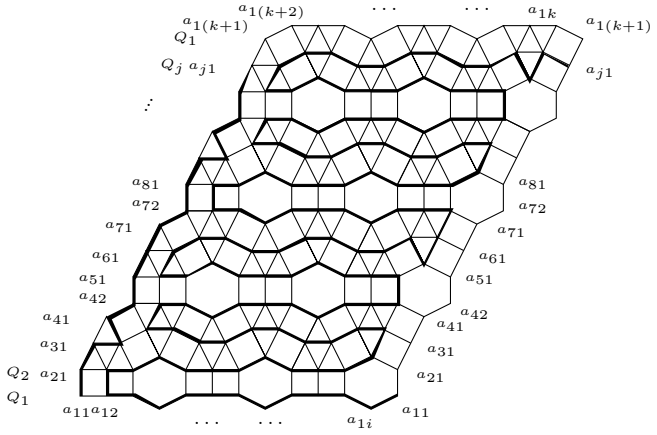


Figure 5.3.3: $M(i, j = 6m + 6, k)$

Lemma 6. *The DSEM M of type T_5 admits an $M(i, j, k)$-representation if*

and only if the following holds: (i) $j = 3m$, *where* $m \in \mathbb{N}$, (ii) $i \geq 4$ *and* $i = 4m$, $m \in \mathbb{N}$, (iii) *number of vertices of* $M(i,j,k) = ij \geq 12$, (iv) $k \in \{4r+3 : 0 \leq r < i/4\}$.

Proof. Let M be a DSEM of type T_5 having n vertices. Clearly, its representation $M(i,j,k)$ has j disjoint horizontal cycles of S_1 type, say $Q_0, Q_1, \ldots, Q_{j-1}$, of length i. Let $Q_0 = C(w_{0,0}, w_{0,1}, \ldots, w_{0,i-1}), Q_1 = C(w_{1,0}, w_{1,1}, \ldots, w_{1,i-1}), \ldots, Q_{j-1} = C(w_{j-1,0}, w_{j-1,1}, \ldots, w_{j-1,i-1})$ be the list of horizontal cycles. Since the vertices of M are in these cycles, the vertices in M is $n = ij$. If $j = 1, 2$ then M is not a map. So $j \geq 3$. If $j \geq 3$ and $j \neq 3m, m \in \mathbb{N}$, then no vertex in the base horizontal cycle follow the face-sequences $(3^3.4^2)$ and $(3.4.6.4)$ after identifying the boundaries of $M(i,j,k)$. So $j = 3m$, where $m \in \mathbb{N}$. Clearly if $i \leq 3$, M is not a map. So $i \geq 4$. If $j = 3m$ and $i \neq 4m$, where $m \in \mathbb{N}$, then the $\mathrm{lk}(w_{0,0})$ is not of type $(3.4.6.4)$. Which is not possible. So, $i = 4m$, where $m \in \mathbb{N}$. Thus, $n = ij \geq 12$.

If $k \in \{r : 0 \leq r \leq i-1\} \setminus (\{4r+3 : 0 \leq r < i/4\})$, then link of some vertices in $M(i,j,k)$ can not be completed. So, $k \in \{4r+3 : 0 \leq r < i/4\}$. The converse follows from Figures 5.3.2 and 5.3.3. This completes the proof. □

5.4 DSEMs of types $T_6 = [f_1^{(f_1 \cdot f_2^6)} : f_2^{(f_1^2 \cdot f_2^3 \cdot f_1 \cdot f_2)}]$, where $f_1 = (3^3.4^2)$, $f_2 = (3^2.4.3.4)$

Let M be a DSEM of type T_6. Then, for the existence of M, we have $2|V_{(3^3.4^2)}| = |V_{(3^2.4.3.4)}|$. Now consider the following path P_1 through a vertex in M.

• A path $P_1 = P(\ldots, w_i, z_j, z_{j+1}, z_{j+2}, z_{j+3}, w_{i+1}, \ldots)$ in M, say of type C_1, indicated by thick black paths in Figure 5.4.1. Here the face-sequences of w_i's and z_j's are $(3^3.4^2)$ and $(3^2.4.3.4)$ respectively.

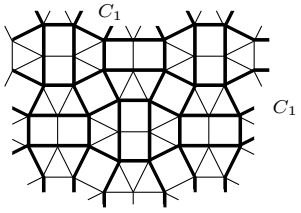


Figure 5.4.1: DSEM of type T_6

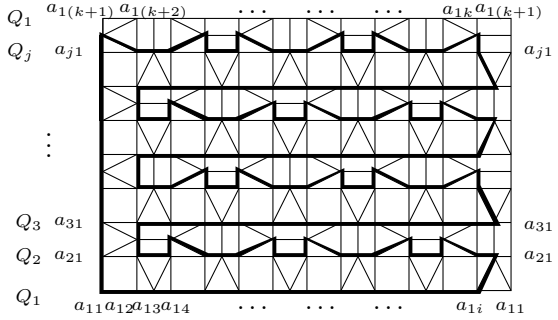


Figure 5.4.2: $M(i, j = 4m+4, k)$ of DSEMs of type T_6

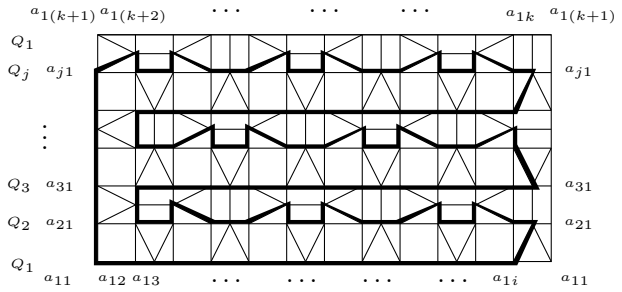


Figure 5.4.3: $M(i, j = 4m + 2, k)$ of DSEMs of type T_6

Note that, through a vertex v with the face-sequence $(3^2.4.3.4)$, we see two paths of type C_1. Now as in the previous section, cutting M along these paths one by one, we get its $M(i, j, k)$ representation. The relations among i, j, k are given by Lemma 7.

Lemma 7. *A DSEM M of type T_6 admits an $M(i, j, k)$ representation if and only if the following holds: (i) $i = 5m$, $m \in \mathbb{N}$ and j is even, (ii) number of vertices of $M(i, j, k) = 6ij/5 \geq 12$, (iii) if $j = 4m + 2$, $m \in \mathbb{N} \cup \{0\}$, then $k \in \{5r+3 : 0 \leq r < i/5\}$, and if $j = 4m$, $m \in \mathbb{N}$, then $k \in \{5r : 0 \leq r < i/5\}$.*

Proof. A representation $M(i, j, k)$ of a DSEM M of type T_6 has j disjoint horizontal cycles of C_1 type having length i. Let $Q_0, Q_1, \ldots, Q_{j-1}$ be the horizontal cycles of type C_1. The number of vertices with face-sequence $(3^3.4^2)$ lying between horizontal cycles $Q_{(2s+1)(mod\, j)}$ and $Q_{(2s+2)(mod\, j)}$, for $0 \leq s \leq j-1$, is $2i/5 \cdot j/2$. Thus, the total number of vertices in M is $n = ij + ij/5 = 6ij/5$. If $j = 1$, then $M(i, 1, k)$ has no vertex with face-sequence $(3^3.4^2)$ or $(3^2.4.3.4)$. So $j \geq 2$. If $j \geq 2$ and j is not an even integer then we get some vertices in the base horizontal cycle which does not have the face-sequence $(3^3.4^2)$ or $(3^2.4.3.4)$. So, j is even.

If j is even and $i < 5$ then again the representation $M(i, j, k)$ has some vertices which do not have the face-sequence $(3^3.4^2)$ or $(3^2.4.3.4)$. So, $i \geq 5$. If $i \geq 5$ and not a multiple of 5, then $2|V(3^3.4^2)| \neq |V(3^2.4.3.4)|$. This is not possible. So $i = 5m$, where $m \in \mathbb{N}$ and $n = 6ij/5 \geq 12$.

If $j = 4m+2$, $m \in \mathbb{N}\cup\{0\}$ and $k \in \{r : 0 \leq r \leq i-1\}\setminus\{5r+3 : 0 \leq r < i/5\}$ then we get some vertices which do not have the face-sequence $(3^3.4^2)$ or $(3^2.4.3.4)$. So, $k \in \{5r + 3 : 0 \leq r < i/5\}$ for $j = 4m + 2$, $m \in \mathbb{N} \cup \{0\}$. Similarly if $j = 4m$, $m \in \mathbb{N}$, then $k \in \{5r : 0 \leq r < i/5\}$. The converse follows by substituting the values of i, j, k in Figures 5.4.2 and 5.4.3. □

5.5 DSEMs of types $T_7 = [f_1^{(f_1.f_2^3.f_1.f_2^2)} : f_2^{(f_1^3.f_2.f_1^2.f_2)}]$, where $f_1 = (3^3.4^2)$, $f_2 = (3^2.4.3.4)$

Let M be a DSEM of type T_7. It is easy to see that, if M exists then $|V_{(3^3.4^2)}| = |V_{(3^2.4.3.4)}|$. We consider following types of paths in M as follows.

