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Interpolative contractions on perturbed metric spaces

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Abstract

In the present paper, we propose a new notion, namely, the interpolative contractions in the context of the perturbed metric spaces. For these mappings, we give a fixed-point result and provide an example to support the advances brought by the new results. Moreover, we give extend our results and provide a Meir-Keeler type fixed-point result in perturbed metric spaces.

1 Introduction and preliminaries

The nature of science, particularly mathematics, forces researchers to improve, extend, and generalize existing results to the best of their knowledge. Our objective is to consider problems within a broad framework to propose solutions that can address a variety of cases and possibilities. Given the finite nature of human existence, this approach is quite prudent. With this motivation, this article aims to explore fixed-point theory in an abstract structure as comprehensively as possible. Following Banach's [2] inventive fixed-point theory, an enormous amount of work has been done and published to develop and generalize this result; see, e.g. [1], [3], [4], [5], [6], [8], [9], [13], [14], [15], [16], [17], [19] and the references cited herein.

In this note, we focus on a recently defined concept, specifically interpolative contraction [10], within the framework of newly established abstract spaces known as perturbed metric spaces [7].

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Before stating the definition of this new abstract structure, we fix some notations. Throughout the paper, each mentioned set is assumed non-empty. For a set U, all functions in the form of $U \times U \to [0, \infty)$ will be denoted by $\mathcal{U}_\delta$ and called distance functions.

Definition 1.1. A function $M \in \mathcal{U}_\delta$ is called *perturbed metric* (PM) over $U \in \mathcal{U}_\delta$ with respect to $R \in \mathcal{U}_\delta$ if

$$d = M - R \in \mathcal{U}_\delta \quad \text{such that} \quad d(x, u) \mapsto M(x, u) - R(x, u)$$

forms a usual metric over U. More precisely, for any $q, u, z \in U$, the upcoming statements are provided:

(i) $(M - R)(q, u) \geq 0$,

(ii) $(M - R)(q, u) = 0$ if and only if $q = u$,

(iii) $(M - R)(q, u) = (M - R)(u, q)$,

(iv) $(M - R)(q, u) \leq (M - R)(q, z) + (M - R)(z, u)$.

In addition, $R \in \mathcal{U}_\delta$ shall be named *perturbed mapping*, where $d = M - R$ is a *standard metric* (SP). The triple (U, M, R) is reserved for *perturbed metric space* (PMS).

The topological basics of PMS are given in the next definition:

Definition 1.2. (See e.g. [7]) Let $\{x_n\}$ be a sequence in PMS (U, M, R) and T be a self-mapping on PMS (U, M, R).

(i) A sequence $\{x_n\}$ is *perturbed convergent* or *p-convergent* in PMS (U, M, R) if $\{x_n\}$ converges in the corresponding standard metric space (SMS) (U, d), with $d = M - R$.

(ii) A sequence $\{x_n\}$ in (U, M, R) is called *perturbed Cauchy* or *p-Cauchy* if $\{x_n\}$ forms a Cauchy sequence with respect to SMS (U, d).

(iii) A triple (U, M, R) is called a *complete perturbed metric space* (in short, CPMS) if the corresponding MS (U, d) is complete.

(iv) A map T is called *perturbed continuous* or *p-continuous* with respect to PMS (U, M, R) if the same mapping T is continuous within the SMS (U, d).

Definition 1.3. A self-mapping T over a complete metric space (U, d) is called an *interpolative Kannan-type contraction*, if for any $x, y \in U \setminus \mathrm{Fix}(T)$

$$d(Tx, Ty) \leq \kappa \big[d(x, Tx)\big]^\alpha \big[d(y, Ty)\big]^{1-\alpha},$$

for some $\kappa \in [0, 1)$ and $\alpha \in (0, 1)$ where $\mathrm{Fix}(T) = \{x \in U \mid Tx = x\}$.

The main goal of this study is to introduce the concept of interpolative contractions in PMSs. For these mappings, we give a fixed-point result and provide an example to support the advances brought about by the new results. Moreover, we extend our results and provide a Meir-Keeler-type fixed-point result in PMS.

