



Fixed point theory in admissible acyclic dominated spaces

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Abstract

In this paper we present Lefschetz type fixed point theorems for maps defined on admissible acyclic dominated spaces.

1 Introduction

In this paper we present fixed point results for two general classes of admissible maps of Gorniewicz type. In particular we establish Lefschetz type fixed point results for these maps on admissible acyclic dominated spaces. Our results improve and complement fixed point theorems in the literature, see [6– 8, 14–17] and the references therein.

First we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q -dimensional Čech homology group with compact carriers of X . For a continuous map $f : X \rightarrow X$, $H(f)$ is the induced linear map $f_* = \{f_{*q}\}$ where $f_{*q} : H_q(X) \rightarrow H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X , Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \rightarrow X$ is called a Vietoris map (written $p : \Gamma \rightrightarrows X$) if the following two conditions are satisfied:

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- (i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic
- (ii). p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \rightarrow Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:

- (i). p is a Vietoris map
- and
- (ii). $q(p^{-1}(x)) \subseteq \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [8]. An upper semicontinuous map $\phi : X \rightarrow 2^Y$ (nonempty subsets of Y) with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ . An example of an admissible map is a Kakutani map. An upper semicontinuous map $\phi : X \rightarrow CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X, Y)$); here Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y . Another example is an acyclic map which we now describe. Let X and Z be subsets of Hausdorff topological spaces and let $F : X \rightarrow K(Z)$ i.e. F has nonempty compact values. Recall a nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now we consider maps $F : X \rightarrow Ac(Z)$ i.e. $F : X \rightarrow K(Z)$ with acyclic values (i.e. F has nonempty acyclic compact values). We say $F \in AC(X, Z)$ (i.e. F is an acyclic map) if $F : X \rightarrow Ac(Z)$ is upper semicontinuous.

For a subset K of a topological space X , we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given a map $F : X \rightarrow 2^X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an α -fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$.

Given two maps $F, G : X \rightarrow 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α -close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$. Of course, given two single valued maps $f, g : X \rightarrow Y$ and $\alpha \in Cov(Y)$, then f and g are α -close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$. Given a map $F : X \rightarrow 2^Y$ and a single valued map $g : X \rightarrow Y$ and $\alpha \in Cov(Y)$, F and g are said to be strongly α -close if for any $x \in X$ there exists a $U_x \in \alpha$ with $F(x) \subseteq U_x$ and $g(x) \in U_x$. We say f and g are α -homotopic if there is a homotopy $h_t : X \rightarrow Y$ ($t \in [0, 1]$) joining f and g such that for each $x \in X$ the values $h_t(x)$ belong to a common $U_x \in \alpha$ for all $t \in [0, 1]$. We say f and g are homotopic if there is a homotopy $h_t : X \rightarrow Y$ ($t \in [0, 1]$) joining f and g .

The following result can be found in [2].

Theorem 1.1. *Let X be a regular topological space, $F : X \rightarrow 2^X$ an upper semicontinuous map with closed values and suppose there exists a cofinal covering $\theta \subseteq \text{Cov}_X(\overline{F(X)})$ such that F has an α -fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

Remark 1.2. From Theorem 1.1 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values it suffices [3, page 298] to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [10, page 199], so such refinements form a cofinal family of open covers). Note also that uniform spaces are regular (in fact completely regular [5]). Also note in Theorem 1.1 if F is compact valued, then the assumption that X is regular can be removed. We note here that when we apply Theorem 1.1 we will assume the space is uniform. Of course one could consider other appropriate spaces (like regular (Hausdorff) spaces) as well.

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \rightarrow Y$ extends to a continuous function $f : X \rightarrow Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in \text{Cov}(Y)$ and any pair (X, K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \rightarrow Y$ there exists a continuous function $f : X \rightarrow Y$ such that $f|_K$ is α -close to f_0 . A space Y is a neighborhood extension space for Q (written $Y \in NES(Q)$) if $\forall X \in Q, \forall K \subseteq X$ closed in X , and any continuous function $f_0 : K \rightarrow Y$ there exists a continuous extension $f : U \rightarrow Y$ of f_0 over a neighborhood U of K in X . A space Y is an approximate neighborhood extension space for Q (written $Y \in ANES(Q)$) if $\forall \alpha \in \text{Cov}(Y), \forall X \in Q, \forall K \subseteq X$ closed in X , and any continuous function $f_0 : K \rightarrow Y$ there exists a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \rightarrow Y$ such that $f_\alpha|_K$ and f_0 are α -close. A space Y is a strongly approximate neighborhood extension space for Q (written $Y \in SANES(Q)$) if $\forall \alpha \in \text{Cov}(Y), \forall X \in Q, \forall K \subseteq X$ closed in X , and any continuous function $f_0 : K \rightarrow Y$ there exists a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \rightarrow Y$ such that $f_\alpha|_K$ and f_0 are α -close and homotopic.

