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Some lattice theoretical results on non-Euclidean graphs

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Abstract

This paper develops a unified framework connecting lattice theory and suborbital graphs, with particular focus on Farey graph. By equipping the Farey graph with lattice structures, we reveal new combinatorial and algebraic properties. Essential element graphs, Hasse diagrams, and integer sequences for vertices and edges are systematically explored. This study distinguishes itself from previous studies by expanding proofs, consolidating definitions, and providing illustrative examples. Our results demonstrate how a lattice perspective can enrich theoretical and applied research in number theory, geometry, and network science.

1 Introduction

In recent years, a number of studies have explored graph structures through lattice-theoretic frameworks, contributing significantly to the intersection of lattice and graph theory. In [1], the concept of the co-maximal graph associated with a finite bounded lattice was introduced, laying the groundwork for further structural analysis of such graphs. In [2], a simple graph was constructed from a lattice, and its planarity was systematically examined, revealing significant connections between lattice-theoretic properties and topological characteristics of associated graph. In [3], the authors associated a simple graph with a lattice containing nonzero elements and zero divisors.

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The resulting zero-divisor graph was analysed with respect to several graph-theoretic parameters, including its diameter and girth. In [4], attention was given to the reduced zero-divisor graph of a poset. The study explored its structural features and examined the conditions under which an ideal of the poset is both annihilating and prime. Furthermore, the authors provided a classification of posets possessing exactly four or seven annihilating prime ideals under specific conditions. In [5], the essential element graph of a lattice was analysed, wherein necessary and sufficient conditions were established for a regular graph to qualify as a complete graph.The study further examined conditions under which such graphs could also be characterized as both complete and bipartite. In [6], the incomparability graphs of certain dismantable lattices, and several properties of these graphs were investigated, shedding light on the interplay between order-theoretic and graph-theoretic concepts. In [7], the focus shifted to zero-divisor graphs and their isomorphic correspondence with incomparable graphs derived from the essential elements; line graphs of these structure were also investigated.The work in [8] examined the Eulerian nature of zero-divisor graphs associated with a particular class of finite pseudo-complemented posets. Moreover, it considered structural properties such as Hamiltonian, vertex-pansyclicity, and edge-pansyclicity of complements of zero-divisor graphs in the context of finite 0-distributive posets. In [9], Beck's conjecture-which posits the equality of the clique number and chromatic number of zero-divisor graphs of posets-was addressed through a direct and elementary proof. The study also demonstrated the conjecture fails in general when the clique number is infinite. Finally, [10] investigated bounded posets and posets with isomorphic zero-divisor graphs, extending the results to finite cases.

Suborbital graphs and lattices are two structures in mathematics that can be related to each other through their applications in group theory, number theory and geometry. Suborbital graphs are a type of graph that emerges especially in the context of modular forms and automorphic functions. These graphs can represent orbitals of pairs of vertices under the influence of a subgroup of a modular group. In this context, the term "lattice" generally refers to discrete subgroup of Euclidean space that includes the vector space on real numbers. In the context of number theory and geometry, a lattice can be a set of points with a periodic structure in $\mathbb{R}^n$. Furthermore, lattices can be discussed in more pure concepts such as Lie groups and algebraic group theory. The relationship between suborbital graphs and lattices can be grouped as follows.

Group Actions: Suborbital graphs are constructed from group actions, specifically actions of the modular group or its subgroups on the upper half plane. Lattices often appear as stabilizers or orbital structures under group actions. The influence of a group on a lattice can lead to the formation of suborbital graphs.

Modular Forms and Automorphic Functions: Lattices are very important in the study of modular forms because they are functions that do not change under the influence of the modular group on the upper half-plane. Suborbital graphs can be used to study the properties of modular forms by visualizing how pairs of vertices match under the action of the group.

Geometry and Flooring: Lattices can be used to tile Euclidean space, and suborbital graphs can be associated with flooring patterns created by group actions on these spaces. For example, flooring of the hyperbolic plane by modular group action can be studied using suborbital graphs.

Number Theory: Both lattices and suborbital graphs are deeply connected to number theory. For example, the integer lattice $\mathbb{Z}^2$ under the influence of the modular group can lead to interesting number theory properties and visualizations via suborbital graphs.

Discrete Structures: Suborbital graphs and lattices are examples of discrete structures that can be studied using combinatorial and algebraic methods. Discrete structures are useful for understanding various features of groups and their actions.

In summary, suborbital graphs and lattices are fundamentally connected through the concept of group actions, their application in studying modular forms and flooring patterns, and their roles in number theory and discrete mathematics. These structures provide descriptive perspectives and tools for exploring the properties and behaviour of mathematical objects under group actions.

This study investigates the structural relationship between suborbital graph and lattice theory. While both areas have been widely studied in discrete mathematics, their direct interaction has received less attention. Especially, although previous studies have examined Farey graphs and lattice structures separately, this study is the first comprehensive work to establish the combinatorial and algebraic bridge between two structures on hyperbolic geometry. In particular, we explore the Farey graph, classical object in number theory and geometry, under the lens of lattice theory. Unlike previous work, by focusing on graphs defined over hyperbolic geometry, specifically the graphs of Farey, $G_{u,N}$ and $F_{u,N}$, we explicitly construct partial orders, lattice structures, and essential element graphs, and present them with illustrative examples and rigorous proofs. Furthermore, leveraging modular group actions, adjacency relations are reformulated as binary relations, facilitating the investigating of poset and lattice conditions within these graphs. Ultimately, our goal is to highlight new algebraic and combinatorial insights, with potential applications in network analysis and hyperbolic tiling problems.