2 Interpolative Kannan perturbed mappings

Definition 2.1. A self-mapping T over a PMS, (U, M, R) is called an *interpolative Kannan perturbed type mapping*, if for each $x, y \in U$ with $x \neq Tx$ we have

$$M(Tx, Ty) \leq \kappa\big[M(x, Tx)\big]^{\alpha}\big[M(y, Ty)\big]^{1-\alpha}, \tag{1}$$

where $\kappa \in [0, 1)$ and $\alpha \in (0, 1)$.

Theorem 2.1. *Let a self-mapping T over a CMS (U, M, R) form an interpolative Kannan perturbed type contraction. Subsequently, then T admits a fixed-point.*

Proof. The proof is constructive by starting with an arbitrary $x_0 \in U$. Employing the Picard operator, we construct a sequence $\{x_n\} \subset U$ (called, Picard sequence) where $x_{n+1} = Tx_n$ for $n \geq 0$.

We presume $x_n \neq x_{n+1}$ for each $n \geq 0$. In fact, if there is an $n_0 \geq 0$ such that $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then x_{n_0} forms the requested fixed-point for T.

Employing $y = x_{n-1}$ and $x = x_n$ in (1) infer that

$$M(x_{n+1}, x_n) = M(Tx_n, Tx_{n-1}) \leq \kappa\big[M(x_n, Tx_n)\big]^{\alpha}\big[M(x_{n-1}, Tx_{n-1})\big]^{1-\alpha},$$

so

$$\big[M(x_n, x_{n+1})\big]^{1-\alpha} \leq \kappa\big[M(x_{n-1}, x_n)\big]^{1-\alpha}.$$

It yields

$$M(x_n, x_{n+1}) \leq \kappa\big[M(x_{n-1}, x_n)\big] \leq \kappa^n M(x_0, x_1) \overset{M(x_0,x_1)=M_0}{=} \kappa^n M_0. \tag{2}$$

Clearly, $d = M - R$ be SM and $d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + R(x_n, x_{n+1})$. Due to the inequality above, we find

$$d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + R(x_n, x_{n+1}) \leq \kappa^n M_0, \quad \forall n \geq 0.$$

Consequently, we deduce

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \kappa^n M_0 + \kappa^{n+1} M_0 + \cdots + \kappa^{n+p-1} M_0 \\ &\leq \frac{\kappa^n}{1-\kappa} M_0. \end{aligned}$$

As a result the sequence $\{x_n\}$ forms Cauchy in SMS (U, d), whence $\{x_n\}$ is a perturbed Cauchy in (U, M, R). Owing to the completeness of PMS (U, M, R), we conclude the existence of $x^* \in U$ so that

$$\lim_{n\to\infty} d(x_n, x^*) = 0.$$

We assert that it is the wanted fixed-point. To demonstrate our assertion, we employ (1):

$$M(Tx_n, Tx) \leq \kappa\big[M(x_n, Tx_n)\big]^{\alpha}\big[M(x, Tx)\big]^{1-\alpha}.$$

By letting $n \to \infty$ in (2) we reach the inequality $M(x^*, Tx^*) \leq 0$. More precisely, $M(x^*, Tx^*) = 0$ and $d(x^*, Tx^*) \leq 0$ subsequently $x = Tx$.

□

Example 2.1. *Using the assumption that $U = \mathbb{R}$, define the perturbed metric $M : U \times U \to [0, \infty)$ by $M(x, y) = |x - y| + x^2y^2$, with the perturbation $R(x, y) = x^2y^2$. Construct a mapping $T : U \to U$ by*

$$T(x) = \begin{cases} 0, & \text{if } x < 2, \\ 1, & \text{if } x \geq 2. \end{cases}$$

We note that $x = 0$ is the only fixed-point in T.
Next, we examine whether T is an interpolative Kannan perturbed mapping. We aim to find $\kappa \in [0, 1)$ and $\alpha \in (0, 1)$ such that for all $x, y \in U$,

$$M(Tx, Ty) \leq \kappa\,[M(x, Tx)]^{\alpha}\,[M(y, Ty)]^{1-\alpha}.$$

Suppose $\kappa = \frac{1}{2}$ and $\alpha = \frac{1}{2}$. Let us consider the following cases:

(i) If $x, y < 2$, then $T(x) = T(y) = 0$, so $M(Tx, Ty) = 0$. Thus, the inequality holds trivially.