Next we describe the maps due to Wu [19]. Let X and Y be subsets lying in Hausdorff topological vector spaces and we say $\Phi \in W(X, Y)$ if $\Phi : X \rightarrow 2^Y$ and there exists a lower semicontinuous map $\theta : X \rightarrow 2^Y$ with $\overline{\text{co}}(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Next we recall a selection theorem [1] (see the proof in Theorem 1.1 there) for Wu maps.

Theorem 1.3. *Let X be a paracompact subset of a Hausdorff topological vector space and Y a metrizable complete subset of a Hausdorff locally convex linear topological space. Suppose $\Phi \in W(X, Y)$ and let $\theta : X \rightarrow 2^Y$ be a lower semicontinuous map with $\overline{\text{co}}(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Then there exists a upper semicontinuous map $\Psi : X \rightarrow CK(Y)$ (collection of nonempty convex compact subsets of Y) with $\Psi(x) \subseteq \overline{\text{co}}(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.*

Remark 1.4. Let X be paracompact and Y a metrizable subset of a complete Hausdorff locally convex linear topological space E and $\Phi \in W(X, Y)$ with $\theta : X \rightarrow 2^Y$ a lower semicontinuous map and $\overline{\text{co}}(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Note [13] that $\overline{\text{co}}\theta : X \rightarrow 2^Y$ (since $\overline{\text{co}}(\theta(x)) \subseteq \Phi(x) \subseteq Y$ for $x \in X$) is lower semicontinuous, so from Michael's selection theorem there exists a continuous (single valued) map $f : X \rightarrow Y$ with $f(x) \in \overline{\text{co}}(\theta(x))$ for $x \in X$, so consequently $f(x) \in \overline{\text{co}}(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.

Let Z be a subset of a Hausdorff topological space Y_1 and W a subset of a Hausdorff topological vector space Y_2 and G a multifunction. We say $F \in HLPY(Z, W)$ [12] if W is convex and there exists a map $S : Z \rightarrow W$ (i.e. $S : Z \rightarrow P(W)$ (collection of subsets of W)) with $\text{co}(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup \{\text{int } S^{-1}(w) : w \in W\}$; here $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in \text{int } S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. For the selection theorem below see [12].

Theorem 1.5. *Let X be a paracompact subset of a Hausdorff topological space, Y a convex subset of a Hausdorff topological vector space and $F \in HLPY(X, Y)$ (let $S : X \rightarrow 2^Y$ with $\text{co}(S(x)) \subseteq F(x)$ for $x \in X$ and $X = \bigcup \{\text{int } S^{-1}(w) : w \in Y\}$). Then there exists a continuous (single-valued) map $f : X \rightarrow Y$ with $f(x) \in \text{co } S(x) \subseteq F(x)$ for all $x \in X$.*

Remark 1.6. These maps are related to the *DKT* maps in the literature and $F \in DKT(Z, W)$ [4] if W is convex and there exists a map $S : Z \rightarrow W$ with $\text{co}(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$. Note these maps were motivated from the Fan maps.

Let X be a subset of a Hausdorff topological space and Y a subset of a Hausdorff topological vector space. We say $T : X \rightarrow 2^Y$ has the strong continuous inclusion property (SCIP) [11] at $x \in X$ if there exists an open set $U(x)$ in X containing x and a $F^x : U(x) \rightarrow 2^Y$ such that $F^x(w) \subseteq T(w)$ for all $w \in U(x)$ and $\text{co } F^x : U(x) \rightarrow 2^Y$ is compact valued and upper semicontinuous. We write $T \in KLU(X, Y)$ if T has the SCIP at every $x \in X$.

In this paper our map T will be a compact map so T has the SCIP is equivalent to T has the CIP [9]. These maps contain as a special case the

Scalzo maps [18] in the literature (see [11 pg12]).