Endowing a Farey graph with a lattice structure and interpreting graph-theoretic properties through lattice-theoretical arguments constitutes a significant advancement both in pure mathematical theory and applied domains. First and foremost, conceptualizing the Farey graph as a poset allows it to be studied within the framework of modular lattices, Galois connections, and order theory. This recharacterization enables the Farey graph to be analysed not only from the perspective of number theory but also through the lens of algebraic and combinatorial structures. Within the context of group actions, the introduction of a lattice structure facilitates a systematic examination of critical concepts such as symmetry, fixed points, and orbit structures in suborbital graphs. Such a lattice-theoretic formalism provides novel perspectives in areas like topological group theory, modular forms, and automorphic functions. Moreover, equipping graphs with lattice structures builds theoretical bridges between distinct fields such as group theory, number theory, and combinatorics. These connections not only enrich the theoretical landscape but also lay the groundwork for new interdisciplinary methodologies. As a direct consequence of the aforementioned theoretical developments, significant implications may also arise in the fields of algorithm design and computational efficiency, cryptography, and numerical applications, as well as in physics and computer science. For more information, see [11, 12].

2 Preliminaries

For clarity, we consolidate all necessary definitions here. We avoid repetition of basic notations in later sections, ensuring a smoother logical flow.

2.1 Lattice Theory

Lattices, which have applications in many different disciplines, essentially perform the ordering of elements defined in a set given by relations such as $\leq$: less than or equal to, $\backslash$: division or $\subseteq$: coverage. Examination of these ordered groups has led to further examination of the lattices. [13, 14] can be given as a reference on this subject. Each subset of the Cartesian product set $A \times B$ is called a relation, where A and B are non-empty sets. If the relation provides the properties of reflection, symmetry and transitivity, it is called an equivalence relation. In this study, we will work with relations that are not equivalence relations.

Let A be a non-empty set and R a binary relation on A. If R is reflexive $(\forall a \in A, aRa)$, antisymmetric $(aRb \wedge bRa \Rightarrow a = b)$, and transitive $(aRb \wedge bRc \Rightarrow aRc)$, then (A, R) is called a partially ordered set or poset. Two elements $a, b \in A$ are said to be comparable if either aRb or bRa. A poset in which every pair of elements is comparable is called a totally ordered set. An element $a \in A$ is minimal if there is no $b \in A$ with $b \neq a$ and bRa; maximal if there is no $b \in A$ with $b \neq a$ and aRb; the smallest element if aRb for all $b \in A$; and the largest if bRa for all $b \in A$. A poset (A, R) is a lattice if every finite non-empty subset of A has both a least upper bound $(supremum, a \vee b)$ and a greatest lower bound $(infimum, a \wedge b)$. A lattice $(A, \vee, \wedge)$ is an algebraic lattice if it satisfy idempotent, commutative, associative, and absorption laws. If the lattice contains a greatest element 1 and least element 0, such that $0 \vee 1 = 1$ and $0 \wedge 1 = 0$, it is called bounded lattice. In a bounded lattice, an atom is an element that covers 0, while a coatom is one covered by 1. An element is join-irreducible if it covers only one element, meet-irreducible if it is covered by only one, and double-irreducible if both conditions hold. Element of $A \setminus \{0, 1\}$ are called proper elements. The horizontal sum of a family of lattices $\{A_i\}_{i \in \mathbb{N}}$ is the disjoint union of the A_i with a shared least element 0 and greatest element 1, where internal ordering is preserved within each A_i and elements from different A_i are incomparable except for their relation 0 and 1. [5, 7, 13, 14] can be given as references.

2.2 Graph Theory

The origins of graph theory are typically traced to two main sources. The first of these is based on the work known as "The Seven Bridges of Königsberg", which Euler (1707-1783) included in his work dated 1736. The second is based on the work of Pythagoras and his students. In graph theory, a graph consists of vertices and edges connecting the vertices. For more detailed information in graph theory, [15] can be examined. Here, some definitions are included about graph theory.

A simple graph is without any geometric or directional data, whereas a directed graph is one where edges have a defined direction. In a strictly directed graph, all edges are directional. A cycle is a path that begins and ends at the same vertex, while a loop is a cycle in which no vertex is visited more than once. The distance between two vertices $x, y \in V(G)$, denoted by $d(x, y)$, is the length of the shortest path between them, and the diameter of a graph G is defined as $\sup\{d(x, y) \mid x, y \in V(G)\}$. A reduced graph is obtained by deleting vertices only. A graph G is called complete if there is an edge between every pair of vertices $x, y \in V(G)$ i.e., $xy \in E(G)$ for all such x, y. A complete graph of k vertices is referred to as a k-clique. A vertex $x \in V(G)$ is said to be isolated if it is not adjacent to any other vertex, i.e., $xy \notin E(G)$ for all $y \in V(G)$. A graph G is bipartite if its vertex set can be partitioned into two subsets X and Y, such that all edges connect vertices from different subsets; if this partitioning can be extended to n disjoint subsets, after that G is called n-bipartite graph. Let A be a lattice. The set of annihilators of an element

$a \in A$ is defined as $\mathrm{Ann}(a) = \{b \in A : a \wedge b = 0\}$. An element $b \in A$ is called an essential element if $a \wedge b \neq 0$ for all $a \in A \setminus \{0\}$. The essential element graph of a lattice A, denoted ϵ_A, is the simple undirected graph whose vertex set is $A \setminus \{0, 1\}$, where two vertices $a, b \in A \setminus \{0, 1\}$ are adjacent if and only if $a \vee b$ is an essential element. The zero-divisor graph Z_A of a lattice A with least element 0 is defined as the simple undirected graph with vertex set $A \setminus \{0\}$, where two vertices $a, b \in A \setminus \{0\}$ are adjacent if and only if $a \wedge b = 0$. It has been show that in a bounded lattice A with a unique atom, every element is essential, and consequently ϵ_A forms a complete graph. Furthermore, if A has a unique coatom, it is clear that this element is essential. Additionally, if an element $a \in A$ is not essential, in that case any element $b \leqslant a$ is not essential.[7] can be examined for detailed information.