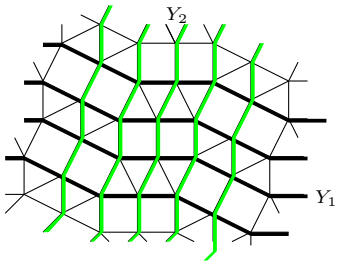


Figure 5.5.1: M of type T_7

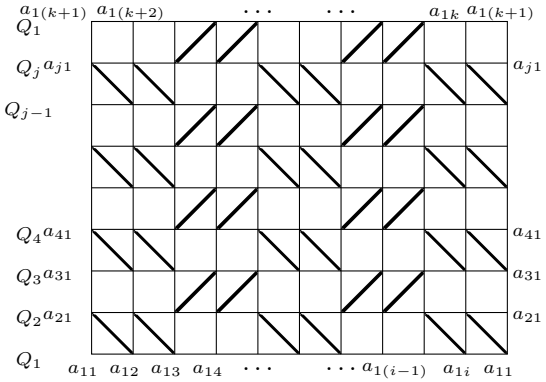


Figure 5.5.2: $M(i, j, k)$ of DSEMs of type T_7

• A path $P_1 = P(\ldots, u_i, v_j, u_{i+1}, v_{j+1}, \ldots)$ in M, say of type Y_1 (indicated by thick black paths in Figure 5.5.1). Here the face-sequences of u_i's and v_j's are $(3^2.4.3.4)$ and $(3^3.4^2)$ respectively.
• A path $P_2 = P(\ldots, u_{i-1}, u_i, u_{i+1}, \ldots)$ in M, say of type Y_2 (indicated by green paths in Figure 5.5.1). Here the face-sequence of u_i's is either $(3^3.4^2)$ or $(3^2.4.3.4)$.

To construct $M(i, j, k)$, we first cut M along a cycle of type Y_1 and then along a cycle of type Y_2. Without loss of generality assume Y_2 whose all the vertices have the face-sequence $(3^2.4.3.4)$ (as in Figure 5.5.2). Following, a similar argument given in Lemma 4, we obtain Lemma 8.

Lemma 8. *A DSEM M of type T_7 admits an $M(i, j, k)$ representation if and only if the following holds: (i) $j \geq 2$ and j even, (ii) if $j = 2$ then $i \geq 8$, and if $j \geq 4$ then $i \geq 4$, also $i = 4m$, where $m \in \mathbb{N}$, (iii) $ij \geq 16$, (iv) if $j = 2$ then $k \in \{4r : 0 < r < i/4\}$, and if $j \geq 4$ then $k \in \{4r : 0 \leq r < i/4\}$.*

5.6 DSEMs of type $T_8 = [f_1^{(f_2^6)} : f_2^{(f_1.f_2^9)}]$, where $f_1 = (3^6) : f_2 = (3^2.6^2)$

Let M be a DSEM of type T_8. In M, we consider a path as follows.
• A path $P_1 = P(\ldots, y_i, z_i, z_{i+1}, y_{i+1}, \ldots)$ in M, say of type H_1 (indicated by thick black or green paths in Figure 5.6.1). Here, the face-sequences of y_i's and z_j's are (3^6) and $(3^2.6^2)$ respectively.

Now, construct $M(i,j,k)$ representation of M by first cutting M along a black colored cycle of type H_1 through a vertex v with face-sequence (3^6) and then cutting along a green colored cycle of type H_1. Note that both these cycles are non-homologous. Then we have the following lemma.

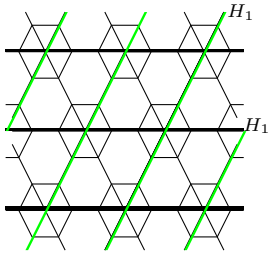


Figure 5.6.1: M of type T_8

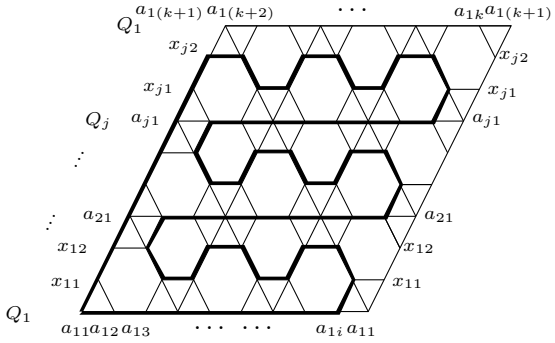


Figure 5.6.2: $M(i,j,k)$

Lemma 9. *A DSEM M of type T_8 admits an $M(i,j,k)$-representation if and only if the following holds: (i) $i \geq 9$ and $i = 3m$, $m \in \mathbb{N}$ for $j = 1$, (ii) $i \geq 6$ and $i = 3m$, $m \in \mathbb{N}$ for $j > 1$, (iii) number of vertices of $M(i,j,k) = 7ij/3$, (iv) if $j = 1$ then $k \in \{3r : 1 < r < i/3\}$, and if $j > 1$ then $k \in \{3r : 0 \leq r < i/3\}$.*

Proof. Let M be a DSEM of type T_8 with n vertices. Clearly $M(i,j,k)$ of M has j disjoint horizontal cycles of H_1 type, say $Q_0, Q_1, \ldots, Q_{j-1}$, of length i. Note that the number of vertices with face-sequence $(3^2.6^2)$ which are above the horizontal cycles Q_s and not belonging to any horizontal cycles Q_s for $0 \leq s \leq j-1$ is $4i/3 \cdot j$. In M, therefore $n = ij + 4ij/3 = 7ij/3$.

If $j = 1$ and $i < 9$, then the link of some vertices in $M(i,1,k)$ can not be completed. This is not possible. So, for $j = 1$, we get $i \geq 9$. Also, if $i \geq 9$ and i is not a multiple of 3, then after identifying the boundaries of $M(i,j,k)$, there is no vertex in the horizontal base cycle having face-sequences (3^6) and $(3^2.6^2)$. Which is not possible. So, $i \geq 9$ and $i = 3m$, $m \in \mathbb{N}$ for $j = 1$. Similarly, as above, for $j > 1$ we get that $i \geq 6$ and $i = 3m$, where $m \in \mathbb{N}$. Thus $n = 7ij/3 \geq 21$.

If $j = 1$ and $k \in \{r : 0 \leq r \leq i-1\} \setminus \{3r : 1 < r < i/3\}$, then we get some vertices whose link can not be completed or does not follow the face-sequences (3^6) and $(3^2, 6^2)$. So, for $j = 1$, we get $k \in \{3r : 1 < r < i/3\}$. Similarly, as above, we see that if $j > 1$, then $k \in \{3r : 0 \leq r < i/3\}$. The converse follows from Figure 5.6.2. Thus the proof. □

5.7 DSEMs of type $T_9 = [f_1^{(f_2^6)} : f_2^{(f_1 \cdot f_2^{13})}]$, where $f_1 = (3^6), f_2 = (3^2.4.12)$

Let M be a DSEM of type T_9. We consider the following paths as follows.
• A path $P_1 = P(\ldots, y_i, z_i, z_{i+1}, z_{i+2}, z_{i+3}, y_{i+1}, \ldots)$ in M, say of type L_1, indicated by thick black or green paths (shown in Figure 5.7.1). The face-sequences of y_i's and z_i's are (3^6) and $(3^2.4.12)$ respectively.

An $M(i, j, k)$ representation of M obtained by first cutting M along a black colored cycle of type L_1 through a vertex with face-sequence (3^6) and then along a green colored cycle of type L_1.

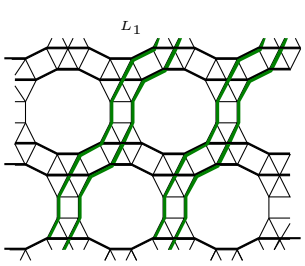


Figure 5.7.1: M of type T_9

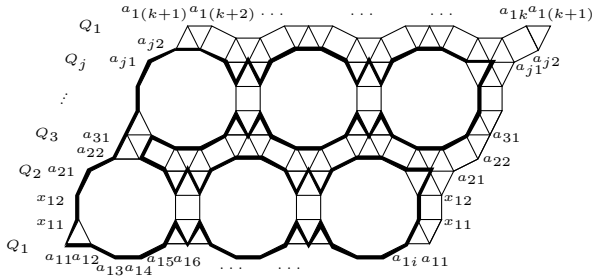


Figure 5.7.2: $M(i, j, k)$

Lemma 10. *A DSEM M of type T_9 admits an $M(i, j, k)$-representation if and only if the following holds: (i) j even and $i = 5m$, $m \in \mathbb{N}$, (ii) $i \geq 15$ for $j = 2$, (ii) $i \geq 10$ for $j \geq 4$, (iii) number of vertices of $M(i, j, k) = 7ij/5 \geq 42$, (iv) if $j = 2$ then $k \in \{5r : 1 < r < i/5\}$, and if $j \geq 4$ then $k \in \{5r : 0 \leq r < i/5\}$.*

Proof. Let M be a DSEM of type T_9 having n vertices. An $M(i, j, k)$ has j disjoint horizontal cycles of L_1 type, say $Q_0, Q_1, \ldots, Q_{j-1}$, of length i. Let $Q_0 = C(w_{0,0}, w_{0,1}, \ldots, w_{0,i-1}), Q_1 = C(w_{1,0}, w_{1,1}, \ldots, w_{1,i-1}), \ldots, Q_{j-1} = C(w_{j-1,0}, w_{j-1,1}, \ldots, w_{j-1,i-1})$ be the list of horizontal cycles. Note that the vertices with face-sequence $(3^2.4.12)$ lying between cycles $Q_{2s(mod\, j)}$ and $Q_{(2s+1)(mod\, j)}$ for $0 \leq s \leq j-1$ is $4i/5 \cdot j/2$. Therefore, $n = ij + 2ij/5 = 7ij/5$.