(ii) If $x, y \geq 2$, then $T(x) = T(y) = 1$, so $M(Tx, Ty) = 1$. Consider,

$$\begin{aligned} \kappa\sqrt{M(x,Tx)M(y,Ty)} &= \frac{1}{2}\sqrt{M(x,1)M(y,1)} \\ &= \frac{1}{2}\sqrt{(|x-1|+x^2)(|y-1|+y^2)} \\ &= \frac{1}{2}\sqrt{(|x-1|+x^2)(|y-1|+y^2)} \\ &\geq \frac{1}{2}.5 = 2.5 > M(Tx,Ty) \end{aligned}$$

Thus, the inequality holds in this case.

(iii) If $y \geq 2$ and $x < 2$ (or contrariwise), then $T(x) = 0$ and $T(y) = 1$, so

$$M(Tx, Ty) = M(0, 1) = |0 - 1| + (0)^2(1)^2 = 1.$$

Also,

$$M(x, Tx) = M(x, 0) = |x - 0| + (x)^2(0)^2 = |x|,$$

and

$$M(y, Ty) = M(y, 1) = |y - 1| + y^2(1)^2 = |y - 1| + y^2.$$

Using $\kappa = \frac{1}{2}$ and $\alpha = \frac{1}{2}$, the right-hand side becomes

$$\frac{1}{2} (|x|)^{1/2} \left(|y - 1| + y^2\right)^{1/2}.$$

Since $y \geq 2$, we have $|y - 1| + y^2 \geq 1 + 4 = 5$, and $|x| \leq 2$ for $x < 2$. Hence,

$$(|x|)^{1/2} \left(|y - 1| + y^2\right)^{1/2} \geq (0)^{1/2}(5)^{1/2} = 0,$$

and in general positive. Therefore, the right-hand side is always at least

$$\frac{1}{2}\sqrt{2}\sqrt{5} = \frac{1}{2}\sqrt{10} \approx 1.581,$$

which is greater than 1. Thus, the inequality holds.

Therefore, T is an interpolative Kannan perturbed mapping under M with parameters $\kappa = \frac{1}{2}, \quad \alpha = \frac{1}{2}$.
Now, presume that $d(x, y) = |x - y|$. Then, for $x < 2$ and $y \geq 2$, we have $d(Tx, Ty) = d(0, 1) = 1$, $d(x, Tx) = |x - 0| = |x|$, $d(y, Ty) = |y - 1|$. The interpolative Kannan inequality becomes

$$d(Tx, Ty) \leq \kappa \left(d(x, Tx)\right)^{\alpha} \left(d(y, Ty)\right)^{1-\alpha},$$

i.e.,

$$1 \leq \kappa \left(|x|\right)^{\alpha} \left(|y - 1|\right)^{1-\alpha}.$$

Take $x = 1$ (so $x < 2$) and $y = 2$ (so $y \geq 2$). Thus, the right-hand side of the inequality becomes

$$\kappa(d(x, Tx))^{\alpha}(d(y, Ty))^{1-\alpha} = \frac{1}{2}(1)^{1/2}(1)^{1/2} = \frac{1}{2}.$$

Therefore, the inequality becomes

$$1 \leq \frac{1}{2},$$

which is clearly false. Hence, the interpolative Kannan inequality is ***not satisfied*** *for $x = 1$ and $y = 2$.*

3 Interpolative Kannan-Meir-Keeler perturbed mappings

Definition 3.1 (see [12]). Let (U, d) be a complete metric space. A mapping $T : U \to U$ is said to be a *Meir-Keeler contraction* on U, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \implies d(Tx, Ty) < \varepsilon,$$

for every $x, y \in U$.

