DOI: 10.2478/auom-2026-0008
An. Şt. Univ. Ovidius Constanţa **Vol. 34(1),2026, 151–167**

On The (De)Homogenization of Sagbi-Gröbner Bases for Modules

Nazish Kanwal and Junaid Alam Khan

Abstract

This article explores the behavior of Sagbi-Gröbner bases for modules over polynomial subalgebras under the process of homogenization and dehomogenization. We prove that the Sagbi-Gröbner basis for submodules over subalgebras behaves well in these processes, as the property of being a Sagbi-Gröbner basis is preserved under homogenization and dehomogenization.

1 Introduction

The concept of *Gröbner bases*, introduced by *Bruno Buchberger* in 1965, is a powerful tool for working with ideals in polynomial rings. Buchberger's algorithm provides an efficient method for computing these bases, significantly aiding in the resolution of systems of polynomial equations and enriching our understanding of polynomial ideal theory and elimination problems [1, 2].

The exploration of Gröbner bases has expanded to include submodules of free modules over polynomial rings, as discussed in various works [4, 11, 19]. For polynomial subalgebras, the concept of Sagbi bases serves as an analogous foundation, with applications in invariant theory and representation theory [10]. Further developments have led to the introduction of Sagbi-Gröbner bases for the ideals of polynomial subalgebras [8, 9]. The extension of these concepts to modules over polynomial subalgebras has been explored in [14],

Key Words: Gradation, Homogeneous vector, Sagbi-Gröbner bases
2010 Mathematics Subject Classification: Primary 13P10.
Received: 10.05.2025.
Accepted: 15.10.2025.

enabling the solution of polynomial equations that exhibit symmetries [7] and addressing problems in invariant theory [17].

Additionally, the transformative process of homogenization plays a crucial role in enhancing the study of ideals and modules within commutative algebra and algebraic geometry. By extending polynomial rings with an extra variable, this process introduces a valuable perspective of homogeneity and symmetry to algebraic structures [3].

Inspires our interest in the subject of this paper [4, 5, 12], where they study the problem of the behavior of Gröbner bases under homogenization and dehomogenization and presented a general principle to compute Gröbner basis for ideal and its homogenized ideal just by passing onto homogenized generators. In [6], they analyzed the behavior of Sagbi bases under the process of (de)homogenization. In this article, we employ the homogenization and dehomogenization process to study the relation between homogenized and dehomogenized Sagbi-Gröbner bases for modules over polynomial subalgebras.

The structure of this article is as follows. This Section 2 gives some basic notions and defines the theory related to Sagbi-Gröbner bases for modules. Section 3 describes the theory of homogenization and dehomogenization for polynomial subalgebras and modules over them. In Section 4, we give the main results (see Theorem 4.2 and Theorem 4.4) related to homogenized and dehomogenized Sagbi-Gröbner bases for modules and illustrate the results with the help of some examples. We will also observe the behavior of the reduced SG-basis for modules under homogenization (see Theorem 4.8 and Example 4.9).

2 Notation & Preliminaries

Here we consider polynomial ring $K[x_1, ..., x_n]$ in n variables $x_1, ..., x_n$ with coefficients from a field K. For short, we write $\mathcal{R} = K[x_1, ..., x_n]$ and a free module $\mathcal{R}^m = (K[x_1, ..., x_n])^m$ over polynomial ring $\mathcal{R}$ equipped with the standard basis $\{e_i \,|\, 1 \leq i \leq m\}$.

Define $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, with the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ representing a monomial $\mathcal{R}$ and $Mon(\mathcal{R})$ denote the set of monomials of $\mathcal{R}$. A monomial in $\mathcal{R}^m$ is a vector $X = x^\alpha e_i$, where x^α is a monomial of $\mathcal{R}$ and $i \in \{1, 2, ..., m\}$, the set of all monomials of $\mathcal{R}^m$ is denoted by $Mon(\mathcal{R}^m)$. For any arbitrary vector $g \in \mathcal{R}^m$, defined as $g = \sum_{i=1}^m g_i e_i$, where $g_i \in \mathcal{R}$, the top-degree of vector g be defined as: top-$deg(g) = max_{\mathrm{i}=1}^m \{deg(g_i)\}$.

Let $>$ be a monomial ordering on $Mon(\mathcal{R})$ (i.e. $>$ a monomial ordering on $\mathcal{R}$). In this paper, we always assume that monomial ordering is a well-ordering (see [4]). For $f = \sum_{i=1}^{k} c_i u_i \in \mathcal{R}$ (where $u_i \in Mon(\mathcal{R})$ and $a_i \in \mathsf{K}$), the $Support(f) = \{u_i | c_i \neq 0\}$, $LM_>(f) = max_>\{u\ u \in Support(f)\}$ is called the leading monomial of f, the coefficient of $LM_>(f)$ is called the leading coefficient of f denoted by $LC_>(f)$, the leading term of f is the product of $LM_>(f)$ and $LC_>(f)$ denote by $LT_>(f) = LC_>(f)LM_>(f)$, and the $Tail(f) = f - LT_>(f)$. For $F \subset \mathcal{R}$, $LM_>(F) = \{LM_>(f) \mid f \in F\}$.

Given a monomial ordering $>$ on $Mon(\mathcal{R})$, the following are two natural ways of obtaining a module ordering $>_M$ on $Mon(\mathcal{R}^m)$. Let $X_1 = x^\alpha e_i$ and $X_2 = x^\beta e_j$ be monomials of $\mathcal{R}^m$.

- The **TOP**(term over position) ordering is defined by

$$X_1 >_M X_2 \Longleftrightarrow \begin{cases} x^\alpha > x^\beta \\ \text{ or} \\ x^\alpha = x^\beta \quad \text{and} \quad i > j \end{cases}$$

- The **POT**(position over term) ordering is defined by

$$X_1 >_M X_2 \Longleftrightarrow \begin{cases} i > j \\ \text{ or} \\ i = j \quad \text{and} \quad x^\alpha > x^\beta \end{cases}$$

From now and on wards, we assume that $>_M$ be a module ordering on $Mon(\mathcal{R}^m)$ (induced by a monomial ordering $>$ on $\mathcal{R}$). Let $g = \sum_{i=1}^{r} a_i X_i$ be any vector in $\mathcal{R}^m$ (where $X_i \in Mon(\mathcal{R}^m)$, $a_i \in \mathsf{K}$), the $Support(g) = \{X_i \mid a_i \neq 0\}$, its leading monomial denoted by $LM_{>_M}(g) = max\{X \mid X \in Support(g)\}$, the coefficient of $LM_{>_M}(g)$ is called the leading coefficient of g and denoted by $LC_{>_M}(g)$, and product of leading coefficient and leading monomial is called the leading term of g, denoted by $LT_{>_M}(g) = LC_{>_M}(g)\ LM_{>_M}(g)$ and the $Tail(g) = g - LT_{>_M}(g)$. For any subset $H \subset \mathcal{R}^m$, $LM_{>_M}(H) = \{LM_{>_M}(h) | h \in H\}$.