If $j = 1$, then no vertex in the base horizontal cycle follows the face-sequence $(3^2.4.12)$ after identifying the boundaries of $M(i, 1, k)$. Therefore, $j \geq 2$. If $j \geq 2$ and j is not an even integer, then similarly as above no vertex in the base horizontal cycle follows the face-sequence $(3^2.4.12)$. So j is an even integer. If j is even and $i \neq 5m$, where $m \in \mathbb{N}$, then the $\mathrm{lk}(w_{0,0})$ is not of type (3^6). Which is not possible. So, $i = 5m$, where $m \in \mathbb{N}$.

If $j = 2$ and $i < 15$, then we get some vertex in $M(i, 2, k)$ whose link can not be constructed. This is not possible. So, for $j = 2$, we get $i \geq 15$. Similarly, $i \geq 10$ for $j \geq 4$. Thus $n = 7ij/5 \geq 42$.

If $j = 2$ and $k \in \{r : 0 \leq r \leq i - 1\} \setminus (\{5r : 1 < r < i/5\})$, then we get some vertex in $M(i, 2, k)$ whose link can not be constructed. So, if $j = 2$,

then $k \in \{5r : 1 < r < i/5\}$. Similarly, as above, we see that if $j \geq 4$, then $k \in \{5r : 0 \leq r < i/5\}$. The converse follows from 5.7.2. This completes the proof. □

5.8 DSEMs of type $T_{10} = [f_1^{(f_2^6)} : f_2^{(f_1 \cdot f_2^6)}]$, where $f_1 = (3^6), f_2 = (3^2.4.3.4)$

Let M be a DSEM of type T_{10}. We consider the following types of paths in M as follows.

• A path $P_1 = P(\ldots u_{i-1}, u_i, u_{i+1}, \ldots)$ in M, say of type E_1 (indicated by thick black paths in Figure 5.8.1). The face-sequence of u_i's is $(3^2.4.3.4)$.

• A path $P_2 = P(\ldots, u_i, v_j, v_{j+1}, v_{j+2}, v_{j+3}, u_{i+1}, \ldots)$ in M, say of type E_2 (indicated by green paths in Figure 5.8.1). The vertices u_i's and v_j's have the face-sequences (3^6) and $(3^2.4.3.4)$ respectively.

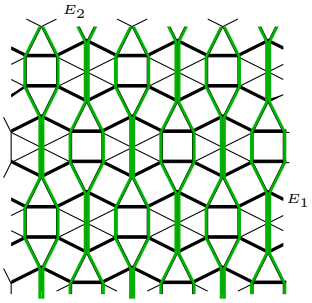


Figure 5.8.1: M of type T_{10}

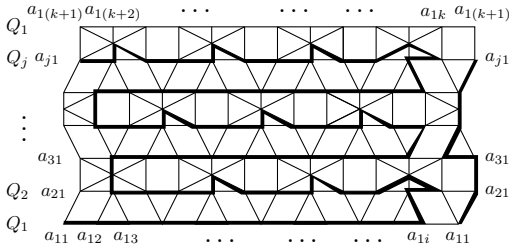


Figure 5.8.2: $M(i, j = 4m + 2, k)$ of DSEMs of type T_{10}

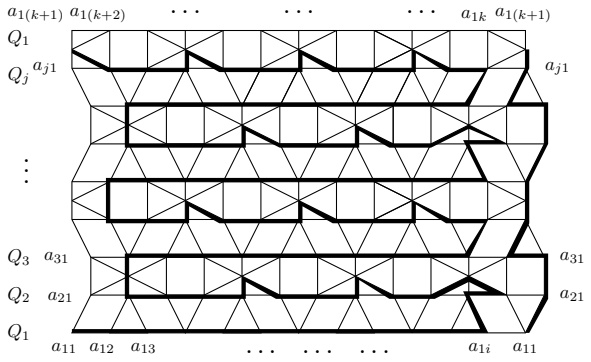


Figure 5.8.3: $M(i, j = 4m + 4, k)$ of DSEMs of type T_{10}

An $M(i, j, k)$ representation of M is obtained by first cutting M along an E_1 type cycle and then along an E_2 type cycle. Proceeding similarly, as in Lemma 7, we get the Lemma 11.

Lemma 11. *A DSEM M of type T_{10} admits an $M(i,j,k)$ representation if and only if the following holds: (i) $j \geq 2$ even, (ii) if $j = 2$ then $i \geq 9$ and if $j \geq 4$ then $i \geq 6$, also $i = 3m$, $m \in \mathbb{N}$, (iii) number of vertices of $M(i,j,k) = 7ij/6 \geq 21$, (iv) if $j = 2$ then $k \in \{3r+2 : 0 < r < (i-3)/3\}$, if $j = 4m+2$ then $k \in \{3r+2 : 0 \leq r < i/3\}$, and if $j = 4m$ then $k \in \{3r : 0 \leq r < i/3\}$, $m \in \mathbb{N}$.*

5.9 DSEMs of type $T_{11} = [f_1^{(f_1^2 \cdot f_2^2 \cdot f_1 \cdot f_2)} : f_2^{(f_2^5 \cdot f_1^3)}]$, where $f_1 = (3^6), f_2 = (3^4.6)$

Let M be a DSEM of type T_{11}. Then for the existence of the map M, we see that $|V_{(3^6)}| = |V_{(3^4.6)}|$. Now, consider a fixed type path in M as follows.
• A path $P_1 = P(\ldots, z_i, y_i, y_{i+1}, y_{i+2}, z_{i+1}, \ldots)$ in M, say of type K_1, indicated by thick black or green paths (shown in Figure 5.9.1). Here either the face-sequences of y_i's and z_j's are $(3^4.6)$ and (3^6) respectively or the face-sequences of y_i's and z_j's is (3^6).

The relations among i, j, k in an $M(i,j,k)$ representation of M are given by the following lemma.

Lemma 12. *A DSEM M of type T_{11} admits an $M(i,j,k)$-representation if and only if the following holds: (i) $j = 3m$ and $i = 4m$, where $m \in \mathbb{N}$, (ii) number of vertices of $M(i,j,k) = ij \geq 12$, (iii) $k \in \{4r+1 : 0 \leq r < i/4\}$.*

Proof. Let M be a DSEM of type T_{11} having n vertices. Its $M(i,j,k)$ has j disjoint horizontal cycles of K_1 type, say $Q_0, Q_1, \ldots, Q_{j-1}$, of length i. Since the vertices of M are in these cycles, the vertices in M is $n = ij$. If $j = 1, 2$ then M is not a map. So $j \geq 3$. If $j \geq 3$ and $j \neq 3m$, $m \in \mathbb{N}$, then no vertex in the base horizontal cycle follows the face-sequence (3^6) after identifying the boundaries of $M(i,j,k)$. So $j = 3m$, where $m \in \mathbb{N}$. Clearly if $i \leq 3$, M is not a map. So $i \geq 4$. If $j = 3m$ and $i \neq 4m$, where $m \in \mathbb{N}$, then either $|V_{(3^6)}| \neq |V_{(3^4.6)}|$ or M is not a map. So, $i = 4m$, where $m \in \mathbb{N}$. Thus $n = ij \geq 12$.

If $k \in \{r : 0 \leq r \leq i-1\} \setminus (\{4r+1 : 0 \leq r < i/4\})$, then the link of some vertex in $M(i,j,k)$ can not be completed. So, $k \in \{4r+1 : 0 \leq r < i/4\}$. The converse follows from Figures 5.9.2 and 5.9.3. This completes the proof. □

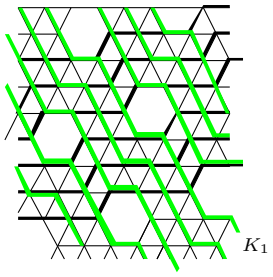


Figure 5.9.1: M of type T_{11}

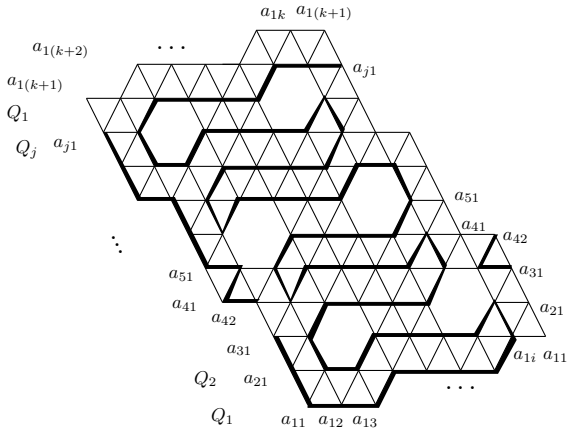


Figure 5.9.2: $M(i, j = 6m + 3, k)$

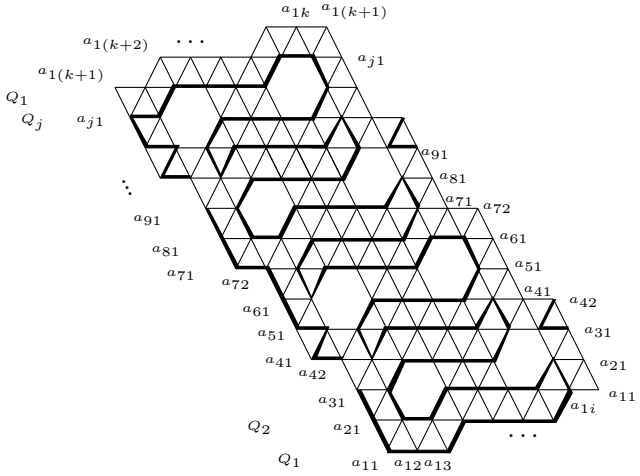


Figure 5.9.3: $M(i, j = 6m + 6, k)$

5.10 DSEMs of type $T_{12} = [f_1^{(f_2^6)} : f_2^{(f_2^5.f_1.f_2.f_1)}]$, where $f_1 = (3^6), f_2 = (3^4.6)$

Let M be a DSEM of type T_{12}. We consider a fixed type path as follows.
• A path $P_1 = P(\ldots, y_i, z_i, z_{i+1}, y_{i+1}, \ldots)$ in M, say of type K_1', indicated by thick black or green paths (shown in Figure 5.10.1). The face-sequences of y_i's and z_j's are (3^6) and $(3^4.6)$ respectively.