Next we recall a selection theorem [11].

Theorem 1.7. *Let X be a paracompact subset of a Hausdorff topological space, Y a subset of a Hausdorff topological vector space and $T \in KLU(X, Y)$. Then there exists an upper semicontinuous map $G : X \rightarrow CK(Y)$ with $G(w) \subseteq coT(w)$ for all $w \in X$.*

Finally we present some preliminaries on the Lefschetz set for Ad maps needed in Section 2. Let $D(X, Y)$ be the set of all pairs $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p, q) . Given two diagrams (p, q) and (p', q') , where $X \xleftarrow{p'} \Gamma' \xrightarrow{q'} Y$, we write $(p, q) \sim (p', q')$ if there are maps $f : \Gamma \rightarrow \Gamma'$ and $g : \Gamma' \rightarrow \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p, q) \in D(X, Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \xleftarrow{p} \Gamma \xrightarrow{q} Y\} : X \rightarrow Y$$

or $\phi = [(p, q)]$ and is called a morphism from X to Y . We let $M(X, Y)$ be the set of all such morphisms. For any $\phi \in M(X, Y)$ a set $\phi(x) = qp^{-1}(x)$ where $\phi = [(p, q)]$ is called an image of x under a morphism ϕ .

Consider vector spaces over a field K . Let E be a vector space and $f : E \rightarrow E$ an endomorphism. Now let $N(f) = \{x \in E : f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the n^{th} iterate of f , and let $\tilde{E} = E \setminus N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \rightarrow \tilde{E}$. We call f admissible if $\dim \tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \rightarrow E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q Tr(f_q).$$

With Čech homology functor extended to a category of morphisms we have the following well known result [7] (note the homology functor H extends over this category i.e. for a morphism

$$\phi = \{X \xleftarrow{p} \Gamma \xrightarrow{q} Y\} : X \rightarrow Y$$

we define the induced map

$$H(\phi) = \phi_* : H(X) \rightarrow H(Y)$$

by putting $\phi_* = q_* \circ p_*^{-1}$. If $\phi : X \rightarrow Y$ and $\psi : Y \rightarrow Z$ are two morphisms (here X , Y and Z are Hausdorff topological spaces) then $(\psi \circ \phi)_* = \psi_* \circ \phi_*$. Two morphisms $\phi, \psi \in M(X, Y)$ are homotopic (written $\phi \sim \psi$) provided there is a morphism $\chi \in M(X \times [0, 1], Y)$ such that $\chi(x, 0) = \phi(x)$, $\chi(x, 1) = \psi(x)$ for every $x \in X$ (i.e. $\phi = \chi \circ i_0$ and $\psi = \chi \circ i_1$, where $i_0, i_1 : X \rightarrow X \times [0, 1]$ are defined by $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$). Recall the following result [10]: If $\phi \sim \psi$ then $\phi_* = \psi_*$.

A map $\phi \in Ad(X, X)$ is said to be a Lefschetz map if for each selected pair $(p, q) \subset \phi$ the linear map $q_* p_*^{-1} : H(X) \rightarrow H(X)$ (the existence of p_*^{-1} follows from the Vietoris Theorem) is a Leray endomorphism. If $\phi : X \rightarrow X$ is a Lefschetz map, we define the Lefschetz set $\Lambda(\phi)$ (or $\Lambda_X(\phi)$) by

$$\Lambda(\phi) = \{\Lambda(q_* p_*^{-1}) : (p, q) \subset \phi\}.$$

A Hausdorff topological space X is said to be a Lefschetz space (for the class Ad) provided every compact $\phi \in Ad(X, X)$ is a Lefschetz map and $\Lambda(\phi) \neq \{0\}$ implies ϕ has a fixed point.

Now we recall the fixed point result in [16] needed in Section 2.

Definition 1.8. We say W is Lefschetz admissible if for all compact subsets K of W , all $\alpha \in Cov_W(K)$, there exists a continuous function $\pi_\alpha : K \rightarrow W$ such that

- (i). π_α and $i : K \hookrightarrow W$ are α -close;
- (ii). $\pi_\alpha(K)$ is contained in a subset $C_\alpha \subseteq W$ and C_α is a Lefschetz space;
- (iii). π_α and $i : K \hookrightarrow W$ are homotopic.