2.3 The Action of Suborbital Graph

The suborbital graphs discussed in this study are suborbital graphs defined on hyperbolic geometry. However, transformation groups that left the vertices and edges invariant were defined on the suborbital graph, and especially the transformation groups were associated with Möbius transformations.

Here, the ideas about suborbital graph action, first put forward by [16] are summarized.

Let (G, Ω) is a permutative group where $g \in G$ and $\alpha, \beta \in \Omega$. The group G acts on $\Omega \times \Omega$ with transform $g : (\alpha, \beta) \to (g(\alpha), g(\beta))$. These transformation orbits formed by the act of the group G are called orbitals of G and are denoted by $O(\alpha, \beta) = \{g(\alpha, \beta) \mid g \in G\}$. The suborbital graph $G(\alpha, \beta)$ compose of suborbitals $O(\alpha, \beta)$. $(\gamma, \delta) \in O(\alpha, \beta)$ is called a directed edge from γ to δ and is symbolized by $\gamma \to \delta$ where the vertices γ and δ are defined on $\widehat{\mathbb{Q}}$. In hyperbolic geometry, an edge between two vertices can be drawn as a hyperbolic geodesic. $O(\beta, \alpha)$ is an orbital as well, it can be equal or different from orbital $O(\alpha, \beta)$. If the orbits are different from each other, the suborbital graphs are opposite to each other and paired suborbital graphs are formed. If they are equal, the graph contains a pair of oppositely directed edges, in which case each pair of edges containing two directions can be replaced by a simple undirected edge. That is, if $O(\beta, \alpha) = O(\alpha, \beta)$ and $(\gamma, \delta) \in O(\alpha, \beta)$, the edge between vertices γ and δ is denoted by $\gamma - \delta$ instead of $\gamma \leftrightarrow \delta$.

Assume that $\approx$ is a congruence relation. If $g(\alpha) \approx g(\beta)$ is satisfied when $\alpha \approx \beta$ for $\forall \alpha, \beta \in \Omega$ and $\forall g \in G$, the $\approx$ relation is called G-invariant equivalence relation and the classes formed in this way are called blocks. The identity relation $\alpha \approx \beta \Rightarrow \alpha = \beta$ for $\forall \alpha, \beta \in \Omega$ and the universal relation $\alpha \approx \beta$ for $\forall \alpha, \beta \in \Omega$ can be given as example for congruence relations. Unlike identity and universal relations, if there is an congruence relation on Ω , the pair (G, Ω)

is called imprimitive, otherwise it is called primitive. In [17], suborbital graphs were studied in case of $G = \Gamma$ and $\Omega = \widehat{\mathbb{Q}}$.

2.4 Modular Group Γ and Action on $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$

The quotient group of the SL $(2, \mathbb{Z})$ group with $\{\mp I\}$ is called the Modular group. Especially, if Modular group is denoted with Γ,

$$\Gamma = PSL\,(2, \mathbb{Z}) \cong SL\,(2, \mathbb{Z})\,/\{\mp I\}$$

is obtained. Beside, Γ is composed of matrix pairs with determinant 1. Here, the difference between + and $\breve{}$ is ignored and considered equal. Γ has got a wide range of congruence subgroups. In this study, especially $\Gamma_0\,(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \pmod{N} \right\}$ will be studied. $\mathbb{H} = \{z \in \mathbb{C} : Imz > 0\}$ Represents hyperbolic plane. Möbius transforms are known transforms in the upper half plane $\mathbb{H}$ with the elements of the Modular group Γ. The Möbius transforms for $\forall z \in \mathbb{C}$ is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z = \frac{az+b}{cz+d}$, $T \in \Gamma$. It is important to note that elements of the Modular group Γ are used to describe the action on $\widehat{\mathbb{Q}}$. $\frac{x}{y} \in \widehat{\mathbb{Q}}$ is reduced rational number for $x, y \in \mathbb{Z}$ and $(x, y) = 1$. Since $\frac{x}{y} = \frac{-x}{-y}$, representation is not unique. If $y = 0$, it is accepted as $\frac{x}{0} = \infty$ for $\frac{x}{y} \in \widehat{\mathbb{Q}}$. Moreover, ∞ is accepted as $\frac{1}{0} = \frac{-1}{0}$. Beside, $(ax + by, cx + dy) = 1$ is obtained for the representation of $\frac{x}{y} \in \widehat{\mathbb{Q}}$ under the $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ where $x, y \in \mathbb{Z}$ and $(x, y) = 1$. Different from the identity and universal relation, the action of the submodular group $\Gamma_0\,(N)$ on $\widehat{\mathbb{Q}}$ and the congruence relation $\approx_N$ can be shown as follows:

Assume that $\frac{r}{s} = g\,(\infty)$ and $\frac{x}{y} = g'\,(\infty)$ is provided for $\frac{r}{s}, \frac{x}{y} \in \widehat{\mathbb{Q}}$ and $g = \begin{pmatrix} r & * \\ s & * \end{pmatrix}, g' = \begin{pmatrix} x & * \\ y & * \end{pmatrix} \in \Gamma$.