Construct $M(i, j, k)$ representation of M by first cutting M along a black

colored cycle of type K_1' through a vertex with face-sequence (3^6) and then along the green colored cycle of type K_1'.

The relations among i, j, k in $M(i, j, k)$ is given by the following lemma (proof is similar as in Lemma 8).

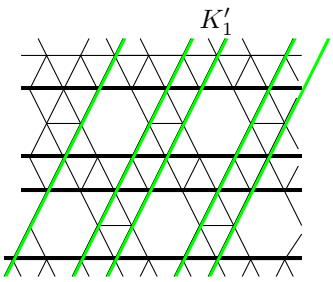


Figure 5.10.1: M of type T_{12}

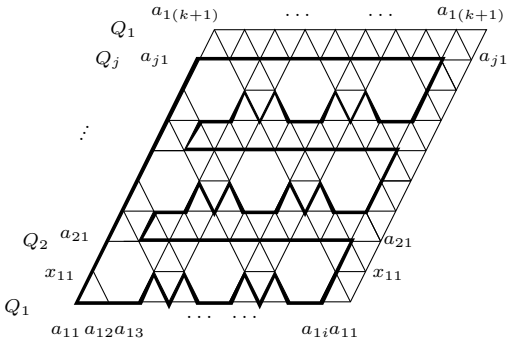


Figure 5.10.2: $M(i, j, k)$

Lemma 13. *A DSEM M of type T_{12} admits an $M(i, j, k)$-representation if and only if the following holds: (i) j even and $i = 3m$, $m \in \mathbb{N}$, (ii) $i \geq 9$ for $j = 2$ and $i \geq 6$ for $j \geq 4$, (iii) number of vertices of $M(i, j, k) = 4ij/3 \geq 24$, (iv) if $j = 2$ then $4 \leq k \leq i - 1$, and if $j \geq 4$ then $0 \leq k \leq i - 1$.*

5.11 DSEMs of type $T_{13} = [f_1^{(f_2{}^6)} : f_2{}^{(f_2{}^7 . f_1)}]$, where $f_1 = (3^6), f_2 = (3^4.6)$

Let M be a DSEM of type T_{13}. We consider a path in M as follows.

• A path $P_1 = P(\ldots, z_{i-1}, z_i, z_{i+1}, \ldots)$ in M, say of type V_1 indicated by thick black colored lines, shown in Figure 5.11.1. Note that, the vertices z_i's have the face-sequence $(3^4, 6)$.

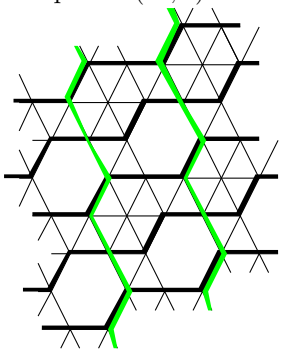


Figure 5.11.1: M **of Type** T_{13}

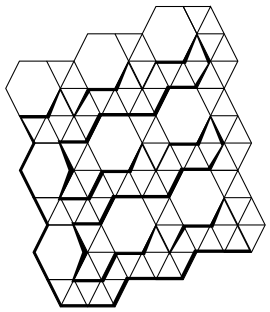


Figure 5.11.2: $M(i, j, k)$

• A path $P_2 = P(\ldots, y_{i-1}, y_i, y_{i+1}, \ldots)$ in M, say of type V_2 indicated by thick green colored lines, shown in Figure 5.11.1. Note that, the vertices y_i's have the face-sequence $(3^4, 6)$.

We construct $M(i, j, k)$ representation of M by first cutting M along a black coloured cycle of type V_1 through a vertex v having face-sequence $(3^4, 6)$ and then cutting it along a green coloured cycle of type V_1 where the beginning adjacent face to black coloured cycle are 3-gons.

We define admissible relations among i, j, k of $M(i, j, k)$ in the next lemma, and its proof follows from the similar arguments as in Lemma 8.

Lemma 14. A DSEM M of type T_{13} admits an $M(i, j, k)$-representation if and only if the following holds: (i) j even, (ii) $i \geq 6$ and $i = 6m$, $m \in \mathbb{N}$, (iii) number of vertices of $M(i, j, k) = 7ij/6 \geq 14$, (iv) if $j = 2$ then $k \in \{3r + 2 : 0 \leq r < (i - 3)/3\}$, and if $j \geq 4$ then $k \in \{3r + 2 : 0 \leq r \leq (i - 3)/3\}$.

5.12 DSEMs of type $T_{14} = [f_1^{(f_2^2 . f_1^7)} : f_2^{(f_2^5 . f_1^4)}]$, where $f_1 = (3.4^2.6), f_2 = (3.4.6.4)$

Let M be a DSEM of type T_{14}. In M, consider a fixed type path $P_1 = P(\ldots, y_i, y_{i+1}, y_{i+2}, z_i, z_{i+1}, y_{i+3}, \ldots)$, say of type R_1, indicated by thick black or green paths (shown in Figure 5.12.1). The vertices y_i's and z_i's have the face-sequences (3.4.6.4) and $(3.4^2.6)$ respectively.

An $M(i, j, k)$ representation is obtained by first cutting M along a black colored cycle of type R_1 through a vertex v with the face-sequence (3.4.6.4) and then along a green colored cycle of type R_1. Assume that the beginning adjacent face to base horizontal cycle Q_1 is triangular. The relations among i, j, k are given in below lemma (its proof follows from similar arguments as in Lemma 8).

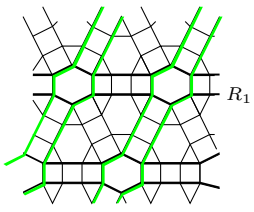


Figure 5.12.1: M of type T_{14}

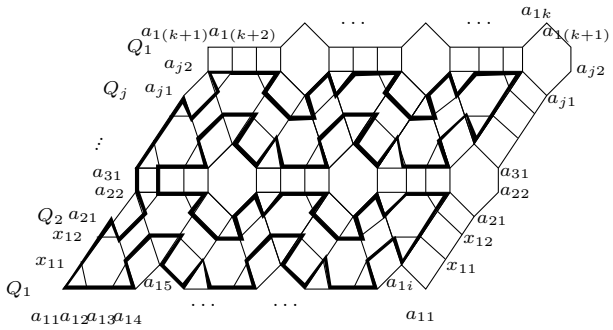


Figure 5.12.2: $M(i, j, k)$

Lemma 15. *A DSEM M of type T_{14} admits an $M(i,j,k)$-representation if and only if the following holds: (i) j even, (ii) $i \geq 5$ and $i = 5m$, $m \in \mathbb{N}$, (iii) number of vertices of $M(i,j,k) = 9ij/5 \geq 18$, (iv) $k \in \{5r : 0 \leq r < i/5\}$.*

5.13 DSEMs of type $T_{15} = [f_1^{(f_2^2 \cdot f_1 \cdot f_2^2 \cdot f_1^2)} : f_2^{(f_1^4 \cdot f_2^5)}]$, where $f_1 = (3^2.4.3.4)$, $f_2 = (3.4.6.4)$

Let M be a DSEM of type T_{15}. We consider a path in M as follows.

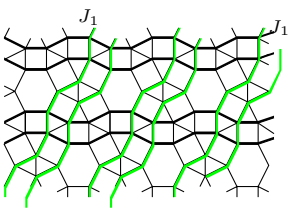


Figure 5.13.1: M of type T_{15}

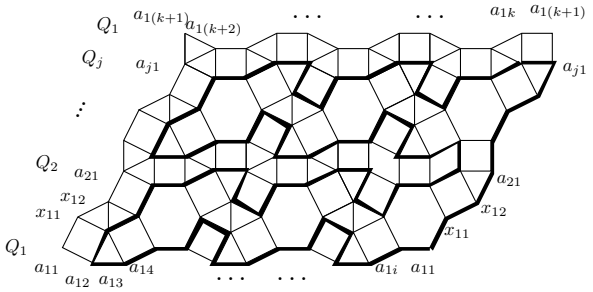


Figure 5.13.2: $M(i,j,k)$

• A path $P_1 = P(\ldots, y_i, y_{i+1}, z_i, z_{i+1}, \ldots)$ in M, say of type J_1 (indicated by thick black or green paths in Figure 5.13.1). The face-sequences of y_i's and z_j's are (3.4.6.4) and $(3^2.4.3.4)$ respectively.

Now, construct $M(i,j,k)$ representation of M by first cutting M along a black colored cycle of type J_1 and then cutting it along the green colored cycle of type J_1, where without loss of generality, let the beginning adjacent face to the base horizontal cycle Q_1 is a quadrangle.