Definition 3.2. Let (U, M, R) be a complete perturbed metric space. A mapping $T : U \to U$ is said to be an *interpolative Kannan-Meir-Keeler perturbed mapping on* U, if there is a $\gamma \in (0, 1)$ so that

(KMK_1) For arbitrary $\varepsilon > 0$, there is a $\delta > 0$ so that:

$$\varepsilon < \left[M(x, Tx)\right]^{\gamma}\left[M(y, Ty)\right]^{1-\gamma} < \varepsilon + \delta \implies M(Tx, Ty) \leq \varepsilon,$$

(KMK_2)

$$M(Tx, Ty) < \left[M(x, Tx)\right]^{\gamma}\left[M(y, Ty)\right]^{1-\gamma},$$

for every $x, y \in U \setminus \mathrm{Fix}(T)$.

Theorem 3.1. *Let (U, M, R) be a complete perturbed metric space and $T : U \to U$ be an interpolative Kannan-Meir-Keeler perturbed mapping. Then T has a fixed-point in U.*

Proof. Let $x_0 \in U$ be fixed and consider the Picard iteration:

$$x_n = Tx_{n-1} = T^n x_0, \quad \text{for all } n \geq 0.$$

By assumption (KMK2). taking $x = x_{n-1}$ and $y = x_n$ we have:

$$M(x_n, x_{n+1}) = M(Tx_{n-1}, Tx_n) < \left[M(x_{n-1}, Tx_{n-1})\right]^{\gamma}\left[M(x_n, Tx_n)\right]^{1-\gamma},$$

subsequently

$$\left[M(x_n, x_{n+1})\right]^{\gamma} < \left[M(x_{n-1}, x_n)\right]^{\gamma}.$$

Thus, we obtain that the sequence $\{M(x_n, x_{n+1})\}$ is strictly decreasing. Since $M(x_n, x_{n+1}) > 0$ for all $n \geq 0$, it follows that the sequence converges to some $L \geq 0$.

We proclaim that $L = 0$. Suppose, contrariwise, that $L > 0$. Then, there exists $N \in \mathbb{N}$ such that:

$$L < M(x_n, x_{n+1}) < L + \delta(L), \quad \text{for all } n \geq N.$$

Then, by (KMK_1), it follows that:

$$M(x_n, x_{n+1}) \le L, \quad \text{for all } n \ge N,$$

a contradiction. Ergo, $L = 0$.

Subsequently, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, since $d(x_n, x_{n+1}) \le M(x_n, x_{n+1})$.

In order to proclaim the sequence $\{x_n\}$ is Cauchy we let $\varepsilon > 0$ and choose $\delta(\varepsilon)$ such that $\delta(\varepsilon) < \varepsilon$. Recalling that both $d(x_n, x_{n+1})$ and $M(x_n, x_{n+1})$ tends to zero, as $n \to \infty$, there is a $k \in \mathbb{N}$ so that:

$$d(x_n, x_{n+1}) < \frac{\varepsilon}{2}, \quad \text{for all } n \ge k,$$

$$M(x_n, x_{n+1}) < \frac{\varepsilon}{2}, \quad \text{for all } n \ge k.$$

We prove that:

$$d(x_n, x_{n+p}) < \varepsilon, \quad \text{for all } p \in \mathbb{N}.$$

If $p = 1$, the inequality holds by assumption. Now, suppose it holds for some $p > 1$. Due to $d < M$ and the condition (KMK_2), the triangle inequality infer that

$$\begin{aligned}
d(x_n, x_{n+p+1}) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p+1}) \\
&\le d(x_n, x_{n+1}) + M(x_{n+1}, x_{n+p+1}) \\
&\le d(x_n, x_{n+1}) + [M(x_n, x_{n+1})]^{\gamma}[M(x_{n+p}, x_{n+p+1})]^{1-\gamma} \\
&< \frac{\varepsilon}{2} + \left(\frac{\varepsilon}{2}\right)^{\gamma}\left(\frac{\varepsilon}{2}\right)^{1-\gamma} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