$g^{-1}g' = \begin{pmatrix} * & -s \\ * & r \end{pmatrix} \begin{pmatrix} x & * \\ y & * \end{pmatrix} = \begin{pmatrix} * & * \\ ry - sx & * \end{pmatrix} \in \Gamma_0\,(N)$ is obtained from congruence relation $\frac{r}{s} \approx_N \frac{x}{y} \Leftrightarrow g^{-1}g' \in \Gamma_0\,(N)$. Thus, congruence relation $\frac{r}{s} \approx_N \frac{x}{y} \Leftrightarrow ry - sx \equiv 0 \pmod{N}$ is attained.

2.5 Suborbital Graphs $G_{u,N}$ and $F_{u,N}$ and Minimal Length Paths

Since the submodular group Γ acts transitively on $\widehat{\mathbb{Q}}$, each orbital covers pair (∞, w). Especially, if $w = \frac{u}{N}$ is taken for $N \geq 0$ and $(u, N) = 1$, suborbital is denoted by $O_{u,N}$ and suborbital graph is denoted by $G_{u,N}$. The theorem that gives the necessary condition to be an edge between two vertices in the suborbital graph $G_{u,N}$ is given below.

Theorem 2.1. *[17]* $\frac{r}{s} \rightarrow \frac{x}{y} \in G_{u,N} \Leftrightarrow$

i. $x \equiv ur \pmod N, y \equiv us \pmod N$ *and* $ry - sx = N$

ii. $x \equiv -ur \pmod N, y \equiv -us \pmod N$ *and* $ry - sx = -N$.

The suborbital graph $G_{u,N}$ consists of discrete unions of the suborbital graph $F_{u,N}$. The vertices of these discrete subgraphs form a single block according to the invariant congruence relation $\approx_N$. The vertices of the suborbital graph $F_{u,N}$ contain ∞ and consist of the block $[\infty] = \left\{\frac{x}{y} \in \widehat{\mathbb{Q}} \mid y \equiv 0 \pmod N\right\}$. The theorem giving the edge condition between two vertices on the suborbital graph $F_{u,N}$ is given below.

Theorem 2.2. *[17]* $\frac{r}{s} \rightarrow \frac{x}{y} \in F_{u,N} \Leftrightarrow$

i. $x \equiv ur \pmod N, y \equiv us \pmod N$ *and* $ry - sx = N$

ii. $x \equiv -ur \pmod N, y \equiv -us \pmod N$ *and* $ry - sx = -N$.

In this section, some informations are given related to minimal length. In the graph $F_{u,N}$, each vertex corresponds to a rational number in reduced forms. Given two such vertices $\frac{r}{s}$ and $\frac{x}{y}$, therefore $\frac{x}{y}$ is the farthest vertex from $\frac{r}{s}$ in the increasing direction, denoted $\frac{r}{s} \overset{\rightarrow}{<} \frac{x}{y} \in F_{u,N}$, if there exists no other vertex greater than $\frac{x}{y}$ that is adjacent to $\frac{r}{s}$ in the graph. Similarly, $\frac{x}{y}$ is defined as the nearest vertex in the decreasing direction, denoted $\frac{r}{s} \overset{\leftarrow}{>} \frac{x}{y} \in F_{u,N}$, if there exists no smaller vertex not connected to $\frac{r}{s}$. These notations of farthest and nearest vertices are used to describe directional extremality among the adjacent vertices of a given vertex in $F_{u,N}$. Building on this concept, a path $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_m$ in $F_{u,N}$ is said to be minimal length if it is satisfies two key conditions. First, the path must not contain shortcuts; that is, for any pair of indices $i < j - 1$, where $i \in \{0, 1, \ldots, m-2\}$ and $j \in \{2, 3, \ldots, m\}$, the vertices v_i and v_j are not directly connected by an edge in the graph. Second, each consecutive vertex v_{i+1} along the path must be selected as the farthest vertex adjacent to v_i, in accordance with the previously defined directional condition. Thus, this process ensures that the path progress in a maximally outward direction at each step, avoiding unnecessary detours and resulting in the shortest possible sequence under the given constraints. For further details, see [18].

2.6 Farey Graph, Diagram and Sequence

If $N = 1$ in the suborbital graph $F_{u,N}$, the Farey graph is obtained. The Farey diagram is obtained from the Farey graph. In the Farey diagram, there are curvilinear triangles that approach to $\widehat{\mathbb{Q}}$ in increasing numbers. A section of the Farey diagram arranged around a circle is given below:

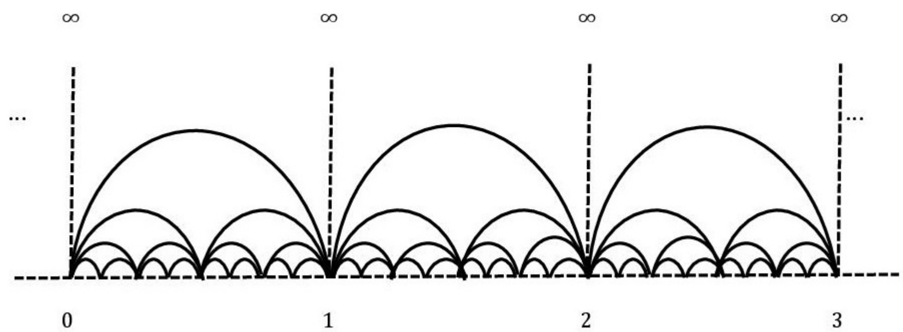

Figure 1: Farey diagram.