Remark 1.9. (i). $NES(\text{compact})$ and $SANES(\text{compact})$ spaces are examples of Lefschetz spaces (see [6, 15]).

(ii). The above definition was motivated from Schauder projections and admissible sets (recall a nonempty subset W of a Hausdorff topological vector space E is said to be admissible if for any nonempty compact subset K of W and every neighborhood V of 0 in E there exists a continuous map $h : K \rightarrow W$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E ; note for example every nonempty convex subset of a locally convex space is admissible).

In [16] the following result was established.

Theorem 1.10. *Let X be Lefschetz admissible with X a uniform space and let $F \in Ad(X, X)$ be a compact map. Then $\Lambda(F)$ is well defined and if $\Lambda(F) \neq \{0\}$ then F has a fixed point i.e. X is a Lefschetz space.*

We now present a result which will be needed in Section 2.

Theorem 1.11. *Let X, Y and Z be subsets of Hausdorff topological spaces.*

(i). *Suppose $F \in AC(X, Z)$, $G \in AC(Y, X)$ and $FG \in Ad(Y, Z)$. Then there exists a selected pair (p, q) of FG with $q_*p_*^{-1} = F_*G_*$.*

(ii). *Suppose $F \in AC(X, Z)$, $G \in AC(Y, X)$ and $FG \in AC(Y, Z)$. Then $(FG)_* = F_*G_*$.*

(iii). *Suppose $F \in AC(X, Z)$ and $g \in C(Y, X)$ (i.e. $g : Y \rightarrow X$ is a continuous single valued map). Then $(Fg)_* = F_*g_*$.*

Proof: (i). Let (p_1, q_1) be a selected pair of F (note from [8, Proposition 40.4] that $F_* = (q_1)_*(p_1)_*^{-1}$) and (p_2, q_2) be a selected pair of G (note $G_* = (q_2)_*(p_2)_*^{-1}$). Now [8, Proposition 40.5] guarantees that there exists a selected pair (p, q) of FG with

$$q_*p_*^{-1} = (q_1)_*(p_1)_*^{-1}(q_2)_*(p_2)_*^{-1} = F_*G_*.$$

(ii). This is immediate from part (i) and [8, Proposition 40.4] since $q_*p_*^{-1} = (FG)_*$ (note $FG \in AC(Y, Z)$ is assumed here).

(iii). This is immediate from part (ii) since $Fg \in AC(Y, Z)$. $\square$

Remark 1.12. Theorem 1.11 (ii) was proved (with a different argument) in [17, Theorem 3.8] while Theorem 1.11 (iii) was proved in [7, Theorem 6.5].

Finally we recall the two classes of maps considered in this paper.

Let X be a subset of a Hausdorff topological vector space. We say $F \in HYAd(X, X)$ if $F : X \rightarrow 2^X$ and there exists a map $\Phi \in Ad(X, X)$ with $\Phi(x) \subseteq co(F(x))$ for $x \in X$.

Let X be a subset of a Hausdorff topological vector space. We say $F \in HYAdC(X, X)$ if $F : X \rightarrow 2^X$ and there exists a map $\Phi \in Ad(X, X)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$.

2 Lefschetz Fixed Point Theory

Let X be a Hausdorff topological space and let $\alpha \in Cov(X)$. X is said to be Lefschetz admissible α -dominated if there exists a Lefschetz admissible space X_α , with X_α a uniform space, and two maps $R_\alpha \in AC(X_\alpha, X)$, $S_\alpha \in AC(X, X_\alpha)$ such that $R_\alpha S_\alpha : X \rightarrow 2^X$ and the identity $i_X : X \rightarrow X$ are strongly α -close and $(R_\alpha)_*(S_\alpha)_* = (i_X)_*$. X is said to be Lefschetz admissible acyclic dominated if X is Lefschetz admissible α -dominated for every $\alpha \in Cov(X)$.

Example 2.1. If $R_\alpha \in C(X_\alpha, X)$, $S_\alpha \in C(X, X_\alpha)$ (i.e. continuous single valued maps) and $R_\alpha S_\alpha : X \rightarrow X$ and $i_X : X \rightarrow X$ are α -close and $R_\alpha S_\alpha \sim i_X$, then immediately $R_\alpha S_\alpha$ and i_X are strongly α -close (same concept for

single valued maps) and $R_\alpha S_\alpha \sim i_X$ guarantees [7, 8] that $(R_\alpha S_\alpha)_* = (i_X)_*$ and so $(R_\alpha)_*(S_\alpha)_* = (R_\alpha S_\alpha)_* = (i_X)_*$.