By induction,

$$d(x_n, x_{n+p}) < \varepsilon, \quad \text{for all } p \in \mathbb{N}.$$

Subsequently, the sequence $\{x_n\}$ is Cauchy in SMS (U, d), so, from Definition 1.2. (iii), $\{x_n\}$ is a p-Cauchy sequence in (U, M, R). Consequently, since PMS (U, M, R) is complete there is $x^* \in U$ so that

$$\lim_{n\to\infty} d(x_n, x^*) = 0.$$

Finally, to show that $x^* = Tx^*$, we suppose that $M(x^*, Tx^*) \ne 0$. By condition (KMK_2), we have:

$$0 < M(Tx_n, Tx^*) < \left[M(x_n, Tx_n)\right]^{\gamma}\left[d(x^*, Tx^*)\right]^{1-\gamma} =$$

$$= \left[M(x_n, x_{n+1})\right]^{\gamma} \left[d(x^*, Tx^*)\right]^{1-\gamma}.$$

Letting $n \to \infty$, we find that $M(x^*, Tx^*) = 0$, which is a contradiction. Hence, $d(x^*, Tx^*) = 0$ so $x^* = Tx^*$, and x^* is a fixed-point of T.

□

Example 3.1. *Assume $Z = \mathbb{R}^2$ and $\mathcal{F} = \{R, Q, O, S\}$, where $R = (1, 0)$, $Q = (0, 1)$, $O = (1, 1)$, $S = (2, 0)$. Define the metric $M : Z \times Z \to [0, \infty)$ as:*

$$M(Z_1, Z_2) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} + p_1 q_2 + p_2 q_1$$

for any $Z_1, Z_2 \in Z$, $Z_1 = (p_1, q_1), Z_2 = (p_2, q_2)$ with $p_1, p_2, q_1, q_2 \in \mathbb{R}$. Define the mapping $\mathsf{T} : Z \to Z$ as follows:

$$\mathsf{T}R = Q,\ \mathsf{T}Q = O,\ \mathsf{T}O = S,\ \mathsf{T}S = P,\ \textit{and}\ \mathsf{T}x = x\ \textit{for}\ x \in Z \setminus \mathcal{R}.$$

Take $\gamma = \frac{1}{2}$. We verify the conditions:

1. *For pair (O, S):*

$$\begin{aligned} M(O, \mathsf{T}O) &= \sqrt{2} + 2 \\ M(S, \mathsf{T}S) &= 1 \\ \sqrt{M(O, \mathsf{T}O) M(S, \mathsf{T}S)} &= \sqrt{\sqrt{2} + 2} > 1.5 \\ M(\mathsf{T}O, \mathsf{T}S) &= 1 \end{aligned}$$

   Choose $\varepsilon = 1.5$, $\delta = 1$. Then:

$$1.5 < \sqrt{\sqrt{2} + 2} < 2.5 \quad \textit{and} \quad M(\mathsf{T}O, \mathsf{T}S) = 1 < 1.5$$

2. *For pair (R, Q):*

$$\begin{aligned} M(R, \mathsf{T}R) &= \sqrt{2} + 1 \\ M(Q, \mathsf{T}Q) &= 2 \\ \sqrt{M(R, \mathsf{T}R) M(Q, \mathsf{T}Q)} &= \sqrt{2\sqrt{2} + 2} > 2 \\ M(\mathsf{T}R, \mathsf{T}Q) &= 2 \end{aligned}$$

   Choose $\varepsilon = 2.1$, $\delta = 1$. Then:

$$2.1 < \sqrt{2\sqrt{2} + 2} < 3.1 \quad \textit{and} \quad M(\mathsf{T}R, \mathsf{T}Q) = 2 < 2.1$$