2.1 Sagbi-Gröbner Bases for Modules

The theory of Sagbi-Gröbner bases for modules over polynomial subalgebras, referred to as SG-bases, has been developed in [9, 8, 14]. In this section, we will define the SG-basis for modules with respect to the module ordering $>_M$ and discuss some of its elementary properties.

For a finite subset $\mathsf{F} \subset \mathcal{R}$, let $\mathcal{A} = \mathsf{K}[\mathsf{F}]$ be a finitely generated subalgebra of $\mathsf{K}[x_1, ..., x_n]$ (generated by F) and $\mathcal{A}^m$ is a free module over $\mathcal{A}$. Let $\mathcal{N} \subset \mathcal{A}^m$ be a submodule of $\mathcal{A}^m$ over subalgebra $\mathcal{A}$.

Definition 2.1. Let $>$ be a monomial ordering on $\mathcal{R}$, induces a module ordering $>_M$.

1. The initial algebra is a subalgebra generated by the leading monomials of $\mathcal{A}$ is defined as $in_>(\mathcal{A}) = \mathsf{K}[LM_>(\mathcal{A})]$. A subset S of $\mathcal{A}$ is called a Sagbi basis of subalgebra $\mathcal{A}$ w.r.t the monomial ordering $>$ if $in_>(\mathcal{A}) = \mathsf{K}[LM_>(\mathsf{S})]$.

2. A subset $\mathcal{G} \subseteq \mathcal{N}$ is called a Sagbi-Gröbner basis (in short *SG*-basis) for $\mathcal{N}$, if $LM_{>_M}(\mathcal{G})$ generates monomial module $\langle LM_{>_M}(\mathcal{N})\rangle$ over $in_>(\mathcal{A})$, i.e.

$$\langle LM_{>_M}(\mathcal{N})\rangle_{in_>(\mathcal{A})} = \langle LM_{>_M}(\mathcal{G})\rangle_{in_>(\mathcal{A})} .^*$$

Remark 2.2. If $\mathcal{A}$ admits a finite Sagbi basis w.r.t a monomial ordering $>$, then any submodule $\mathcal{G}$ of $\mathcal{A}^m$ has a finite *SG*-basis w.r.t $>_M$ (For further details see [8, 14]).

The following is an example that illustrates the concept of *SG*-bases for modules.

Example 2.3. *Let $>$ be the lexicographical ordering $x > z > y$ on $\mathbb{Q}[x, y, z]$ induced the module ordering $>_M$, which is* ***POT*** *ordering with $e_2 > e_1$ on $\mathcal{R}^m$. Let $\mathcal{A} = \mathbb{Q}[\mathsf{S}] \subseteq \mathbb{Q}[x, y, z]$, where $\mathsf{S} = \{x^2y + x,\ z^2,\ xy^2\}$ and $\mathcal{N} = \big\langle (x^2y + x + z^2, x^2y + x + 1),\ (xy^2 - 2,\ xy^2 + z^4)\big\rangle \subset \mathcal{A}^2$. Then, S is a Sagbi basis of $\mathcal{A}$ w.r.t $>$ and $\mathcal{G} = \{(x^2y + x + z^2, x^2y + x + 1),\ (xy^2 - 2,\ xy^2 + z^4),\ (x^2yz^4 + xy^2z^2 + 2x^2y - xy^2 + xz^4 + 2x + z^6 + 2, 0)\}$ is an SG-basis w.r.t $>_M$ for $\mathcal{N}$ over $\mathcal{A}$* †

The following theorem illustrates the properties of *SG*-bases.

Proposition 2.4. *([8, 14]) For a subset $\mathcal{G} \subseteq \mathcal{N}$ of a submodule of $\mathcal{A}^m$ over subalgebra $\mathcal{A}$, the following conditions are equivalent:*

(i) $\mathcal{G}$ is an SG-basis for $\mathcal{N}$ w.r.t $>_M$.

(ii) For every $h \in \mathcal{N}$, every final si-reductum‡ of h via $\mathcal{G}$ is 0.

*For each $f \in \mathcal{N}$, there exist $g \in \mathcal{G}$, $h \in \mathcal{A}$ we have $LM_{>_M}(f) = LM_>(h)\, LM_{>_M}(g)$.

†For computing SG-bases, refer to the libraries "**sagbigrob.lib**" [15] and "**stdmodule.lib**", [16] in the computer algebra system SINGULAR[18].

‡A vector $h \in \mathcal{N}$ is said to *si*-reduce in one step via $\mathcal{G}$ to h_1 if there exists $a \in \mathcal{A}$ and $g \in \mathcal{G}$ such that $LM_{>_M}(ag) = LM_{>_M}(h)$ and $h_1 = h - ag$. If there is a chain of one-step reductions from h to h_k via $\mathcal{G}$, and h_k cannot be *si*-reduced any further, we call h_k a final *si*-reductum via $\mathcal{G}$.

(iii) Every $h \in \mathcal{N}$ has a unique SG-representation with respect to $\mathcal{G}$, that is,

$$h = \sum_{i=1}^{t} a_i\, g_i, \quad a_i \in \mathcal{A},\ g_i \in \mathcal{G},\ \textit{with} \quad LM_{>_M}(h) \geq (LM_{>_M}(a_i\, g_i))\,.$$

3 (De)Homogenization of Modules

In this section, we introduce the concepts related to homogenization and dehomogenization. To define a homogenized module over a homogenized subalgebra, we first establish the notion of natural grading on $\mathcal{R}$ and $\mathcal{R}^m$.

Definition 3.1. Let f and g be the polynomial and vector in $\mathcal{R}$ and $\mathcal{R}^m$ respectively.

1. A polynomial $f = \sum c_\alpha x^\alpha \in \mathcal{R}$ is homogeneous of degree γ if $deg(x^\alpha) = \gamma$, for all $x^\alpha \in Support(f)$.
2. A vector $g = (g_1, ..., g_m) \in \mathcal{R}^m$ is called homogeneous vector of top-degree γ, if either g_i is homogeneous polynomial of degree γ or $g_i = 0$, where $1 \leq i \leq m$.

The polynomial ring $\mathcal{R}$ is naturally graded, and it can be decomposed into the direct sum as:

$$\mathcal{R} = \bigoplus_{\gamma \in \mathbb{N}} \mathcal{R}_\gamma$$

such that

$$\mathcal{R}_\gamma = \{\text{homogeneous polynomial of degree } \gamma\} \cup \{0\}.$$

where for all $\alpha, \beta \in \mathbb{N},\ f \in \mathcal{R}_\alpha,\ g \in \mathcal{R}_\beta \Longrightarrow f \cdot g \in \mathcal{R}_{\alpha+\beta}$ and $\mathcal{R}_0 = \mathsf{K}$. Each $\mathcal{R}_\gamma$ is said to be a graded component of $\mathcal{R}$ of degree γ.