Example 2.2. If $R_\alpha \in AC(X_\alpha, X)$, $S_\alpha \in AC(X, X_\alpha)$ and $R_\alpha S_\alpha \in AC(X, X)$ then Theorem 1.10 guarantees that $(R_\alpha S_\alpha)_* = (R_\alpha)_*(S_\alpha)_*$ so if $R_\alpha S_\alpha \sim i_X$ (in the sense of [8, Definition 32.5] (see also [7]) i.e. an acyclic homotopy) then [8, Proposition 32.6] (see also [7]) guarantees that $(R_\alpha S_\alpha)_* = (i_X)_*$ and so $(R_\alpha)_*(S_\alpha)_* = (R_\alpha S_\alpha)_* = (i_X)_*$. For other comments see Remark 2.4.

Theorem 2.3. *Let X be Lefschetz admissible acyclic dominated (and X a subset of a Hausdorff topological vector space) and $F \in HYAd(X, X)$ with $co F$ a compact map (so in particular there exists a compact map $\Phi \in Ad(X, X)$ with $\Phi(x) \subseteq co(F(x))$ for $x \in X$). Then $\Lambda(\Phi)$ is well defined and if $\Lambda(\Phi) \neq \{0\}$ then Φ (so consequently $co F$) has a fixed point.*

Proof: Let Φ be as in the statement above. Let $\alpha \in Cov(X)$ and let $p, q : \Gamma \rightarrow X$ be a selected pair of Φ . Now since X is Lefschetz admissible acyclic dominated then there exists a Lefschetz space X_α with X_α a uniform space and two maps $R_\alpha \in AC(X_\alpha, X)$, $S_\alpha \in AC(X, X_\alpha)$ such that $R_\alpha S_\alpha : X \rightarrow 2^X$ and the identity $i_X : X \rightarrow X$ are strongly α -close and $(R_\alpha)_*(S_\alpha)_* = (i_X)_*$. Let $\Phi_\alpha = S_\alpha \Phi R_\alpha$ and note $\Phi R_\alpha \in Ad(X_\alpha, X)$ and $\Phi_\alpha \in Ad(X_\alpha, X_\alpha)$. Let (p'_α, q'_α) be a selected pair of R_α (note [8] that $(R_\alpha)_* = (q'_\alpha)_*(p'_\alpha)^{-1}$ since $R_\alpha \in AC(X_\alpha, X)$). Then [8, Proposition 40.5] guarantees that there exists a selected pair (p''_α, q''_α) of ΦR_α with

$$(q''_\alpha)_*(p''_\alpha)^{-1} = q_* p_*^{-1} (q'_\alpha)_*(p'_\alpha)^{-1},$$

so

$$(2.1) \quad (q''_\alpha)_*(p''_\alpha)^{-1} = q_* p_*^{-1} (R_\alpha)_*.$$

Let $(p'''_\alpha, q'''_\alpha)$ be a selected pair of S_α (note [8] that $(S_\alpha)_* = (q'''_\alpha)_*(p'''_\alpha)^{-1}$ since $S_\alpha \in AC(X, X_\alpha)$). Then [8, Proposition 40.5] guarantees that there exists a selected pair (p_α, q_α) of Φ_α with

$$(q_\alpha)_*(p_\alpha)^{-1} = (q'''_\alpha)_*(p'''_\alpha)^{-1} (q''_\alpha)_*(p''_\alpha)^{-1},$$

so

$$(2.2) \quad (q_\alpha)_*(p_\alpha)^{-1} = (S_\alpha)_*(q''_\alpha)_*(p''_\alpha)^{-1}.$$

Since X_α is Lefschetz admissible then from Theorem 1.10 we have that $(q_\alpha)_*(p_\alpha)^{-1}$ is a Leray endomorphism. Next note since $(R_\alpha)_*(S_\alpha)_* = (i_X)_*$ that

$$(R_\alpha)_*(S_\alpha)_* q_* p_*^{-1} = (i_X)_* q_* p_*^{-1} = q_* p_*^{-1}$$

and from (2.1) and (2.2) we have

$$(S_\alpha)_* q_* p_*^{-1} (R_\alpha)_* = (S_\alpha)_* (q''_*) (p''_*)^{-1} = (q_\alpha)_* (p_\alpha)_*^{-1}.$$