3. For pair (Q, O):

$$M(Q, \mathsf{T}Q) = 2$$
$$M(R, \mathsf{T}O) = \sqrt{2} + 2$$
$$\sqrt{M(Q, \mathsf{T}Q)M(O, \mathsf{T}O)} = \sqrt{4 + 2\sqrt{2}} > 2.5$$
$$M(\mathsf{T}Q, \mathsf{T}O) = \sqrt{2} + 2$$

Choose $\varepsilon = 3.5$, $\delta = 1$. Then:

$$3.5 < \sqrt{4 + 2\sqrt{2}} < 4.5 \quad \textit{and} \quad M(\mathsf{T}Q, \mathsf{T}O) = \sqrt{2} + 2 < 3.5$$

Thus, T satisfies Theorem 2.2 with simple $\delta = 1$ in all cases, and has fixed-points $U \setminus \mathcal{P}$.

For pairs (O, S) and (R, Q):

$$\sqrt{d(R, \mathsf{T}R)d(S, \mathsf{T}S)} = \sqrt{\sqrt{2} \cdot 1} \approx 1.189 > d(\mathsf{T}O, \mathsf{T}S) = 1.$$

$$\sqrt{d(P, \mathsf{T}R)d(Q, \mathsf{T}Q)} = \sqrt{\sqrt{2} \cdot 1} \approx 1.189 > d(\mathsf{T}O, \mathsf{T}Q) = 1.$$

But for pair (Q, O): $\sqrt{d(Q, \mathsf{T}Q)d(O, \mathsf{T}O)} = \sqrt{1 \cdot \sqrt{2}} \approx 1.189 < d(\mathsf{T}Q, \mathsf{T}O) = \sqrt{2} \approx 1.414$.
The mapping T satisfies all conditions under metric M (with $\delta = 1$ throughout), but fails for the pair (Q, O) under standard metric d. This demonstrates the necessity of the perturbed metric M. The fixed point $Z \setminus \mathcal{R}$ exists under M by Theorem 3.1.

4 Application

One of the significant applications of the main results can be solving nonlinear matrix equations. First, we set up the terminology and indicate the impact of the main result.

The notion $\|.\|_{tr}$ is reserved for trace norm. In particular, $tr(A) = \|A\|_{tr}$ is the sum of singular values of A such that singular values are the roots of eigenvalues of $A{*}A$. Further, $\|A\|_F$ is the square root of the sum of the squares of the singular values of A, where the singular values are the square roots of the eigenvalues of $A * A$. Formally,

$$\|A\|_F = \sqrt{tr(A * A)}$$

The letters M_n, H_n, P_n, H_n^+ denote the set of $n \times n$ matrices, hermitian matrices, positive definite matrices, and positive definite hermitian matrices, respectively.

Lemma 4.1. *If $S, T \in H_n^+$, then $0 \leq tr(ST) \leq ||S||tr(T)$.*

Consider the following nonlinear matrix equation

$$A = B + \sum_{i=1}^{m} U_i^* \mathcal{H}(A) U_i \tag{3}$$

where each $U_i \in M_n$ for each $i = 1, 2, ..., m$. Respectively, $B \in H_n^+$ is a positive definite hermitian matrix. $\mathcal{H}$ is a continuous map from H_n into P_n such that $H(0) = 0$. Moreover, H_n endowed with the trace norm is a Banach space. Thus, it is necessarily a complete metric space.
Let $\mathcal{T} : H_n \to H_n$ be a continuous mapping such that

$$\mathcal{T}(A) = B + \sum_{i=1}^{m} U_i^* \mathcal{H}(A) U_i \tag{4}$$

for all $A \in H_n$. As it is expected, a fixed-point of $\mathcal{T}$ forms a solution of (4).

Theorem 4.1. *Regarding* (3)*, there are $\alpha \in (0, 1)$ and $\kappa \in [0, 1)$ for $U, Y \in H_n$ having the next estimations:*

(i) $\sum_{i=1}^{m} U_i^* U_i \leq \frac{\kappa}{2} I_n$.