The middle vertex between these two vertices is obtained using the mediant, where $\frac{r}{s}$ and $\frac{x}{y}$ are the two vertices on $\widehat{\mathbb{Q}}$. Subsequently, mediant is derived as $\frac{r+x}{s+y}$. By the mediant, the Farey sequence is reached as follows. For $\frac{r}{s}$ and $\frac{x}{y}$ are two vertices, Farey sequence $\left\{\frac{r}{s}, \cdots, \frac{3r+x}{3s+y}, \frac{2r+x}{2s+y}, \frac{r+x}{s+y}, \frac{r+2x}{s+2y}, \frac{r+3x}{s+3y}, \cdots, \frac{x}{y}\right\}$ is reached. Let the Farey sequence consisting of n elements is denoted by $\mathscr{F}_n$ where $n \geq 1$. The elements of the Farey sequence $\mathscr{F}_n$ consist of rational numbers $\frac{x}{y} \in \widehat{\mathbb{Q}}$ such that $(x, y) = 1$. There is a ordering $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \mathscr{F}_3 \subset \cdots$ among the subsequences of the Farey sequence and $\bigcup_{n \geq 1} \mathscr{F}_n = \widehat{\mathbb{Q}}$ is attained. Edge conditions are satisfied between the elements of the Farey sequence $\mathscr{F}_n$, as in the suborbital graphs $G_{u,N}$ and $F_{u,N}$. Let the Farey graph created by the elements of the Farey sequence $\mathscr{F}_n$ be symbolized as $\mathbf{F}$.

Lemma 2.3. *[17] Assume that $\frac{r}{s}$ and $\frac{x}{y}$ are reduced rational numbers in Farey graph $\boldsymbol{F}$. Subsequently, three conditions in the following are equivalent to each other in Farey graph $\boldsymbol{F}$.*

i. $\frac{r}{s}$ and $\frac{x}{y}$ are neighbour vertices in Farey graph $\boldsymbol{F}$.

ii. ry-sx=$\pm$ 1;

iii. $\frac{r}{s}$ and $\frac{x}{y}$ are neighbour vertices in Farey sequence $\mathscr{F}_n$ for $n \in \mathbb{N}$.

2.7 Hasse Diagram

Since partial orders are a binary relationship, they can be represented by a directed graph. However, for a partial ordered relation must satisfy the reflection, antisymmetry and transitivity properties, many edges can be neglected in obtaining the Hasse diagram of a graph. Here, the vertices in the graph are ordered from down to up. For example, let $\backslash$: division relation and $V = \{1, 2, 3, 4, 6\}$. Let's draw the Hasse diagram of poset $(V, \backslash)$. Thus, relation $\{(1,1), (1,2), (1,3), (1,4), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (6,6)\}$ is attained.

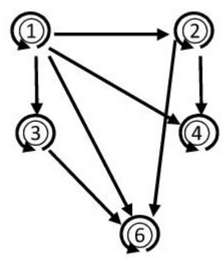

Figure 2: Relation graph.

The following steps are applied to obtain the Hasse diagram from the relation graph:

1. Step: Remove all self-loops.
2. Step: Remove all edges that provide the transitivity.
3. Step: Arrange so that all edges points upwards.
4. Step: Remove redirects on edges.

In Step 1, by removing the elements $(1,1), (2,2), (3,3), (4,4), (6,6)$ from the relation, the relation remains as $\{(1,2), (1,3), (1,4), (1,6), (2,4), (2,6), (3,6)\}$. In step 2, the elements $(1,4)$ and $(1,6)$ that provide the transitivity property of the relation are removed and the relation remains as $\{(1,2), (1,3), (2,4), (2,6), (3,6)\}$. By step 3 and step 4, relation graph can be represented as follows.

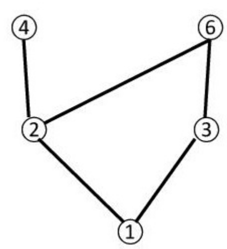

Figure 3: the Hasse diagram of partial ordered $(V, \backslash)$.

In this study, by using the literature defined above, we aim to show that relation connecting the vertices and edges of a suborbital graph defined in a non-Euclidean space is a partial ordered relation. Later, we aimed to demonstrate that graphs $G_{u,N}$, $F_{u,N}$ and Farey graphs $\mathbf{F}$ are lattice. Especially, the essential element graph of the Farey graph $\mathbf{F}$ was investigated and the sequences obtained by the numbers of vertices and edges were given.

3 Main Results

We expand on the lattice structures obtained from suborbital graphs. Proofs of Lemma 3.5, Lemma 3.7, and Theorem 3.15 are now presented in full detail, including intermediate steps. We also provide explicit illustrative examples for small Farey graphs to illustrate the essential element graphs and Hasse diagrams. Tables of integers sequences for vertices and edges are clarified. While general closed forms are not always derivable, we emphasize that the presented sequences follow from observed patterns and conjectural formulas.

3.1 Posets and Lattices Obtained by Set of Vertices of Suborbital Graphs

Example 3.1. *Let V is the set of vertices in the suborbital graph and the relation $\leq$:less than or equal to. For $V = \{v_1, v_2, \cdots, v_n\}$ and $i, j, m \in \{1, 2, \cdots, n\}$, it is provided that reflection for $v_i \leq v_i$, antisymmetry $v_i = v_j$ for $v_i \leq v_j$ and $v_j \leq v_i$ and transitivity $v_i \leq v_m$ for $v_i \leq v_j$ and $v_j \leq v_m$. Consequently, $(V, \leq)$ is a poset.*

Example 3.2. *Assume V is the set of vertices in the suborbital graph and the relation $\leq$: less than or equal to. For $V = \{v_1, v_2, \cdots, v_n\}$ and $i, j \in \{1, 2, \cdots, n\}$, $v_i \vee v_j = \sup\{v_i, v_j\} = \max\{v_i, v_j\} \in V$,$v_i \wedge v_j = \inf\{v_i, v_j\} = \min\{v_i, v_j\} \in V$ are satisfied. Then, $(V, \leq)$ is a lattice.*