Lemma 16. *A DSEM M of type T_{15} admits an $M(i,j,k)$-representation if and only if the following holds: (i) j even and $i = 4m$, $m \in \mathbb{N}$, (ii) number of vertices of $M(i,j,k) = 3ij/2 \geq 12$, (iii) $k \in \{4r : 0 \leq r < i/4\}$.*

Proof. Let M be a DSEM of type T_{15} having n vertices. Then its $M(i,j,k)$ has j disjoint horizontal cycles of J_1 type, say $Q_0, Q_1, \ldots, Q_{j-1}$, of length i. Let $Q_0 = C(w_{0,0}, w_{0,1}, \ldots, w_{0,i-1}), Q_1 = C(w_{1,0}, w_{1,1}, \ldots, w_{1,i-1}), \ldots, Q_{j-1} = C(w_{j-1,0}, w_{j-1,1}, \ldots, w_{j-1,i-1})$ be the list of horizontal cycles. Observe that the vertices having face-sequence $(3^2, 4, 3, 4)$ lying between horizontal cycles $Q_{2s(mod\, j)}$ and $Q_{(2s+1)(mod\, j)}$ for $0 \leq s \leq j-1$ is $4i/4 \cdot j/2$. Therefore, $n = ij + ij/2 = 3ij/2$.

If $j = 1$, then no vertex in the base horizontal cycle follow the face-sequence $(3^2.4.3.4)$ after identifying the boundaries of $M(i,1,k)$. Therefore, $j \geq 2$. If $j \geq 2$ and j is not an even integer, then as above no vertex in the base horizontal cycle follows the face-sequence $(3^2.4.3.4)$. So j is an even integer. If

j is even and $i \neq 4m$, where $m \in \mathbb{N}$, then the $\mathrm{lk}(w_{0,0})$ is not of type (3.4.6.4). Which is not possible. So, j is an even integer and $i = 4m$, where $m \in \mathbb{N}$. Thus, $3ij/ \geq 12$.

If j is an even integer and $k \in \{r : 0 \leq r \leq i-1\} \setminus (k \in \{4r : 0 \leq r < i/4\})$, then some vertices in $M(i,j,k)$ have face-sequences neither $(3^2.4.3.4)$ nor (3.4.6.4). So, $k \in \{4r : 0 \leq r < i/4\}$. The converse follows from Figure 5.13.2. This completes the proof. □

5.14 DSEMs of type $T_{16} = [f_1^{(f_2^3.f_1^2.f_2^3.f_1)} : f_2^{(f_2^{11}.f_1.f_2^2.f_1^2)}]$, where $f_1 =$ (3.4.6.4), $f_2 =$ (4.6.12)

Let M be a DSEM of type T_{16}. In M, consider a fixed type path $P_1 = P(\ldots, y_i, y_{i+1}, y_{i+2}, y_{i+3}, z_i, z_{i+1}, \ldots)$ in M, say of type O_1, indicated by thick black or green paths (shown in Figure 5.14.1). The vertices y_i's and z_i's have the face-sequences (4.6.12) and (3.4.6.4) respectively.

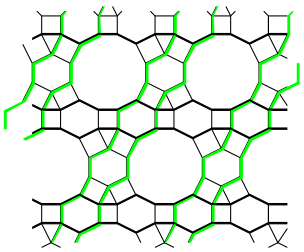


Figure 5.14.1: M of type T_{16}

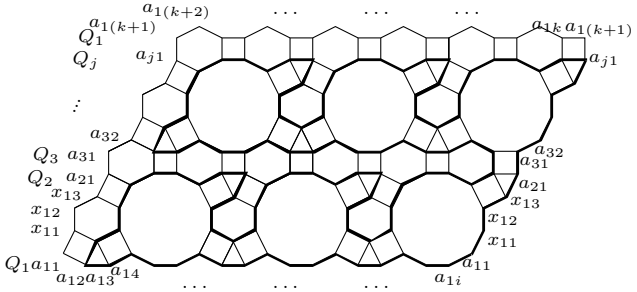


Figure 5.14.2: $M(i,j,k)$

We construct $M(i,j,k)$ representation of M by first cutting M along a black coloured cycle of type O_1 through a vertex v having face-sequence (4.6.12) and then cutting it along a green coloured cycle of type O_1 where the beginning adjacent face to black colored cycle is a 4-gon. The relations among i, j, k are given in below lemma (proof follows from a similar arguments as in Lemma 10).

Lemma 17. *A DSEM M of type T_{16} admits an $M(i,j,k)$-representation if and only if the following holds: (i) j even, (ii) $i \geq 12$ and $i = 6m$, $m \in \mathbb{N}$, (iii) number of vertices of $M(i,j,k) = 3ij/2 \geq 36$, (iv) if $j = 2$ then $k \in \{3r+2 : 1 < r < (i-3)/3\}$, and if $j \geq 4$ then $k \in \{3r+2 : 0 \leq r < i/3\}$.*

5.15 DSEMs of type $T_{17} = [f_1^{F_1} : f_2^{F_2}]$, where $f_1 = (3.4.3.12)$, $f_2 = (3.12^2)$, $F_1 = (f_2^2.f_1.f_2^2.f_1.f_2^2.f_1.f_2^2.f_1^3)$, $F_2 = (f_1.f_2^2.f_1.f_2^2.f_1.f_2^2.f_1.f_2.f_1.f_2^2.f_1.f_2^2.f_1.f_2^2.f_1)$

Let M be a DSEM of type T_{17}. In M, consider a fixed type path $P_1 = P(\ldots, y_i, z_i, z_{i+1}, y_{i+1}, \ldots)$, say of type N_1, indicated by thick black or green paths (shown in Figure 5.15.1). The vertices y_i's and z_i's have the face-sequences $(3.4.3.12)$ and (3.12^2) respectively.

Now construct an $M(i, j, k)$ representation of M, as in previous subsections, by cutting M through a vertex v with the face-sequence $(3, 12^2)$ one by one along the cycles of type N_1. Assume that the beginning adjacent face to the base horizontal cycle Q_1 is 3-gon.

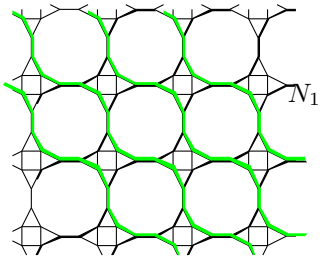


Figure 5.15.1: M of type T_{17}

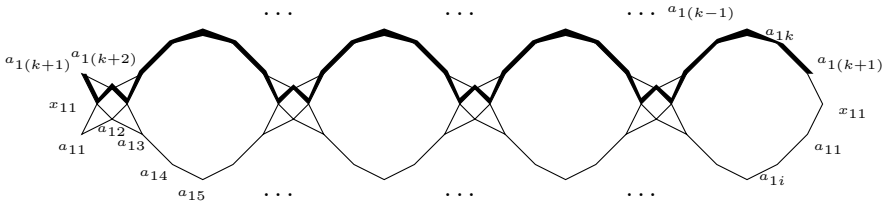


Figure 5.15.2: $M(i, 1, k)$

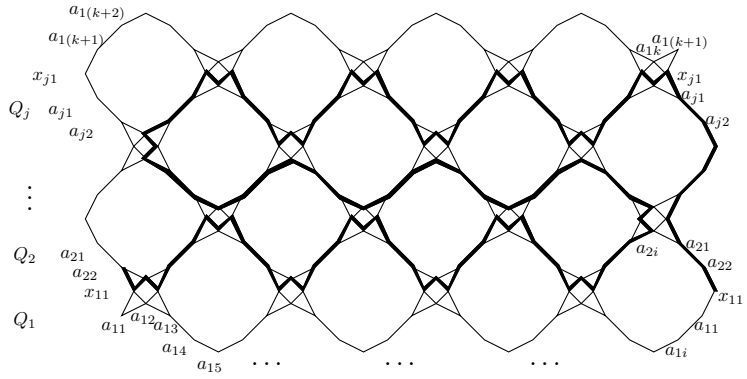


Figure 5.15.3: $M(i, j = 2m, k)$

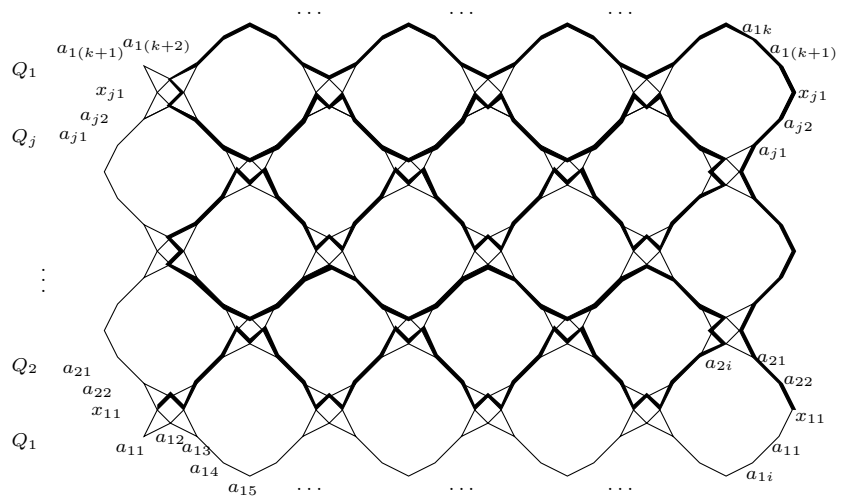


Figure 5.15.4: $M(i, j = 2m + 1, k)$

Lemma 18. *A DSEM M of type T_{17} admits an $M(i, j, k)$-representation if and only if the following holds: (i) $j \geq 1$ and $i = 6m$, $m \in \mathbb{N}$, (ii) $i \geq 24$ for $j = 1$, and $i \geq 18$ for $j = 2$, (iii) $i \geq 12$ for $j > 2$, (iv) number of vertices of $M(i, j, k) = 8ij/6 \geq 32$, (v) if $j = 1$ then $k \in \{6r + 3 : 1 \leq r < (i - 6)/6\}$, if $j = 2$ then $k \in \{6r + 5 : 1 \leq r < (i - 12)/6\}$, if $j = 2m + 1$ then $k \in \{6r + 3 : 0 \leq r \leq (i - 6)/6\}$, and if $j = 2m + 2$ then $k \in \{6r + 5 : 0 \leq r \leq (i - 6)/6\}$.*