A free module $\mathcal{R}^m$ can also be seen as a naturally graded module, and it can be decomposed into the direct sum as

$$\mathcal{R}^m := \bigoplus_{\gamma \in \mathbb{N}} (\mathcal{R}^m)_\gamma$$

such that

$$(\mathcal{R}^m)_\gamma = (\mathcal{R}_\gamma)^m = \{g \in \mathcal{R}^m |\ \text{every component of } g \text{ is either a homogeneous polynomial of degree } \gamma \text{ or zero}\}.$$

where for all $\alpha, \beta \in \mathbb{N},\ f \in \mathcal{R}_\alpha,\ g \in (\mathcal{R}^m)_\beta \implies f \cdot g \in (\mathcal{R}^m)_{\alpha+\beta}$. Each $(\mathcal{R}^m)_\gamma$ is called a graded component of $\mathcal{R}^m$ and contains homogeneous vector of top-*deg* γ.

Let $\mathcal{R}[t] = \mathsf{K}[x_1, \dots, x_n, t]$ represent a polynomial ring that includes an additional variable t. To define the process of homogenization for an arbitrary polynomial in $\mathcal{R}$ with respect to the additional variable t, we need to consider the natural grading on $\mathcal{R}[t]$. This grading is defined in relation to the graded components of $\mathcal{R}$ as follows:

$$\mathcal{R}[t] = \bigoplus_{\gamma \in \mathbb{N}} \mathcal{R}[t]_\gamma, \text{ where } \quad \mathcal{R}[t]_\gamma = \left\{ \sum_{p+q=\gamma} \mathsf{F}_p t^q :\ \mathsf{F}_p \in \mathcal{R}_p,\ \gamma \geq 0 \right\}.$$

Let h be the polynomial in $\mathcal{R}$ of degree γ, then h can be decomposed in the natural graded component as $h = h_\gamma + h_{\gamma-1} + ... + h_{\gamma-s}$ with $h_{\gamma-q} \in \mathcal{R}_{\gamma-q}, 0 \leq q \leq s$ and $h_\gamma \neq 0$. We define h^* as the homogenization of h with respect to the variable t as:

$$h^* = h_\gamma + t\, h_{\gamma-1} + ... + t^s f_{\gamma-s}$$

Note that $h^* \in \mathcal{R}[t]$ is a homogeneous polynomial of degree γ therefore, $h^* \in \mathcal{R}[t]_\gamma$. Moreover, we denote the leading homogeneous polynomial of h by $LH(h) = h_\gamma$.

Consider an onto ring homomorphism $\Lambda : \mathcal{R}[t] \longrightarrow \mathcal{R}$ defined by $\Lambda(t) = 1$. For a homogeneous polynomial $\mathcal{F} \in \mathcal{R}[t]$, we denote the dehomogenization of $\mathcal{F}$ with respect to the variable t as $\mathcal{F}_* = \Lambda(\mathcal{F}) = \mathcal{F}_{t=1}$.

To define the process of homogenization for an arbitrary vector of free module $\mathcal{R}^m$, we need to consider a natural grading on free module $(\mathcal{R}[t])^m$ in relation to the graded component of $\mathcal{R}^m$ which could be defined as:

$$\mathcal{R}[t]^m = \bigoplus_{\gamma \in \mathbb{N}} \mathcal{R}[t]^m_\gamma, \text{ where } \quad \mathcal{R}[t]^m_\gamma = \left\{ \sum_{p+q=\gamma} \mathsf{G}_p t^q :\ \mathsf{G}_p \in \mathcal{R}^m_p,\ \gamma \geq 0 \right\}.$$

Let g be a vector in $\mathcal{R}^m$ of top degree γ. The decomposition of g into its natural graded components is given by: $g = g_\gamma + g_{\gamma-1} + ... + g_{\gamma-s} \in \mathcal{R}^m$ with $g_{\gamma-q} \in (\mathcal{R}^m)_{\gamma-q}, 0 \leq q \leq s$ and $g_\gamma \neq 0$. We define g^*, the homogenization of g with respect to the variable t, as follows:

$$g^* = g_\gamma + t g_{\gamma-1} + ... + t^s g_{\gamma-s}$$

Note that $g^* \in \mathcal{R}[t]^m$ is a homogeneous vector of top-degree γ therefore, $g^* \in (\mathcal{R}[t]^m)_\gamma$. Moreover, the leading homogeneous vector of g is the homogeneous vector of top-degree γ and we denote it by $LH(g) = g_\gamma$.

Consider an onto module homomorphism $\Upsilon : \mathcal{R}[t]^m \longrightarrow \mathcal{R}^m$ defined as:

$$\Upsilon(\mathcal{G}) = ((\mathcal{G}_1)_{t=1}, ..., (\mathcal{G}_m)_{t=1})$$

where $\mathcal{G} = (\mathcal{G}_1, ..., \mathcal{G}_m) \in \mathcal{R}[t]^m$. For a homogeneous vector $\mathcal{G}$ of $\mathcal{R}[t]^m$, the vector $\mathcal{G}_* = \Upsilon(\mathcal{G})$ denote the dehomogenization of $\mathcal{G}$ with respect to t.

A monomial ordering $>$ is said to be a degree ordering if for $u, v \in Mon(\mathcal{R})$, $deg(u) > deg(v)$, implies $u > v$. We can extend the degree monomial ordering $>$ on $Mon(\mathcal{R})$ to $Mon(\mathcal{R}[t]) = \{u\, t^\alpha \mid u \in Mon(\mathcal{R}), \alpha \geq 0\}$, which we denote by $>_t$, as follows:

$$u\, t^a >_t vt^b \quad \text{if and only if} \quad u > v \quad \text{or} \quad u = v \text{ and } a > b.$$

The module ordering on $(\mathcal{R}[t])^m$ induced by $>_t$ is denoted by $>_{M,t}$.

The following lemma demonstrates the relationship between the leading monomials of homogenized and dehomogenized polynomial vectors.

Lemma 3.2. *Let $>$ be a degree ordering on R. The following statements hold.*

(i) If $g \in \mathcal{R}^m$, then $LM_{>_M}(g) = LM_{>_M}(LH(g))$.

(ii) If $g \in \mathcal{R}^m$, then $LM_{>_{M,t}}(g^*) = LM_{>_M}(g)$.

(iii) If $\mathcal{H}$ is a non-zero homogeneous vector of $(\mathcal{R}[t])^m$, then $LM_{>_M}(\mathcal{H}_*) = LM_{>_{M,t}}(\mathcal{H})_*$.