Example 3.3. *Let $P(V)$ is the power set of the vertices in the suborbital graph and $\subseteq$: coverage relation. For $P(V) = \{V_1, V_2, \cdots, V_n\}$ and $i, j, m \in \{1, 2, \cdots, n\}$, it is provided that reflection for $V_i \subseteq V_i$, antisymmetry $V_i = V_j$ for $V_i \subseteq V_j$ and $V_j \subseteq V_i$ and transitivity $V_i \subseteq V_m$ for $V_i \subseteq V_j$ and $V_j \subseteq V_m$. Then, $(P(V), \subseteq)$ is a poset.*

Example 3.4. *Let $P(V)$ is the power set of the vertices in the suborbital graph and $\subseteq$: coverage relation. For $P(V) = \{V_1, V_2, \cdots, V_n\}$ and $i, j \in \{1, 2, \cdots, n\}$, $V_i \vee V_j = \sup\{V_i, V_j\} = V_i \cup V_j \in V$,$V_i \wedge V_j = \inf\{V_i, V_j\} = V_i \cap V_j \in V$ are provided. Thus, $(P(V), \subseteq)$ is a lattice.*

At the moment, unlike the $\leq$: less than or equal and $\subseteq$: coverage relations, the relation

$$\triangle\colon ry - sx = \begin{cases} 0 & \frac{r}{s} = \frac{x}{y} \\ 1 & \frac{r}{s} \neq \frac{x}{y} \end{cases}$$

is defined by the transformation used to define the action between vertices in suborbital graphs, where $\frac{r}{s} \to \frac{x}{y}$ is an edge in suborbital graph for $\frac{r}{s}, \frac{r}{s} \in V$. Beside, the poset and lattice $(V, \triangle)$ are examined.
In the suborbital graphs, the action is either right-directional or left-directional. If action is right direction, $ry - sx = -1$ otherwise is left direction, $ry - sx = 1$. In this relation, the distance between two vertices is taken into account, regardless of direction.

Lemma 3.5. *Assume V is a set of vertices of Farey graph $\boldsymbol{F}$ and $\triangle$ is a partial ordered relation. For $\frac{r}{s}, \frac{x}{y}, \frac{a}{b} \in V$ and $\frac{a+r}{x} = \frac{b+s}{y} = 1$, $(V, \triangle)$ is a poset.*

Proof. We aim to show that $(V, \triangle)$ satisfy the properties of a partial order: reflexivity, antisymmetry, and transitivity.
First, $\frac{r}{s} \triangle \frac{r}{s} = rs - rs = 0 \in \triangle$ is provided for $\frac{r}{s} \in V$. Therefore, $\frac{r}{s} \triangle \frac{r}{s}$ holds, and reflexivity is satisfied.
Second, we now proceed to verify that the condition of antisymmetry is fulfilled. If it is taken as $\frac{r}{s} \neq \frac{x}{y}$ for $\frac{r}{s}, \frac{x}{y} \in V$, $\frac{r}{s} \triangle \frac{x}{y}$ or $\frac{x}{y} \triangle \frac{r}{s}$ are achieved. Next, by definition of $\triangle$, $ry - sx = 1$ and $sx - ry = 1$. Adding both equations yields $0 \neq 2$, which is a contradiction. Hence, both relations cannot hold simultaneously unless $\frac{r}{s} = \frac{x}{y}$. Thus, antisymmetry holds.
Third, it remains to verify that the transitivity condition holds. Let $\frac{r}{s} \triangle \frac{x}{y}$ and $\frac{x}{y} \triangle \frac{a}{b}$ for $\frac{r}{s}, \frac{x}{y}, \frac{a}{b} \in V$ and $\frac{r}{s} \neq \frac{x}{y} \neq \frac{a}{b}$. Later, by $\triangle$, $ry - sx = 1$ for $\frac{r}{s} \triangle \frac{x}{y}$ and $xb - ay = 1$ for $\frac{x}{y} \triangle \frac{a}{b}$. Then, $ry - sx = xb - ay$ is reached from the equations $ry - sx = 1$ and $xb - ay = 1$. After that, $\frac{x}{y} = \frac{a+r}{s+b}$ is derived for $(x, y) = 1$ via fundamental elementary operations. Subsequently, $(a + r, s + b) = 1$. Multiplying the equation $ry - sx = 1$ by a and the equation $xb - ay = 1$ by r, $a(ry - sx) = a$ and $r(xb - ay) = r$ are reached, respectively. If the equations are added side by side, $x\,(rb - as) = a + r$ is obtained. Therefore, $rb - as = \frac{a+r}{x} = 1$. Consequently, $rb - as = \frac{r}{s} \triangle \frac{a}{b}$ and transitivity is provided. □

Lemma 3.6. *Assume V is a set of vertices of Farey graph $\boldsymbol{F}$ and $\triangle$ is a partial ordered relation. Clearly, $(V, \triangle)$ is a lattice.*

Proof. Assume that $V = \{v_1, v_2, \cdots, v_n\}$ for $i, j \in \{1, 2, \cdots, n\}$ and $(x, y) = (z, t) = 1$ for $v_i = \frac{x}{y} \in V, v_j = \frac{z}{t} \in V$. By Example 3.2, $v_i \vee v_j = \sup\{v_i, v_j\} = \max\{v_i, v_j\} \in V$ is provided. Consequently, $(V, \triangle)$ is a lattice. □

Here, in addition to relation $\triangle$, relation $\triangle_N$ can be given for graphs $G_{u,N}$ and $F_{u,N}$. Note that, relation $\triangle_N$ is used to define action between vertices of these graphs. In this section, $\triangle_N$ is considered as binary relation. Then, the poset and lattice $(V, \triangle_N)$ are investigated.