Proof. Let M be a DSEM of type T_{17} having n vertices. An $M(i, j, k)$ of M has j disjoint horizontal cycles of N_1 type, say $Q_0, Q_1, \ldots, Q_{j-1}$, of length i. Let $Q_0 = C(w_{0,0}, w_{0,1}, \ldots, w_{0,i-1}), Q_1 = C(w_{1,0}, w_{1,1}, \ldots, w_{1,i-1}), \ldots, Q_{j-1} =$

$C(w_{j-1,0}, w_{j-1,1}, \ldots, w_{j-1,i-1})$ be the list of horizontal cycles. Observe that the vertices having face-sequence (3.4.3.12) which are lie above the cycles Q_s and not belonging to any of these cycles Q_s for $0 \leq s \leq j-1$ is $2i/6 \cdot j$. So, the total number of vertices in M is $n = ij + 2ij/6 = 8ij/6$.

If $j = 1$ and $i \neq 6m$, where $m \in \mathbb{N}$, then the face-sequence of $w_{0,0}$ is not of the type (3.12^2). This is not possible. So, if $j = 1$, then $i = 6m$, where $m \in \mathbb{N}$. Similarly, if $j > 1$, then $i = 6m$.

If $j = 1$ and $i < 24$, then link of some vertices can not be completed. So, for $j = 1$, we get $i \geq 24$. Similarly, as above, we get that $i \geq 18$ if $j = 2$ and $i \geq 12$ for $j > 2$. Thus, $n = 8ij/6 \geq 32$.

If $j = 1$ and $k \in \{r : 0 \leq r \leq i-1\} \setminus (k \in \{6r+3 : 1 \leq r < (i-6)/6\})$ then some vertices do not have the given face-sequences. So, $k \in \{6r + 3 : 1 \leq r < (i-6)/6\}$ for $j = 1$. Proceeding similarly, we see if $j = 2$ then $k \in \{6r + 5 : 1 \leq r < (i-12)/6\}$, if $j = 2m+1$ then $k \in \{6r+3 : 0 \leq r \leq (i-6)/6\}$, and if $j = 2m + 2$ then $k \in \{6r + 5 : 0 \leq r \leq (i-6)/6\}$. The converse follows from Figures 5.15.2,5.15.3 and 5.15.4. □

5.16 DSEMs of type $T_{18} = [f_1^{F_1} : f_2^{F_2}]$, where $f_1 = (3^4.6), f_2 = (3^2.6^2)$, $F_1 = (f_2^2.f_1.f_2.f_1.f_2^2.f_1)$, $F_2 = (f_2^2.f_1^3.f_2^2.f_1.f_2.f_1)$

Let M be a DSEM of type T_{18}. Consider the following types of paths as follows.

• A path $P_1 = P(\ldots, y_i, z_i, y_{i+1}, z_{i+1}, \ldots)$ in M, say of type I_1 (indicated by thick black paths in Figure 5.16.1). Here, the face-sequences of y_i's and z_j's are $(3^2.6^2)$ and $(3^4.6)$ respectively.

• A path $P_2 = P(\ldots, y_i, y_{i+1}, z_i, z_{i+1}, y_{i+2}, y_{i+3}, \ldots)$ in M, say of type I_2 (indicated by green paths in Figure 5.16.1). Here, the face-sequences of y_i's and z_j's are $(3^2.6^2)$ and $(3^4.6)$ respectively.

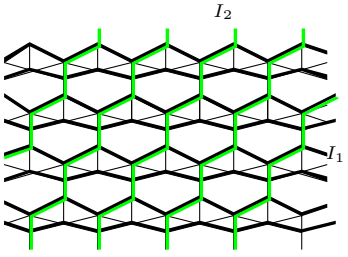


Figure 5.16.1: M of type T_{18}

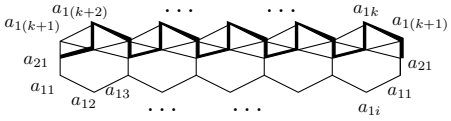


Figure 5.16.2: $M(i, 2, k)$

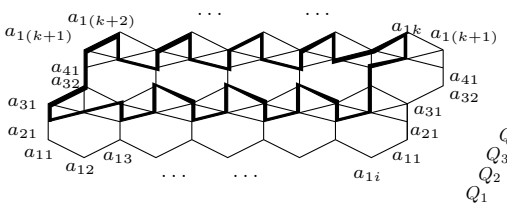


Figure 5.16.3: $M(i, 4, k)$

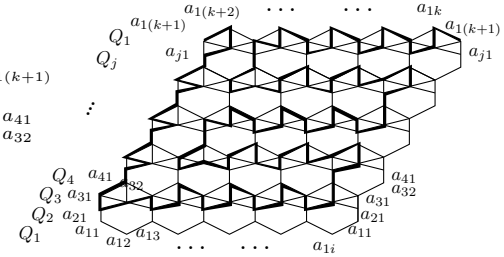


Figure 5.16.4: $M(i, j = 4m + 2, k)$

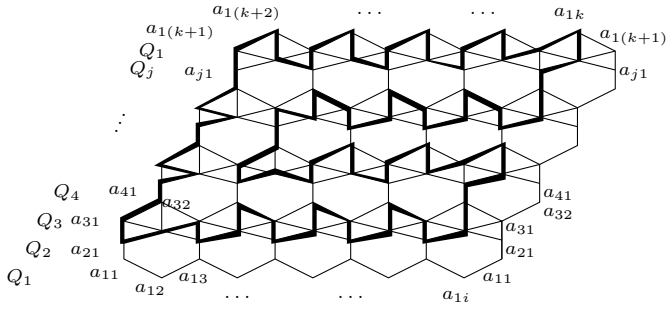


Figure 5.16.5: $M(i, j = 4m + 4, k)$

An $M(i, j, k)$ representation of M is obtained here by first cutting M along a cycle of type I_1 and then along a cycle of type I_2. Without loss of generality, assume that the beginning adjacent face to the cycle I_1 is a hexagon (as shown in Figures 5.16.2, 5.16.3, 5.16.4, and 5.16.5). Then we have:

Lemma 19. *A DSEM M of type T_{18} admits an $M(i, j, k)$-representation if and only if the following holds: (i) $i \geq 6$ and i, j even, (ii) number of vertices in $M(i, j, k) = ij \geq 12$, (iii) if $j = 2$ then $k \in \{2r + 3 : 0 \leq r \leq (i - 6)/2\}$, and if $j \geq 4$ then $k \in \{2r + 1 : 0 \leq r \leq (i - 2)/2\}$.*

Proof. Let M be a DSEM of type T_{18} with n vertices. Then its $M(i, j, k)$ has j disjoint horizontal cycles of I_1 type having length i. Thus the number of vertices in M is $n = ij$.

If $j = 1$, then $M(i, 1, k)$ has no vertex with face-sequence $(3^4.6)$. It is not possible. Therefore, $j \geq 2$. If $j \geq 2$ and j is not an even integer, then no vertex in the base horizontal cycle follows the face-sequence $(3^4.6)$ after identifying the boundaries of $M(i, j, k)$. Therefore, j is even.

If j is an even integer and $i < 6$, then we get some vertex in $M(i, 1, k)$ whose link can not be constructed. So, $i \geq 6$. Also, if $i \geq 6$ and i is not an even integer, then similarly as above, there is no vertex in the horizontal base cycle having face-sequences $(3^4.6)$ and $(3^2.6^2)$. Which is not possible. So, $i \geq 6$ and i, j are even integers. Thus, $n = ij \geq 12$.

If $j = 2$ and $k \in \{r : 0 \leq r \leq i-1\} \setminus (\{2r+3 : 0 \leq r \leq (i-6)/2\}$, then some vertices in the lower horizontal cycle do not follow the face-sequences $(3^4.6)$ and $(3^2.6^2)$. Which is a contradiction. So, for $j = 2$, we get $k \in \{2r+3 : 0 \leq r \leq (i-6)/2\}$. Similarly, as above, we see that if $j \geq 4$, then $k \in \{2r+1 : 0 \leq r \leq (i-2)/2\}$. The converse follows from Figures 5.16.2, 5.16.3, 5.16.4, and 5.16.5. This completes the proof. □

5.17 DSEMs of type $T_{19} = [f_1^{(f_1^4.f_2.f_1^2.f_2)} : f_2^{(f_1^4.f_2.f_1^4.f_2)}]$, where $f_1 = (3^4.6), f_2 = (3.6.3.6)$

Let M be a DSEM of type T_{19} on the torus. We consider a path in M as follows.

• A path $P_1 = P(\ldots, z_{i-1}, z_i, z_{i+1}, \ldots)$ in M, say of type T_1 indicated by thick colored lines, shown in Figure 5.17.1. The vertices z_i's of the path have the face-sequence $(3^4, 6)$.