Proof. (i) and (ii) are obvious from the notations and definitions.
(iii): Let $\mathcal{H} \in (\mathcal{R}[t])^m$ be non-zero homogeneous vector of top-degree γ, then $\mathcal{H} \in (\mathcal{R}[t])^m)_\gamma$ and

$$\mathcal{H} = H_1 + H_2 + ... + H_k \tag{3.1}$$

where each H_i is a monomial of $(\mathcal{R}[t])^m$ such that $H_i = t^{\alpha_i} x^{\alpha_i} e_{\alpha_i} \in (\mathcal{R}[t])^m)_\gamma$, $1 \leq i \leq k$, $H_1 >_{M,t} H_2 >_{M,t} ... >_{M,t} H_k$ and $LM_{>_{M,t}}(\mathcal{H}) = H_1 = t^{\alpha_1} x^{\alpha_1} e_{\alpha_1}$. After dehomogenization of of $\mathcal{H}$ given in (3.1) we get

$$\mathcal{H}_* = (H_1)_* + (H_2)_* + ... + (H_k)_* = x^{\alpha_1} e_{\alpha_1} + x^{\alpha_2} e_{\alpha_2} + ... + x^{\alpha_k} e_{\alpha_k}$$

If $x^{\alpha_1}e_{\alpha_1} <_M x^{\alpha_p}e_{\alpha_p}$ for some $p \neq 1$, then $t^{\alpha_1}x^{\alpha_1}e_{\alpha_1} <_{M,t} t^{\alpha_p}x^{\alpha_p}e_{\alpha_p}$ therefore $H_1 <_{M,t} H_p$, which is not possible since $H_1 >_{M,t} H_i$, $2 \leq i \leq k$. It implies that

$$LM_{>_M}(\mathcal{H}_*) = x^{\alpha_1}e_{\alpha_1} = LM_{>_{M,t}}(\mathcal{H})_*.$$

□

Now, we define the process of homogenization and dehomogenization for a polynomial subalgebra. Let $\mathcal{A}$ be the finitely generated subalgebra of the polynomial ring $\mathcal{R}$. The subalgebra $\mathcal{A}^* = \mathsf{K}[\{f^* \mid f \in \mathcal{A}\}]$ is a natural graded subalgebra of $\mathcal{R}[t]$ and its gradation is given as

$$\mathcal{A}^* := \bigoplus_{\gamma \in \mathbb{N}} \mathcal{A}^*_\gamma \quad \text{where each} \quad \mathcal{A}^*_\gamma = \mathcal{A}^* \cap \mathcal{R}[t]_\gamma.$$

For a given subalgebra $\mathcal{A}$ of $\mathcal{R}$, its homogenization is defined as: $\mathcal{A}^*[t] = \mathsf{K}[\mathcal{A}^*, t]$, with the natural grading:

$$\mathcal{A}^*[t] = \bigoplus_{\alpha \in \mathbb{N}} \mathcal{A}^*[t]_\alpha$$

where each $\mathcal{A}^*[t]_\alpha$ is a homogeneous component of degree α given as

$$\mathcal{A}^*[t]_\alpha = \left\{ \sum_{p+q=\alpha} \mathcal{F}_p\, t^q : \mathcal{F}_p \in \mathcal{A}^*_p,\ q \geq 0 \right\}.$$

Note that for a polynomial $f \in \mathcal{A}$, its homogenization $f^* \in \mathcal{A}^*[t]$[§]. Also, for $F \in \mathcal{A}^*[t]$, its dehomogenization $F_* \in \mathcal{A}$.

Remark 3.3. For a homogeneous subalgebra $\mathcal{B}$ of $\mathcal{R}[t]$,[¶] $\mathcal{B}_* = \{\mathcal{F}_* | \ \mathcal{F} \in \mathcal{B}\}$ is a dehomogenized subalgebra of $\mathcal{R}$.

The homogenization of a free module $\mathcal{A}^m$ over a subalgebra $\mathcal{A}$ is $(\mathcal{A}^*[t])^m$ (a free module over $\mathcal{A}^*[t]$), which is naturally graded in the following way:

$$(\mathcal{A}^*[t])^m := \bigoplus_{\alpha \in \mathbb{N}} ((\mathcal{A}^*[t])^m)_\alpha \quad \text{where} \quad ((\mathcal{A}^*[t])^m)_\alpha = ((\mathcal{A}^*[t])_\alpha)^m.$$

Note that, for a polynomial vector $h \in \mathcal{A}^m$, its homogenization $h^* \in (\mathcal{A}^*[t])^m$. Also, for $H \in (\mathcal{A}^*[t])^m$, its dehomogenization $H_* \in \mathcal{A}^m$.

[§] In general, $f^* \notin \mathcal{A}^*$, therefore it is appropriate to define the homogenization of $\mathcal{A}$ as $\mathcal{A}^*[t]$

[¶] A subalgebra of $\mathcal{R}[t]$ is homogeneous if it is generated by homogeneous polynomials.

Example 3.4. *Let $\mathcal{A} = \mathbb{Q}[F]$ where $F = \{x^2 + yz,\ xz + z^2,\ y^2 - z^2\}$. Let $g = (x^4 + 2x^2yz + y^2z^2 + 3,\ x^3z^3 + 3x^2z^4 + 3xz^5 + y^6 + 3y^2z^4 - 3y^4z^2,\ xz + y^2)$ be a vector of $\mathcal{A}^3$, then its homogenization,*

$$\begin{aligned} g^* = (t^2x^4 + 2t^2x^2yz + t^2y^2z^2 + 3t^6,\\ x^3z^3 + 3x^2z^4 + 3xz^5 + y^6 + 3y^2z^4 - 3y^4z^2,\\ t^4xz + t^4y^2) \end{aligned}$$

contained in $\mathcal{A}^*[t]^3$.

Now, we define the homogenization (resp, dehomogenization) of a submodule of a free module $\mathcal{A}^m$ over a subalgebra $\mathcal{A}$.

Definition 3.5. Let $\mathcal{M}$ be a submodule of a free module $\mathcal{A}^m$ over subalgebra $\mathcal{A}$, and $\mathcal{N}$ be a homogeneous submodule[‖] of $(\mathcal{A}^*[t])^m$ over homogenized subalgebra $\mathcal{A}^*[t]$.

1. For $\mathcal{M}^* = \{g^* : g\ \in \mathcal{M}\} \subseteq (\mathcal{A}^*[t])^m$, the homogeneous submodule $\langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}$ is called the homogenization of $\mathcal{M}$.
2. The set $\mathcal{N}_* = \{\ell_* : \ell\ \in \mathcal{N}\} \subseteq \mathcal{A}^m$ is an $\mathcal{A}$-module, called the dehomogenization of $\mathcal{N}$.

Remark 3.6. For any arbitrary submodule $\mathcal{M} = \langle \mathcal{G} \rangle_{\mathcal{A}}$, in general $\langle \mathcal{G}^* \rangle_{\mathcal{A}^*[t]} \neq \langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}$.

The following example illustrates this fact.

Example 3.7. *Let $\mathcal{A} = \mathbb{Q}[\mathcal{F}]$ where $\mathcal{F} = \{x^2 - y,\ y^2\}$ and $\mathcal{G} = \{g_1,\ g_2\} \subset \mathcal{A}^2$ with $g_1 = (x^2 - y,\ y^4 + y^2)$ and $g_2 = (x^2 + y^2 - y,\ y^4)$. Let $\mathcal{M} = \langle \mathcal{G} \rangle_{\mathcal{A}}$ be a submodule of $\mathcal{A}^2$ and $\langle \mathcal{G}^* \rangle_{\mathcal{A}^*[t]} = \langle g_1^*,\ g_2^* \rangle_{\mathcal{A}^*[t]}$ where $g_1^* = (x^2t^2 - yt^3,\ y^4 + y^2t^2)$ and $g_2^* = (x^2t^2 + y^2t^2 - yt^3,\ y^4)$. Note that $\langle g_1^*,\ g_2^* \rangle \subsetneq \mathcal{M}^*$, since $(g_1 - g_2)^* = (-y^2,\ y^2) \in \langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}$ is an element of top-degree 2, but this element does not belong to $\langle g_1^*,\ g_2^* \rangle_{\mathcal{A}^*[t]}$ since top-deg$(g_1^*) = 4 =$top-deg(g_2^*).*

4 De(homogenized) SG-Basis for Modules

In this section, we discuss the behavior SG-basis for modules under the (de)homogenization process. Firstly, we give the properties of vectors and modules under homogenization and dehomogenization.