Now [6, pp 214 (see (1.3))] (here $E' = H(X_\alpha)$, $E'' = H(X)$, $u = (R_\alpha)_*$, $v = (S_\alpha)_* q_* p_*^{-1}$, $f' = (q_\alpha)_* (p_\alpha^{-1})_*$, $f'' = q_* p_*^{-1}$ and note $u f' = (R_\alpha)_* (q_\alpha)_* (p_\alpha^{-1})_* = (R_\alpha)_* (S_\alpha)_* q_* p_*^{-1} (R_\alpha)_* = q_* p_*^{-1} (R_\alpha)_* = f'' u$) guarantees that $q_* p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_* p_*^{-1}) = \Lambda((q_\alpha)_* (p_\alpha^{-1})_*)$ i.e. $\Lambda(\Phi)$ is well defined and $\Lambda(\Phi) \subseteq \Lambda(\Phi_\alpha)$.

Now assume $\Lambda(\Phi) \neq \{0\}$. Then $\Lambda(\Phi_\alpha) \neq \{0\}$. Now since X_α is Lefschetz admissible then Theorem 1.10 guarantees that there exists a $x \in X_\alpha$ with $x \in \Phi_\alpha(x) = S_\alpha \Phi R_\alpha(x)$. Then $x \in S_\alpha(y)$ for some $y \in \Phi R_\alpha(x)$. Now note $y \in \Phi(z)$ for some $z \in R_\alpha S_\alpha(y)$. Since $R_\alpha S_\alpha$ and i_X are strongly α -close then there exists a $U \in \alpha$ with $R_\alpha S_\alpha(y) \subseteq U$ and $i_X(y) \in U$ i.e. $z \in U$ and $y \in U$. As a result $z \in U$ and $\Phi(z) \cap U \neq \emptyset$ (note $y \in U$ and $y \in \Phi(z)$) Thus Φ has an α -fixed point (for each $\alpha \in Cov_X(X)$). Now Theorem 1.1 and Remark 1.2 (note Hausdorff topological vector spaces are uniform spaces) guarantee that Φ (so consequently $co(F)$) has a fixed point. $\square$

Remark 2.4. (i). Note from the above analysis we only need X to be Lefschetz admissible α -dominated for every $\alpha \in Cov_X(\overline{\Phi(X)})$ (here Φ is as in the proof of Theorem 2.3) to deduce Theorem 2.3.

(ii). Also from the analysis in Theorem 2.3 we see in the definition of Lefschetz admissible α -dominated one could replace " $R_\alpha S_\alpha : X \rightarrow 2^X$ and the identity $i_X : X \rightarrow X$ are strongly α -close" with the condition "for each $y \in X$ and each $z \in R_\alpha S_\alpha(y)$ there exists a $U \in \alpha$ with $y \in U$ and $z \in U$ ". In fact in Theorem 2.3 we only need a condition to guarantee that "for each $y \in X$ and each $z \in R_\alpha S_\alpha(y)$ with $y \in \Phi(z)$ there exists a $U \in \alpha$ with $y \in U$ and $z \in U$ ".

(iii). If $R_\alpha S_\alpha \in C(X, X)$ (i.e. $R_\alpha S_\alpha : X \rightarrow X$ is a continuous single valued map) then $R_\alpha S_\alpha$ and i_X are strongly α -close means $R_\alpha S_\alpha$ and i_X are α -close.

The same argument as in Theorem 2.3 establishes the following result.

Theorem 2.5. *Let X be Lefschetz admissible acyclic dominated with X a uniform space and $F \in HYAdC(X, X)$ with F a compact map (so in particular there exists a compact map $\Phi \in Ad(X, X)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$). Then $\Lambda(\Phi)$ is well defined and if $\Lambda(\Phi) \neq \{0\}$ then Φ (so consequently F) has a fixed point.*

A special case of Theorem 2.5 (with $\Phi = F$) is the following.

Theorem 2.6. *Let X be Lefschetz admissible acyclic dominated with X a uniform space and $F \in Ad(X, X)$ a compact map. Then $\Lambda(F)$ is well defined and if $\Lambda(F) \neq \{0\}$ then F has a fixed point i.e. X is a Lefschetz space.*

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