• A path $P_2 = P(\ldots, y_i, x_i, x_{i+1}, y_{i+1}, , \ldots)$ in M, say of type T_2 indicated by thick colored lines, shown in Figure 5.17.2. The vertices y_i's and x_i's of the path have the face-sequences $(3^4, 6)$ and $(3, 6, 3, 6)$ respectively.

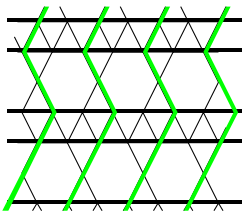


Figure 5.17.1: M **of Type** T_{19}

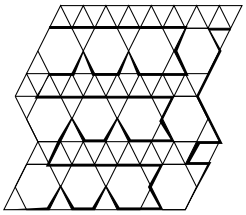


Figure 5.17.2: $M(i, j, k)$

Here, we construct $M(i, j, k)$ representation of M by first cutting M along a black coloured cycle of type T_1 through a vertex v having face-sequence $(3^4, 6)$ and then cutting it along a green coloured cycle of type T_2 where the beginning adjacent face to black coloured cycle is a 3-gon.

We define admissible relations among i, j, k of $M(i, j, k)$ in the next lemma, and its proof follows from the similar arguments as in Lemma 8.

Lemma 20. A DSEM M of type T_{19} admits an $M(i, j, k)$-representation if and only if the following holds: (i) j even, (ii) $i \geq 6$ and i even, (iii) number of vertices of $M(i, j, k) = 5ij/2 \geq 15$, (iv) if $j = 2$ then $4 \leq k \leq i-1$, and if $j \geq 4$ then $0 \leq k < i-1$.

5.18 DSEMs of type $T_{20} = [f_1^{(f_1^7.f_2)}, f_2^{(f_1^{10})}]$, where $f_1 = (3^4.6), f_2 = (3.6.3.6)$

Let M be a DSEM of type T_{20}. Consider a path in M as follows.
• A path $P_1 = P(\ldots, z_{i-1}, z_i, z_{i+1}, \ldots)$ in M, say of type U_1 indicated by thick black colored lines, shown in Figure 5.18.1. Observe that, the vertices z_i's have the face-sequence $(3^4, 6)$.
• A path $P_2 = P(\ldots, y_{i-1}, y_i, y_{i+1}, \ldots)$ in M, say of type U_2 indicated by thick green colored lines, shown in Figure 5.18.2. Note that, the vertices y_i's have the face-sequence $(3^4, 6)$.

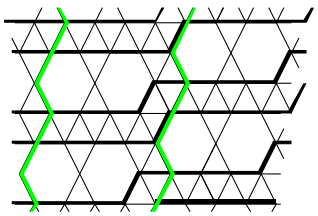


Figure 5.18.1: M **of Type** T_{20}

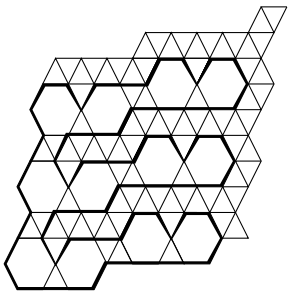


Figure 5.18.2: $M(i, j, k)$

We construct $M(i, j, k)$ representation of M by first cutting M along a black coloured cycle of type U_1 through a vertex v having face-sequence $(3^4, 6)$ and then cutting it along a green coloured cycle of type U_1 where the beginning adjacent face to black coloured cycle is a 6-gon.

We define admissible relations among i, j, k of $M(i, j, k)$ in the next lemma, and its proof follows from the similar arguments as in Lemma 16.

Lemma 21. A DSEM M of type T_{20} admits an $M(i, j, k)$-representation if and only if the following holds: (i) j even, (ii) $i = 5m$, $m \in \mathbb{N}$, (iii) number of vertices of $M(i, j, k) = 11ij/10 \geq 22$, (iv) if $j = 2$ then $k \in \{5r : 1 \leq r < i/5\}$, and if $j \geq 4$ then $k \in \{5r : 0 \leq r < i/5\}$.

5.19 DSEMs of type $T_{21} = [f_1^{G_1} : f_2^{G_2}]$, where $f_1 = (3^2.6^2)$, $f_2 = (3.6.3.6)$, $G_1 = (f_1^2.f_2.f_1.f_2.f_1^2.f_2.f_1.f_2)$, $G_2 = (f_1^4.f_2.f_1^4.f_2)$

Let M be a DSEM of type T_{21}. We consider the following types of paths in M as follows.
• A path $P_1 = P(\ldots, y_{i-1}, y_i, y_{i+1}, \ldots)$ in M, say of type F_1 (indicated by thick black paths in Figure 5.19.1). The vertices y_i's have the face-sequence $(3^2, 6^2)$.
• A path $P_2 = P(\ldots, y_i, z_i, y_{i+1}, z_{i+1}, \ldots)$ in M, say of type F_2 (indicated by green paths in Figure 5.19.1). The vertices y_i's and z_i's have the face-sequences

$(3^2.6^2)$ and $(3.6.3.6)$ respectively.

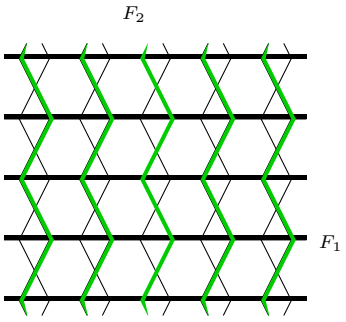


Figure 5.19.1: M of type T_{21}

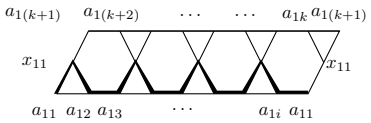


Figure 5.19.2: $M(i,1,k)$

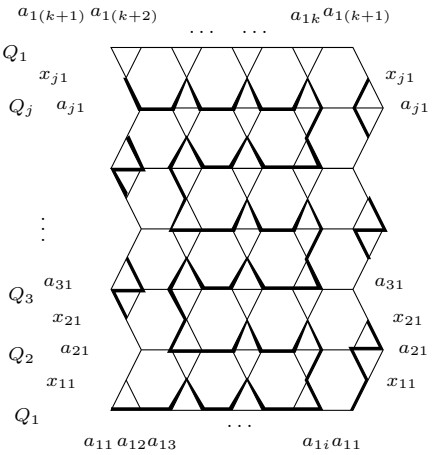


Figure 5.19.3: $M(i, j = 2m, k)$

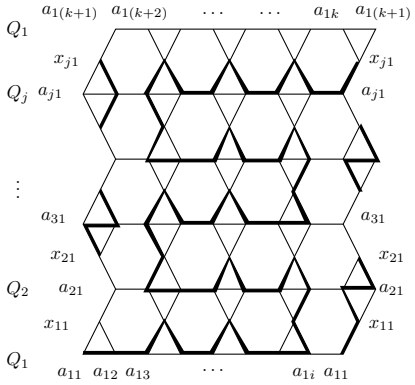


Figure 5.19.4: $M(i, j = 2m + 1, k)$

A construction of an $M(i, j, k)$ representation for M follows by first cutting M along a cycle of type F_1 and then along a cycle of type F_2. This gives the following result.

Lemma 22. *A DSEM M of type T_{21} admits an $M(i, j, k)$-representation if and only if the following holds: (i) $j \geq 1$ and i even, (ii) number of vertices of $M(i, j, k) = 3ij/2 \geq 15$, (iii) $i \geq 10$ if $j = 1$, and $i \geq 6$ if $j \geq 2$, (iv) if $j = 1$ then $k \in \{2r + 5 : 0 \leq r < (i - 8)/2\} \setminus \{(i - 8)/2 + 5\}$, if $j = 2$ then $k \in \{2r : 0 < r < i/2\}$, if $j = 2m + 1$, where $m \in \mathbb{N}$ then $k \in \{2r + 1 : 0 \leq r < i/2\}$, and if $j = 2m + 2$, where $m \in \mathbb{N}$ then $k \in \{2r : 0 \leq r < i/2\}$.*

Proof. Let M be a DSEM of type T_{21} with n vertices. Then $M(i, j, k)$ of M has j disjoint horizontal cycles of type F_1, say $Q_0, Q_1, \ldots, Q_{j-1}$, of length i. Note

that the number of adjacent vertices with face-sequence (3.6.3.6) which are lying on one side of horizontal cycles Q_s and not belonging to any horizontal cycles Q_s for $0 \leq s \leq j-1$ is $i/2 \cdot j$. So, $n = ij + ij/2 = 3ij/2$.

If $j = 1$ and i is not an even integer, then some vertex in the base horizontal cycle do not have face-sequence $(3^2.6^2)$, which is not possible. So, if $j = 1$, then i is an even integer. Similarly, if $j > 1$, then i is an even integer.

If $j = 1$ and $i < 10$, then we get some vertex in $M(i, 1, k)$ whose link can not be constructed. So, for $j = 1$, we get $i \geq 10$. Similarly, as above, we get that $i \geq 6$ if $j \geq 2$. Thus $n = 3ij/2 \geq 15$.