Lemma 4.1. *(i) For $\mathcal{G}, \mathcal{H} \in (\mathcal{A}^*[t])^m$, $(\mathcal{G} + \mathcal{H})_* = \mathcal{G}_* + \mathcal{H}_*$ and $(\mathcal{G}\mathcal{H})_* = \mathcal{G}_*\mathcal{H}_*$.*

[‖]A submodule $\mathcal{N}$ of $(\mathcal{A}^*[t])^m$ is homogeneous if it is generated by homogeneous vectors.

(ii) *For any vector* $g \in \mathcal{A}^m$, $(g^*)_* = g$.

(iii) *For an* $\mathcal{A}$*-module* $\mathcal{M} \subseteq \mathcal{A}^m$, $\left(\langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}\right)_* = \mathcal{M}$.

(iv) *If* $\mathcal{G} \in (\mathcal{A}^*[t])^m_\gamma$ *and* $(\mathcal{G}_*)^* \in (\mathcal{A}^*[t])^m)_\eta$, *then* $\gamma \geq \eta$ *and* $t^\delta(\mathcal{G}_*)^* = \mathcal{G}$ *with* $\delta = \gamma - \eta$.

(v) *If* $\mathcal{M}$ *is a submodule of* $\mathcal{A}^m$ *over* $\mathcal{A}$, *then each homogeneous element* $\mathcal{G} \in \langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}$ *is of the form* $t^\delta H^*$ *for some* $\delta \in \mathbb{N}$ *and* $H \in \mathcal{M}$.

(vi) *If* $\mathcal{N}$ *is a homogeneous submodule of* $(\mathcal{A}^*[t])^m$, *then for each* $L \in \mathcal{N}_*$ *there is some homogeneous element* $\mathcal{H} \in \mathcal{N}$ *such that* $\mathcal{H}_* = L$.

Proof. The proof of Parts (i) and (ii) is obvious from the definitions of homogenization and dehomogenization.

Part (iii): Let $f \in \mathcal{M}$, then $f^* \in \langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}$ and $f = (f^*)_* \in \left(\langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}\right)_*$, therefore $\mathcal{M} \subset \left(\langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}\right)_*$. On the other hand, let $g \in \left(\langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}\right)_*$, then there exist $h \in \langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}$ with $h_* = g$. Since $h \in \mathcal{M}^*$, then h can be decomposed as $h = a_1 h_1^* + ... + a_s h_s^*$ where $a_i \in \mathcal{A}^*[t]$, $h_i^* \in \mathcal{M}^*$, for $h_i \in \mathcal{M}$. Then the dehomogenization of h could be given as:

$$g = h_* = (a_1 h_1^*)_* + ... + (a_s h_s^*)_* = a_{1_*} h_1 + ... + a_{s_*} h_s \in \mathcal{M}$$

where $a_{i_*} \in \mathcal{A}$, which implies $g \in \mathcal{M}$.

Part (iv): Let $\mathcal{G} \in (\mathcal{A}^*[t]^m)_\gamma$, and $H = \mathcal{G}_* \in \mathcal{A}^m$. Suppose that top-*deg*$(H) = \eta$, then we can assume that

$$\mathcal{G} = t^\delta H_\eta + t^{\delta+1}\ H_{\eta-1} + ... + t^{\delta+s}\ H_{\eta-s}$$

where $H_{\eta-i} \in (\mathcal{A}^m)_{\eta-i},\ 0 \leq i \leq s$ such that $\delta + \eta = \gamma$, and

$$\mathcal{G}_* = H = H_\eta + H_{\eta-1} + ... + H_{\eta-s} \in \mathcal{A}^m$$

after the homogenization of H, we have

$$(\mathcal{G}_*)^* = H^* = H_\eta + t\ H_{\eta-1} + ... + t^s\ H_{\eta-s} \in (\mathcal{A}^*[t])^m)_\eta,$$

which is a homogeneous vector of top-degree η. Multiplying both side by t^δ, we get

$$t^\delta(\mathcal{G}_*)^* = t^\delta H^* = t^\delta H_\eta + t^{\delta+1}\ H_{\eta-1} + ... + t^{\delta+s}\ H_{\eta-s} \in (\mathcal{A}^*[t])^m$$

which implies $\mathcal{G} = t^{\delta}(\mathcal{G}_*)^*$ where $\delta = \gamma - \eta$.

Part (v): It follows from (ii) of this Lemma that for $\mathcal{G} \in \langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}$, $H = \mathcal{G}_* \in \left(\langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}\right)_* = \mathcal{M}$. From (iv) of this Lemma we obtain $\mathcal{G} = t^{\delta} H^*$ for $\delta \in \mathbb{N}$.

Part (vi): Let $L \in \mathcal{N}_*$, where $\mathcal{N}_*$ is the homomorphic image of $\mathcal{N}$, therefore there exist a vector $\mathcal{H} \in \mathcal{N}$ such that $\mathcal{H}_* = L$. Further multiplying it with the homogeneous components by monomials in "t". We can consider $\mathcal{H}$ to be a homogeneous vector in $\mathcal{N}$ satisfying $\mathcal{H}_* = L$ for $L \in \mathcal{N}_*$. $\square$

The following theorem illustrates the behavior of SG-bases for modules (w.r.t module ordering induced by a degree ordering) under homogenization.

Theorem 4.2. *Let $\mathcal{M} = \langle \mathcal{G} \rangle_{\mathcal{A}}$ be a submodule of $\mathcal{A}^m$ over subalgebra $\mathcal{A}$ generated by a subset $\mathcal{G} \subset \mathcal{A}^m$, and $\langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}$ be a homogenization of $\mathcal{M}$ in $(\mathcal{A}^*[t])^m$ w.r.t the variable t. The following two statements are equivalent.*

(A) $\mathcal{G}$ is an SG-basis for $\mathcal{M}$ in $\mathcal{A}^m$ w.r.t module ordering $>_M$ (induced by a degree ordering $>$ on $\mathcal{R}$).

(B) $\mathcal{G}^* = \{g^* \mid g \in \mathcal{G}\}$ is an SG-basis for $\langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}$ in $(\mathcal{A}^*[t])^m$ w.r.t extended module ordering $>_{M,t}$.