$$\triangle_N\colon ry - sx = \begin{cases} 0 & \frac{r}{s} = \frac{x}{y} \\ N & \frac{r}{s} \neq \frac{x}{y} \end{cases}$$

Lemma 3.7. *Let V is a set of vertices of suborbital graph $G_{u,N}$ and $\triangle_N$ is a partial ordered relation. For $\frac{r}{s}, \frac{x}{y}, \frac{a}{b} \in V$ and $\frac{a+r}{x} = \frac{b+s}{y} = 1$,$(V, \triangle_N)$ is a poset, where $\triangle_N$ is a relation.*

Proof. By Lemma 3.5, it is clear that reflection and antisymmetry are satisfied for $(x, y) = 1$.
Let us show that the relation $\triangle_N$ satisfies the transitivity property. To this end, assume that $\frac{r}{s} \triangle_N \frac{x}{y}$ and $\frac{x}{y} \triangle_N \frac{a}{b}$ for $\frac{r}{s}, \frac{x}{y}, \frac{a}{b} \in V$ and $\frac{r}{s} \neq \frac{x}{y} \neq \frac{a}{b}$. Initially, $ry - sx = N$ is obtained for $\frac{r}{s} \triangle_N \frac{x}{y}$ and $xb - ay = N$ is obtained for $\frac{x}{y} \triangle_N \frac{a}{b}$. Moreover, $ry - sx = xb - ay$ is reached by equations $ry - sx = N$ and $xb - ay = N$. If the equation $ry - sx = xb - ay$ is rearranged, $\frac{x}{y} = \frac{a+r}{s+b}$ is attained for $(x, y) = 1$. Hence, $(a + r, s + b) = 1$ is achieved. Multiplying the equation $ry - sx = N$ by a and the equation $xb - ay = N$ by r, $ary - asx = aN$ and $rxb - ray = rN$ are obtained, respectively. If the two equations are added together, the result yields $x(rb - as) = N(a + r)$ is obtained. Since $\frac{a+r}{x}$ is assumed, it follows that $rb - as = \frac{r}{s} \triangle_N \frac{a}{b}$. Therefore, transitivity is satisfied. □

Theorem 3.8. *Let V is a set of vertices of suborbital graph $G_{u,N}$ and $\triangle_N$ is a partial ordered relation. $(V, \triangle_N)$ is a lattice.*

Proof. Let $V = \{v_1, v_2, \cdots, v_n\}$ for $i, j \in \{1, 2, \cdots, n\}$ and $(x, y) = (z, t) = 1$ for $v_i = \frac{x}{y}, v_j = \frac{z}{t} \in V$. By Example 3.2, $v_i \vee v_j = \sup\{v_i, v_j\} = \max\{v_i, v_j\} \in V$ is attained. Therefore, $(V, \triangle_N)$ is a lattice. □

Since the suborbital graph $G_{u,N}$ consists of disjoint combinations of the suborbital graph $F_{u,N}$, the same results are reached for the suborbital graph $F_{u,N}$.

Lemma 3.9. *Let V is a set of vertices of suborbital graph $F_{u,N}$ and $\triangle_N$ is a partial ordered relation. For $\frac{r}{s}, \frac{x}{y}, \frac{a}{b} \in V$ and $\frac{a+r}{x} = \frac{b+s}{y} = 1$, $(V, \triangle_N)$ is a poset.*

Proof. The proof is similar to that of Lemma 3.7. □

Theorem 3.10. *Let V is a set of vertices of suborbital graph $F_{u,N}$ and $\triangle_N$ is a partial ordered relation. $(V, \triangle_N)$ is a lattice.*

Proof. The proof is similar to that of Theorem 3.8. □

3.2 Investigation the Essential Element Graph $\epsilon_{\mathbf{F}}$ of the Farey Graph F

We extend the discussion of essential element graphs of the Farey lattice by comparing our construction with recent research on zero-divisor graphs and essential element graphs [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. This comparison highlights novelty in our framework and situate the contribution within the broader literature.

The essential element graph of the Farey graph $\mathbf{F}$ is symbolized with $\epsilon_{\mathbf{F}}$. An example of the Farey diagram, which consists of an increasing number of curvilinear circles with decreasing diameter on the $\widehat{\mathbb{Q}}$, is given in Figure 1. The Farey diagram given in Figure 1 is a discrete combination of subgraphs. The graph given in Figure 4 is a subgraph of the Farey graph.

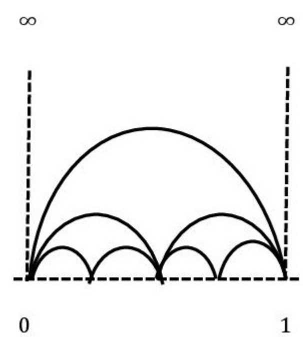

Figure 4: Suborbital graph of Farey graph.

Farey essential element graph $\epsilon_{\mathbf{F}}$ is a simple undirected graph consisting of a set of vertices $\widehat{\mathbb{Q}} - \{0, 1\}$. At that time, some of the Farey graphs $\mathbf{F}_n$ and essential element graphs $\epsilon_{\mathbf{F}_n}$ can be visualized as follows.
For $n = 1$,

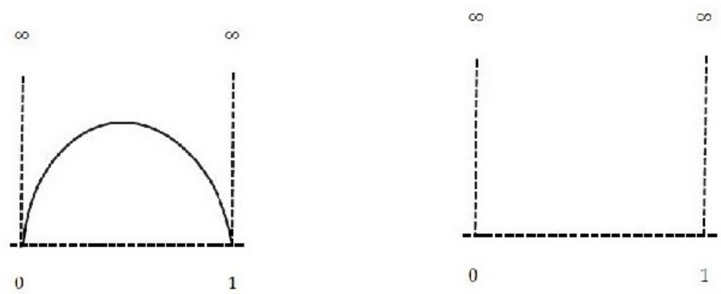

Figure 5: Farey graph $\mathbf{F}_1$ and Farey essential element graph $\epsilon_{\mathbf{F}_1}$.