If $j = 1$ and $k \in \{r : 0 \leq r \leq i-1\} \setminus (\{2r+5 : 0 \leq r < (i-8)/2\} \setminus \{(i-8)/2+5\})$, then we get some vertices whose link can not be constructed. So, for $j = 1$, we get $k \in \{2r+5 : 0 \leq r < (i-8)/2\} \setminus \{(i-8)/2+5\}$. If $j = 2$ and $k \in \{r : 0 \leq r \leq i-1\} \setminus \{2r : 0 < r < i/2\}$, then some vertices in the lower horizontal cycle do not follow the face-sequence $(3^2.6^2)$ after identifying the boundaries of $M(i, j, k)$. Which is a contradiction. So, for $j = 2$, we get $k \in \{2r : 0 < r < i/2\}$. Similarly as above, we see that if $j = 2m+1$, where $m \in \mathbb{N}$, then $k \in \{2r+1 : 0 \leq r < i/2\}$, and if $j = 2m+2$, where $m \in \mathbb{N}$, then $k \in \{2r : 0 \leq r < i/2\}$. The converse follows from the Figures 5.19.2, 5.19.3, and 5.19.4. This completes the proof. □

5.20 DSEMs of type $T_{22} = [f_1^{(f_1^5.f_2.f_1^2.f_2)} : f_2^{(f_1^4.f_2.f_1^4.f_2)}]$, where $f_1 = (3.4^2.6), f_2 = (3.6.3.6)$

Let M be a DSEM of type T_{22}. In M, we consider following types of paths as follows.

• A path $P_1 = P(\ldots, y_{i-1}, y_i, y_{i+1}, \ldots)$ in M, say of type D_1 (indicated by thick black paths in Figures 5.20.1 - 5.20.4). The face-sequence of y_i's is $(3.4^2.6)$.

• A path $P_2 = P(\ldots, y_i, y_{i+1}, z_i, y_{i+2}, y_{i+3}, z_{i+1}, \ldots)$ in M, say of type D_2 (indicated by green paths in Figures 5.20.1 - 5.20.4). The face-sequences of y_i's and z_i's are $(3.4^2.6)$ and $(3.6.3.6)$ respectively.

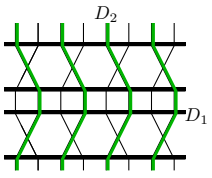


Figure 5.20.1

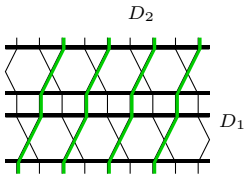


Figure 5.20.2

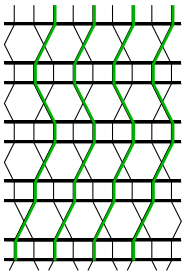


Figure 5.20.3

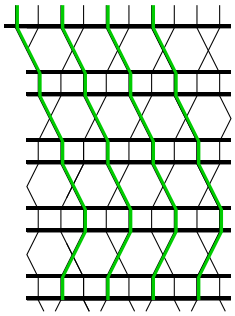


Figure 5.20.4

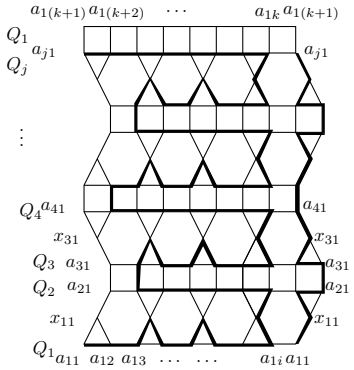


Figure 5.20.5: $M(i,j,k)$

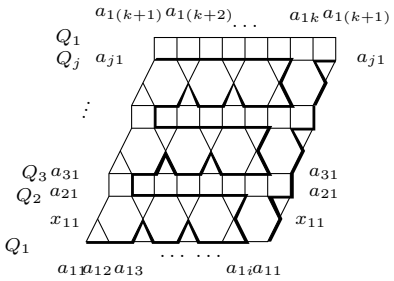


Figure 5.20.6: $M(i,j,k)$

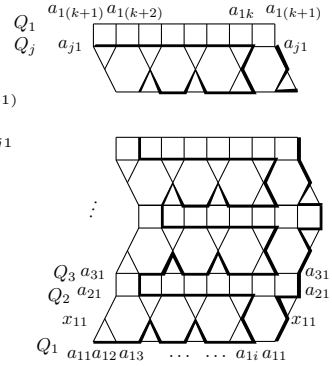


Figure 5.20.7: $M(i,j,k)$

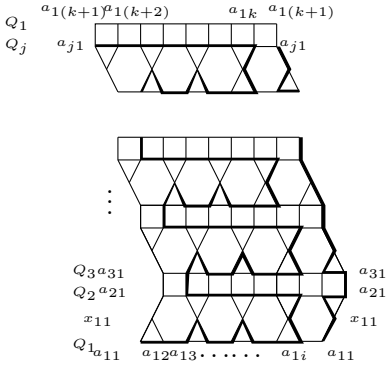


Figure 5.20.8: $M(i,j,k)$

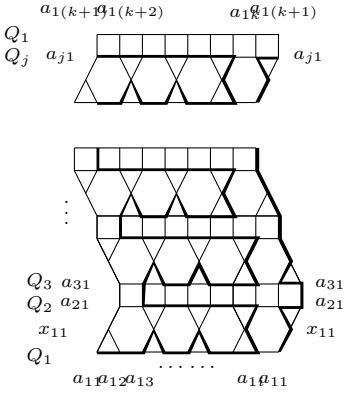


Figure 5.20.9: $M(i,j,k)$

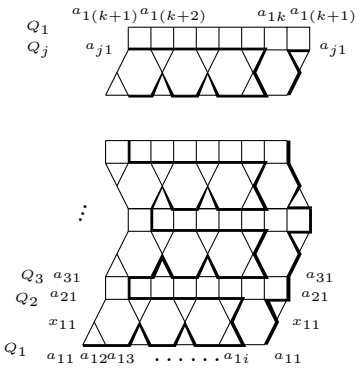


Figure 5.20.10: $M(i, j, k)$

Note that, there are infinitely many DSEMs of this type. In each of the type, we consider paths of type D_2 and there movement is either as in Figure 5.20.1 or either as in Figure 5.20.2 or they follow both movement as in Figures 5.20.1 and 5.20.2 (for example see Figures 5.20.3 and 5.20.4).

An $M(i, j, k)$ representation of M is obtained here by cutting M first along a cycle of type D_1 and then along a cycle of type D_2. Without loss of generality, let the starting adjacent face to the base horizontal cycle Q_1 (of type D_1) is a 3-gon (as shown in Figures 5.20.5-5.20.10).

Lemma 23. *A DSEM M of type T_{22} admits an $M(i, j, k)$-representation if and only if the following holds: (i) $i \geq 6$ and i, j even, (ii) number of vertices of $M(i, j, k) = 5ij/4 \geq 15$, (iii) if $j = 2$ then $4 \leq k \leq i - 2$, and if $j \geq 4$ then $0 \leq k \leq i - 1$.*

Proof. Let M be a DSEM of type $[3.4^2.6 : 3.6.3.6]$ with n vertices. Its $M(i, j, k)$ representation has j disjoint horizontal cycles of D_1 type, say $Q_0, Q_1, \ldots, Q_{j-1}$, of length i. Note that the number of vertices between the horizontal cycles $Q_{2s(mod\, j)}$ and $Q_{(2s+1)(mod\, j)}$ for $0 \leq s \leq j-1$ with the face-sequence (3.6.3.6) is $i/2 \cdot j/2$. Therefore, the number of vertices in M is $n = ij + ij/4 = 5ij/4$.

If $j = 1$, then $M(i, 1, k)$ has no vertex with face-sequence $(3, 4^2, 6)$. This is not possible. Thus, $j \geq 2$. If $j \geq 2$ and j is not an even integer, then we see that there is no vertex in the base horizontal cycle which has face-sequence $(3.4^2.6)$ after identifying the boundaries of $M(i, j, k)$. Thus j is even.

For $i \leq 4$, M is not a map. So $i > 4$. If $i > 4$ and not an even integer, M is not a map. So $i \geq 6$, and i, j are even. Thus $n = 5ij/4 \geq 15$.

If $j = 2$ and $k \in \{r : 0 \leq r \leq i - 1\} \setminus \{0, 1, 2, 3, i - 1\}$, then we get some vertices whose link can not be constructed. So, for $j = 2$, we get $4 \leq k \leq i-2$. Similarly, if $j \geq 4$, then $0 \leq k \leq i - 1$. The converse follows from Figures 5.20.5-5.20.10. Thus the proof. □

Proof of Theorem 1 Let M be a DSEM of type T_1, T_2, T_3, T_4, and T_7. Then every M has its $M(i,j,k)$ representation (shown in Figures 5.2.1.1, 5.2.1.2, 5.2.1.3, 5.2.1.4 and 5.2.4.2). Now, if we remove the diagonal edges (indicated by thick black edges), we get a map, say M', on the torus of type $[4^4]$. Clearly, $V(M) = V(M')$ and $E(M) \subset E(M')$. By [Theorem 4, [1]] the map M' contains a Hamiltonian cycle, say H. The cycle H is also a Hamiltonian cycle in M as $E(M') \subset E(M)$. Hence, M is Hamiltonian. If M is a DSEM on the torus of remaining type T_i then it has its $M(i,j,k)$ representation of type T_i, shown in respective subsections. The Hamiltonian cycle (thick black colored cycle) in $M(i,j,k)$ is also a Hamiltonian cycle in M. Thus M is Hamiltonian. □

Thus, we have shown that there are twenty two types of DSEMs, $T_1, T_2, \ldots, T_{22}$ with curvature 0 which provide infinitely many DSEMs of respective types on the torus and every such DSEMs contains Hamiltonian cycle. Also, apart from DSEMs of types T_{16}, T_{17} all the DSEMs are 4-connected. This establish the Nash-Williams conjecture for 4-connected DSEMs on the torus.

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