Proof. $(A) \Rightarrow (B)$
Given that $LM_{>_{M,t}}(\mathcal{G}^*) \subset LM_{>_{M,t}}\left(\langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}\right)$, we need to prove $\mathcal{G}^*$ is an SG-basis for $\langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}$, i.e.

$$\left\langle LM_{>_{M,t}}(\mathcal{G}^*) \right\rangle_{in_{>_t}(\mathcal{A}^*[t])} = \left\langle LM_{>_{M,t}}\left(\langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}\right) \right\rangle_{in_{>_t}(\mathcal{A}^*[t])},$$

where the initial subalgebra $in_{>_t}(\mathcal{A}^*[t]) = \mathsf{K}[LM_{>_t}(\mathcal{A}^*[t])]$. Let $H \in \langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}$, by Lemma 3.2(ii) $LM_{>_{M,t}}(H) = LM_{>_{M,t}}(LH_{>_{M,t}}(H))$. So, without loss of generality, we can assume that H is a homogeneous vector. From Lemma 4.1(v), $H = t^{\delta}(\tilde{H})^*$ for some $\delta \geq 0$ and $\tilde{H} \in \mathcal{M}$. By using Lemma 3.2

$$LM_{>_{M,t}}(H) = LM_{>_{M,t}}(t^{\delta}(\tilde{H})^*) = t^{\delta} LM_{>_{M,t}}((\tilde{H})^*) = t^{\delta} LM_{>_M}(\tilde{H}). \quad (4.1)$$

Since $\mathcal{G}$ is an SG-basis for $\mathcal{M}$, therefore for $\tilde{H} \in \mathcal{M}$, there exists some $g \in \mathcal{G}$ and $h \in \mathcal{A}$ such that

$$LM_{>_M}(\tilde{H}) = LM_{>}(h) LM_{>_M}(g) = LM_{>_t}(h^*) LM_{>_{M,t}}(g^*). \quad (4.2)$$

From (4.1) and (4.2),

$$LM_{>_{M,t}}(H) = t^{\delta} LM_{>_t}(h^*) LM_{>_{M,t}}(g^*) = LM_{>_t}(t^{\delta} h^*) LM_{>_{M,t}}(g^*)$$

Note that, $t^{\delta} h^* \in \mathcal{A}^*[t]$, therefore, $LM_{>_t}(t^{\delta} h^*) \in in_{>_t}(\mathcal{A}^*[t])$. Also, $g^* \in \mathcal{G}^*$, this implies $LM_{>_{M,t}}(H) \in \left\langle LM_{>_{M,t}}(\mathcal{G}^*) \right\rangle_{in_{>_t}(\mathcal{A}^*[t])}$.

$(B) \Rightarrow (A)$

Let $f \in \mathcal{M}$, which implies $f^* \in \langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}$. Also $\mathcal{G}^*$ is an SG-basis of $\langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}$, therefore, there exists $g^* \in \mathcal{G}^*$ and $L \in \mathcal{A}^*[t]$ such that

$$LM_{>_{M,t}}(f^*) = LM_{>_t}(L) LM_{>_{M,t}}(g^*)$$

After dehomogenization and using Lemma 3.2, we get

$$\begin{aligned}
\left(LM_{>_{M,t}}(f^*)\right)_* &= (LM_{>_t}(L))_* \left(LM_{>_{M,t}}(g^*)\right)_* \\
LM_{>_M}((f^*)_*) &= LM_{>}(L_*) LM_{>_M}((g^*)_*) \\
LM_{>_M}(f) &= LM_{>}(L_*) LM_{>_M}(g) \quad \text{where } L_* \in \mathcal{A} \text{ and } g \in \mathcal{G}.
\end{aligned}$$

This shows that $\mathcal{G}$ is an SG-basis of $\mathcal{M}$ over subalgebra $\mathcal{A}$. □

Example 4.3. *Let $\mathcal{A} = \mathbb{Q}[x^2, xy] \subset \mathbb{Q}[x, y]$ equipped with the degree lexicographical ordering $x > y$ and $\mathcal{M} = \left\langle (x^2 + xy,\ 1),\ (x^2y^2 - 2,\ xy) \right\rangle_{\mathcal{A}}$ be the submodule of $\mathcal{A}^2$. The set $\mathcal{G} = \{(x^2 + xy,\ 1),\ (x^2y^2 - 2,\ xy),\ (2x^2 + 2xy,\ -x^3y)\}$ is an SG-basis for $\mathcal{M}$ w.r.t **TOP** module ordering $>_M$ (induced by $>$) with $e_1 > e_2$. By theorem 4.2, $\mathcal{G}^* = \{(x^2 + xy,\ t^2),\ (x^2y^2 - 2t^4,\ xyt^2),\ (2x^2t^2 + 2xyt^2,\ -x^3y)\}$ is an SG-basis for $\langle \mathcal{M}^* \rangle_{\mathcal{A}^*[t]}$ w.r.t $>_{M,t}$.*

Next, we have a result that gives a general approach to obtaining SG-basis for a submodule by dehomogenizing a homogeneous SG-basis of a homogeneous submodule.

Theorem 4.4. *Let $\mathcal{N}$ be a homogeneous submodule of $(\mathcal{A}^*[t])^m$. If $\mathcal{G}$ is a homogeneous SG-basis for $\mathcal{N}$ w.r.t $>_{M,t}$, then $\mathcal{G}_* = \{g_* \mid g \in \mathcal{G}\}$ is an SG-basis for submodule $\mathcal{N}_*$ in $\mathcal{A}^m$ w.r.t $>_M$ over subalgebra $\mathcal{A}$.*

Proof. Assume $\mathcal{G}$ is a homogeneous SG-basis for $\mathcal{N}$ which implies $\langle \mathcal{G} \rangle_{\mathcal{A}^*[t]} = \mathcal{N}$, therefore $\mathcal{G}_* = \Upsilon(\mathcal{G})$ generates $\mathcal{N}_* = \Upsilon(\mathcal{N})$. Let $L \in \mathcal{N}_*$ by Lemma 4.1 (vi) there exist homogeneous vector $H \in \mathcal{N}$ such that $H_* = L$. Also, $LM_{>_M}(L) = LM_{>_{M,t}}(L^*) = LM_{>_{M,t}}((H_*)^*)$. By Lemma 4.1 (v), we can write $H = t^{\delta}(H_*)^*$, therefore

$$LM_{>_{M,t}}(H) = LM_{>_{M,t}}(t^{\delta}(H_*)^*) = t^{\delta} LM_{>_{M,t}}(H_*)^*) = t^{\delta}(LM_{>_{M,t}}(L)^*).$$

Since $\mathcal{G}$ is an SG-basis for $\mathcal{N}$, so $LM_{>_{M,t}}(H) = LM_{>_t}(a)LM_{>_{M,t}}(g)$ for some $a \in \mathcal{A}^*[t]$ and $g \in \mathcal{G}$. Hence $t^\delta LM_{>_{M,t}}(L^*) = t^\delta LM_{>_t}(a)LM_{>_{M,t}}(g)$, by dehomogenization,

$$\left(t^\delta LM_{>_{M,t}}(L^*)\right)_* = \left(t^\delta LM_{>_t}(a)LM_{>_{M,t}}(g)\right)_*.$$

Hence by Lemma 3.2 (iii) and Lemma 4.1 (i) we obtain

$$LM_{>_M}(L) = LM_{>}(a_*)LM_{>_M}(g_*) \tag{4.3}$$

Therefore, $\mathcal{G}_*$ is an SG-basis for $\mathcal{N}_*$ w.r.t $>_M$ over subalgebra $\mathcal{A}$. □

We call $\mathcal{G}_*$ obtained in 4.4 the dehomogenized SG-basis w.r.t $>_M$.