(ii) $tr((\mathcal{H}(Y) - \mathcal{H}(U)) \leq \|U - \mathcal{T}(U)\|_{tr}^{\alpha} \|Y - \mathcal{T}(Y)\|_{tr}^{1-\alpha}$

Then, (3) *has a solution.*

Proof. Let $M : H_n \times H_n \to [0, \infty)$ as

$$M(A, B) = \|A - B\|_{tr} + \|A - B\|_F.$$

Thus, (H_n, M, P) is a PMS with $d(A, B) = \|A - B\|_{tr}$ and the perturbation function $P(A,B) = \|A-B\|_F$. Let $A, B \in H_n$. Consider

$$\begin{aligned}
M(\mathcal{T}(Y),\mathcal{T}(U)) &= \|\sum_{i=1}^{m} U_i^*\mathcal{H}(Y)U_i - \sum_{i=1}^{m} U_i^*\mathcal{H}(U)U_i\|_{tr} \\
&+ \|\sum_{i=1}^{m} U_i^*\mathcal{H}(Y)U_i - \sum_{i=1}^{m} U_i^*\mathcal{H}(U)U_i\|_{F} \\
&\leq \|\sum_{i=1}^{m} U_i^*\mathcal{H}(Y)U_i - \sum_{i=1}^{m} U_i^*\mathcal{H}(U)U_i\|_{tr} \\
&+ \|\sum_{i=1}^{m} U_i^*\mathcal{H}(Y)U_i - \sum_{i=1}^{m} U_i^*\mathcal{H}(U)U_i\|_{tr} \\
&= 2.\left(tr\left(\sum_{i=1}^{m} U_i^*\mathcal{H}(Y)U_i - U_i^*\mathcal{H}(U)U_i\right)\right) \\
&= 2.\left(tr\left(\sum_{i=1}^{m} (U_i^*(\mathcal{H}(Y) - \mathcal{H}(U))U_i)\right)\right) \\
&= 2.\left(\sum_{i=1}^{m} (tr(U_i^*(\mathcal{H}(Y) - \mathcal{H}(U))U_i))\right) \\
&= 2.\left(tr\left(\sum_{i=1}^{m} ((U_i^*(\mathcal{H}(Y) - \mathcal{H}(U))U_i))\right)\right) \\
&= 2.\left(tr\left(\sum_{i=1}^{m} (U_i^*U_i)(\mathcal{H}(Y) - \mathcal{H}(U))\right)\right) \\
&\leq 2.\|(\sum_{i=1}^{m} (U_i^*U_i)\|\|\mathcal{H}(Y) - \mathcal{H}(U))\|_{tr} \quad (By\ Lemma\ 4.1) \\
&\leq 2.\frac{\kappa}{2}\|\mathcal{H}(Y) - \mathcal{H}(U))\|_{tr} \quad (By\ \ (i)) \\
&\leq \kappa.\|\mathcal{H}(Y) - \mathcal{H}(U))\|_{tr}
\end{aligned}$$

Now, because of (ii) and Theorem 2.1, we find that $\mathcal{T}$ has a fixed-point. Hence, the nonlinear matrix equation (3) has a solution. □

5 Conclusion

The idea of usage of perturbation in the construction of metric space is new, and it seems an interesting candidate to solve several unsolved problems that have a non-smooth nature. A metric space is an ideal structure, and it is based on perfection. On the other hand, in the real world, we confront several problems that can not be stated in our perfect construction. This is where the idea of "perturbation" comes into play. We can separate the problems we encounter in daily life that cannot be expressed within the perfect metric structure into the perfect part and the perturbed part. We can transfer the knowledge, experience, and techniques we have gained in the metric space to this new structure, the perturbed metric space, and increase the potential to propose solutions to problems. In this article, roughly speaking, we aim to emphasize that the idea of "perturbed metric space" has great potential in applied fields based on the fact that daily life problems can be expressed and solved in this new abstract construction, easily.

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