Below is the example that illustrates the procedure to compute the homogenized SG-basis for a submodule of $\mathcal{A}^m$ by passing through homogenized generators.

Example 4.5. *Let $\mathcal{A} = \mathbb{Q}[x^2 - y, y^2]$ be a subalgebra of $\mathbb{Q}[x, y]$, equipped with the degree lexicographical ordering $x > y$ and $\mathcal{M} = \langle \mathcal{G} \rangle_{\mathcal{A}}$ be a submodule of $\mathcal{A}^2$ where $\mathcal{G} = \{(x^2+xy,\ 1),\ (x^2y^2-2,\ xy)\}$. Note that $\mathcal{A}^*[t] = \mathbb{Q}[x^2-yt,\ y^2, t] \subset \mathcal{R}[t] = \mathbb{Q}[x, y, t]$ and $\mathcal{H} = \{(x^2y^2-y^3t, y^4),\ (x^4-2x^2yt+y^2t^2, 2y^2t^2),\ (0, x^2y^4-y^5t-2y^4t^2)\}$ is an SG-basis for $\langle \mathcal{G}^* \rangle_{\mathcal{A}^*[t]}$ w.r.t extended **TOP** module ordering $>_{M,t}$ (induced by extended degree lexicographical ordering $>_t$ on $\mathcal{R}[t]$), with $e_1 > e_2$. On dehomogenizing $\mathcal{H}$ we get*

$$\mathcal{H}_* = \{(x^2y^2 - y^3, y^4),\ (x^4 - 2x^2y + y^2, 2y^2),\ (0, y^5 - x^2y^4 + 2y^4)\}$$

*By Theorem 4.4, $\mathcal{H}_*$ is an SG-basis for $\mathcal{M} = \langle \mathcal{G}^* \rangle_*$ over subalgebra A w.r.t **TOP** module ordering $>_M$ (with $e_1 > e_2$).*

4.1 Reduced SG-basis & Homogenization

In this section, we examine the behavior of a reduced SG-basis of the module during the processes of homogenization and dehomogenization. For this purpose, we first define the reduced SG-basis for the submodule of $\mathcal{A}^m$. We then present the result for the homogenized reduced SG-basis.

Definition 4.6. (Reduced SG-Basis)
An SG-basis $\mathcal{G} = \{g_1, g_2, ..., g_t\} \subset \mathcal{A}^m$ over subalgebra $\mathcal{A}$ is said to be reduced SG-basis if the following conditions hold:

(i) $LC_{>_M}(g_j) = 1$ for all j where $j = \{1, ..., t\}$.

(ii) $LM_{>_M}(g_i) \neq \ell\, LM_{>_M}(g_j)$, for $i \neq j$, for all $\ell \in in_>(\mathcal{A})$.

(iii) For all j, $Support(Tail(g_j)) \cap \langle LM(\mathcal{G})\rangle_{in(\mathcal{A})} = \phi$. [‡]

Following is an example of a reduced SG-basis.

Example 4.7. *Let $\mathcal{A} = \mathbb{Q}[L] \subseteq \mathbb{Q}[x, y, z]$, where $L = \{x^2 - z, y^2 + z,\ yz\}$ and $\mathcal{G} = \langle g_1, g_2, g_3\rangle_{\mathcal{A}}$ be the submodule of $\mathcal{A}^3$, where $g_1 = (x^2 + y^2, 0, yz + 1)$ $g_2 = (y^2z^2,\ x^2yz - yz^2, 2)$, $g_3 = (y^2z, 2yz, yz^2 + y^3z)$ is the SG-basis w.r.t* ***TOP*** *module ordering $>_M$ (induced by the degree reverse lexicographical ordering $x > z > y$). Observe that $LC_>(g_i) = 1$ for $1 \le i \le 3$, we have $LM_>(g_1) = x^2e_1$, $LM_>(g_2) = x^2yze_2$, $LM_>(g_3) = y^3ze_3$, and $LM_>(g_i) \neq aLM_>(g_j)$ for all $a \in in_>(\mathcal{A})$ and $i \neq j$. Also, no monomial of $Support(Tail(g_i))$ written as product of $LM(g_j)$ and $LM(h)$ for all $g_i \in \mathcal{G}$ and $h \in \mathcal{A}$. Hence $\mathcal{G}$ is reduced SG-basis over subalgebra $\mathcal{A}$.*

Now, we define the result for a reduced SG-basis under homogenization.

Theorem 4.8. *Let $\mathcal{M} = \langle \mathcal{G}\rangle$ be the submodule of $\mathcal{A}^m$ and $\mathcal{G}$ be a reduced SG-basis for $\mathcal{M}$ w.r.t $>_M$, then $\mathcal{G}^* = \{h^* \mid h \in \mathcal{M}\}$ is the reduced SG-basis of $\langle \mathcal{M}^*\rangle_{\mathcal{A}^*[t]}$ w.r.t $>_{M,t}$.*

Proof. We assume that $\mathcal{G}$ is an SG-basis for $\mathcal{M}$, then by Theorem 4.2 (b), $\mathcal{G}^*$ is an SG-basis of $\langle \mathcal{M}^*\rangle_{\mathcal{A}^*[t]}$. We need to show that $\mathcal{G}^*$ is reduced. For all $g \in \mathcal{G}$, we have $LC_{>_M}(g) = 1$ therefore $LC_{>_{M,t}}(g^*) = 1$.

If there exist $g_i^*, g_j^* \in \mathcal{G}^*$ such that $LM_{>_{M,t}}(g_i^*) = \ell\ LM_{>_{M,t}}(g_j^*)$ with $i \neq j$ for some $\ell \in in_{>_t}(\mathcal{A}^*[t])$. Then by dehomogenization we get $LM_{>_M}(g_i) = \ell_*\ LM_{>_M}(g_j)$, where $\ell_* \in in_>(\mathcal{A})$. This is a contradiction since $\mathcal{G}$ is the reduced SG-basis.

If for some $g_j^* \in \mathcal{G}^*$, $Support\left(Tail(g_j^*)\right) \cap \left\langle LM_{>_{M,t}}(\mathcal{G}^*)\right\rangle_{in(\mathcal{A}^*[t])} \neq \phi$. It implies that there exist $\mu \in Support(Tail(g_j^*))$ such that $\mu = q\ LM_{>_{M,t}}(g_i^*)$ for some $q \in in_{>_t}(\mathcal{A}^*[t])$ and $g_i^* \in \mathcal{G}^*$. The monomial $\mu \in Support(Tail(g_j^*))$ if, and only if $\mu = t^\delta\nu$ for $\nu \in Support(Tail(g))$ and $\delta \in \mathbb{N}$. Therefore, $t^\delta\nu = q\ LM_{>_{M,t}}(g_i^*))$. By dehomogenization, we will obtain $\nu = q_*\ LM_{>_M}(g_i)$, where $q_* \in in_>(\mathcal{A})$. This is a contradiction since $\mathcal{G}$ is the reduced SG-basis. $\square$

Example 4.9. *In Example 4.7, $\mathcal{G}$ is the reduced SG-basis over $\mathcal{A}$. By theorem 4.8.*

$$\begin{aligned} \mathcal{G}^* = \{&\left(x^2 + y^2, 0, yz + t^2\right), \\ &\left(y^2z^2, x^2yz - yz^2y, 2t^4\right), \\ &\left(y^2t^2 + zt^3, 2yzt^2, y^3z + yz^2t + y\right)\} \end{aligned}$$

‡No monomial in the support of $Tail(g_j)$ can be written as a product of any $LM_{>_M}(g_i)$ and $LM_>(h)$, for all $g_i \in \mathcal{G}$ and $h \in \mathcal{A}$.

is the reduced SG-basis of $\langle \mathcal{G}^* \rangle_{\mathcal{A}^*[t]}$ *over subalgebra* $\mathcal{A}^*[t] = \mathbb{Q}[x^2 - zt, y^2 + zt, yz, t]$.

The next example shows that for a homogeneous submodule $\mathcal{N} \subset (\mathcal{A}^*[t])^m$, the dehomogenization of the reduced *SG*-basis of $\mathcal{N}$ is not necessarily the reduced *SG*-basis of $\mathcal{N}_*$.

Example 4.10. *Let* $\mathcal{A}^*[t] = \mathbb{Q}[xy - yt,\ y^2,\ t] \subset \mathbb{Q}[x, y, t]$ *be the homogenized subalgebra of* $\mathcal{A} = \mathbb{Q}[xy - y,\ y^2]$ *equipped with extended ordeing* $>_t$ *(induced by degree lexicographical ordering with* $x > y$*) on* $\mathbb{Q}[x, y]$*. Let* $\mathcal{G} = \{(xy^3 - y^3t,\ y^4),\ (xyt - yt^2,\ y^2t)\} \subset (\mathcal{A}^*[t])^2$*. Observe that* $\mathcal{G}$ *is the reduced SG-basis for the submodule* $\mathcal{N} = \langle \mathcal{G} \rangle_{\mathcal{A}^*[t]}$ *w.r.t the extended* ***TOP*** *module ordering* $>_{M,t}$ *(induced by* $>_t$*). Then* $\mathcal{G}_* = \{(xy^3 - y^3,\ y^4),\ (xy - y,\ y^2)\}$ *is a SG-basis for* $\mathcal{N}_* = \langle \mathcal{G}_* \rangle_{\mathcal{A}}$ *w.r.t the* $>_M$ *over subalgebra* $\mathcal{A}$*. But* $\mathcal{G}_*$ *is not reduced since* $LM_{>_M}(g_1) = a\ LM_{>_M}(g_2)$ *for* $a = y^2 \in in_>(\mathcal{A})$*, i.e.* $(xy^3, 0) = y^2(xy, 0)$.

References

[1] B. Buchberger, *An Algorithm for finding the Bases of the Residue Class Ring Modulo a Zero Dimensional Polynomial Ideal.* PhD thesis, Universitat Innsbruck, Austria, 1965.

[2] B. Buchberger, *Gröbner Bases: An Algorithmic method in Polynomial Ideal Theory.* In N.K.Bose, ed, Recent Trends in Multidimensional System Theory, Chapter 6. Dordrecht, Holland: D. Riedel, pp. 184-232, 1985.

[3] Charles C. Hanna & Jon L. Johnson, *Homogenization and dimension in* Λ*-graded rings*, Communications in Algebra, Vol 12 (3), P P: 259-269, 1984.

[4] G-M Greuel, G. Pfister, *A SINGULAR Introduction to Commutative Algebra*, Springer, second edition, 2008.

[5] H. Li, Note on (De)homogenize Gröbner Bases. *Journal of Algebra, Number Theory: Advances and Applications,*volume 1(3), 2010, pp: 35-70.

[6] J. A. Khan, On the (De)homogenization of Sagbi Bases. An. Şt. Univ. Ovidius Constant, Vol. 21(2),2013, pp:173-180.

[7] J-C. Faugere & S. Rahmany, *Solving systems of polynomial equations with symmetries using SAGBI-Gröbner bases*, Proceedings of the International Symposium on Symbolic and Algebraic Computation, ISSAC. PP: 151-158, 2009.

[8] J. Lyn. Miller, *Analogues of Gröbner bases in polynomial rings over a ring*, Journal of Symbolic Computation, 11, Pages 1-16, 2008.

[9] J. Lyn. Miller, *Effective Algorithms for Intrinsically computing SAGBI-Gröbner bases in polynomial ring over a field.* In Groebner bases and applications (Linz, 1998),251 of London Math. Soc. Lecture Notes Ser., Pages 421 -433. Cambridge Univ. Press, Cambridge, 1998.

[10] L. Robbiano; & M. Sweedler, *Subalgebra Bases.* W. Bruns and A. Simis, Lecture Notes in Mathematics Series, Volume 1430, Springer-Verlag Berlin Heidelberg, 42, pp. 61-87, 1988.

[11] M. Kreuzer, & L. Robbbiano, *Computational Commutative Algebra 1*, Springer-Verlag. Berlin Heidelberg, 2000.

[12] M. Kreuzer, L. Robbiano, *Computational Commutative Algebra 2*, Springer-Verlag, 2005.

[13] M. Stillman, H. Tsai, *Using SAGBI bases to compute invariants*, Journal of Pure and Applied Algebra, Vol 139 (1–3), PP: 285-302, 1999.

[14] N. Kanwal, N. Jabeen, and J. A. Khan, *Standard bases for modules over polynomial subalgebras*, Communications in Algebra, Vol 49 (1), PP: 58-72 Taylor & Francis 2021.

[15] N. Kanwal and J. A. Khan "`sagbigrob.lib`": A SINGULAR 4-3-0 library to compute Sagbi-Gröbner Basis of an Ideal of a Subalgebra, 2022.

[16] N. Kanwal "`stdmodule.lib`": A SINGULAR 4-4-0 library to compute Standard Bases of a submodule of a free module over polynomial subalgebra w.r.t Natural Grading, 2024.

[17] Nicolas M. Thiéry, *Computing Minimal Generating Sets of Invariant Rings of Permutation Groups with SAGBI-Gröbner Basis*, Discrete Models: Combinatorics, Computation, and Geometry, DM-CCG 2001, Paris, France. PP: 315-328, 2001.

[18] W. Decker, G-M Greuel, G. Pfister, H, Schönemann. SINGULAR 4-3-0 — A computer algebra system for polynomial computations, 2022.

[19] W. W. Adams and P. Loustauneau, *An Introduction to Gröbner Bases*, American Mathematical Society. Providence, RI. Graduate Studies in Mathematics, Vol. 3, 1994.

Nazish Kanwal,
Department of Sciences & Humanities,
FAST-National University of Computer & Emerging Sciences,
Karachi, Pakistan.
Email: nazish.kanwal@nu.edu.pk

Junaid Alam Khan,
School of Mathematics & Computer Science,
Institute of Business Administration,
Karachi, Pakistan.
Email: jakhan@iba.edu.pk