For $n = 2$.

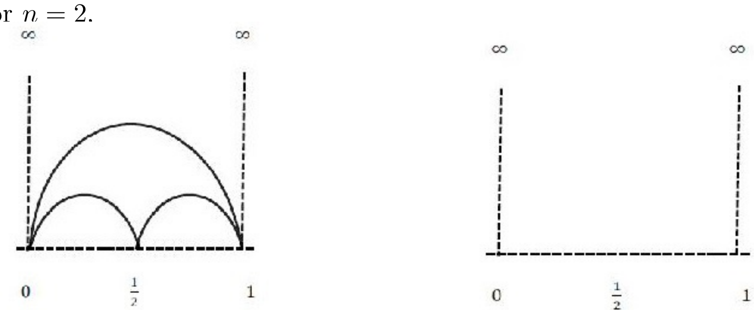

Figure 6: Farey graph $\mathbf{F}_2$ and Farey essential element graph $\epsilon_{\mathbf{F}_2}$.

For $n = 3$.

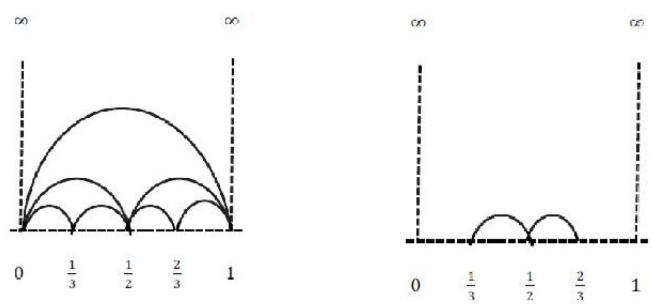

Figure 7: Farey graph $\mathbf{F}_3$ and Farey essential element graph $\epsilon_{\mathbf{F}_3}$.

Moreover, related to sequences of constituted of the vertices and edges of the Farey graph $\mathbf{F}$ and the Farey essential element graph $\epsilon_{\mathbf{F}}$ are given in the table below:

No. of $\mathbf{F}_n$	$E(\mathbf{F}_n)$	$V(\mathbf{F}_n)$	$E(\epsilon_{\mathbf{F}_n})$	$V(\epsilon_{\mathbf{F}_n})$
$\mathbf{F}_1$	1	2	0	0
$\mathbf{F}_2$	3	3	0	0
$\mathbf{F}_3$	7	5	2	3
$\mathbf{F}_4$	15	9	8	7
$\mathbf{F}_5$	31	17	22	15
$\mathbf{F}_6$	63	33	52	31
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{F}_n$	$2n+1$	$2^{n-1}+1$	$\frac{n(n^2-1)}{3}$	$2n+1$

Table 1: Sequences of constitued of the vertices and edges of the Farey graph $\mathbf{F}_n$ and the Farey essential element graph $\epsilon_{\mathbf{F}_n}$.

For all $n \geqslant 1$, it is clear that the general terms of the sequences $E(\mathbf{F}_n)$ and $V(\mathbf{F}_n)$ are $2n+1$ and $2^{n-1}+1$, respectively. Similarly, for $n \geqslant 3$, it is evident that $2n+1$ is the general term of the sequence $V(\epsilon_{\mathbf{F}_n})$. Note that the derivation of these general terms is conjectural.

The general term of the sequence $E(\epsilon_{\mathbf{F}_n})$ is now considered. The initial six terms of the sequence are given $x_0 = 0, x_1 = 0, x_2 = 2, x_3 = 8, x_4 = 22, x_5 = 52$. The general term of the sequence can be written as a polynomial in terms of $x_n = \alpha n^3 + \beta n^2 + \gamma n + \delta$. The first four terms of the sequence can be given as polynomials as follows:

$$x_0 = \alpha 0^3 + \beta 0^2 + \gamma 0 + \delta = \delta = 0$$
$$x_1 = \alpha 1^3 + \beta 1^2 + \gamma 1 + \delta = \alpha + \beta + \gamma = 0$$
$$x_2 = \alpha 2^3 + \beta 2^2 + \gamma 2 + \delta = 8\alpha + 4\beta + 2\gamma = 2$$
$$x_3 = \alpha 3^3 + \beta 3^2 + \gamma 3 + \delta = 27\alpha + 9\beta + 3\gamma = 8$$

Substituting $\gamma = -\alpha - \beta$ into the equation for x_2 yields value is $3\alpha + \beta = 1$. Similarly, substituting the value of $\gamma = -\alpha - \beta$ into x_3 yields $4\alpha + \beta = \frac{4}{3}$. $\alpha = \frac{1}{3}$ is derived from solving these equations. Furthermore, $\beta = 0$ is obtained. Hence, it is clear that the result is $\gamma = \frac{-1}{3}$. It is evident that substituting the determined coefficients into the general term of the polynomial results in $x_n = \frac{1}{3}n(n^2-1)$.

In this section, some results related to Farey graph $\mathbf{F}_n$ for $n = 2$ are presented and symbolized with $\mathbf{F}_2$.

Theorem 3.11. *Assume that $\mathbf{F}_2$ is Farey lattice with single atom and bounded. Therefore, upper bound of Farey lattice $\mathbf{F}_2$ is an essential element.*

Proof. Farey lattice $\mathbf{F}_2$ is an bounded lattice by 0 from the bottom and 1 from the top. For $v = \frac{1}{2}$, $0 \leqslant \frac{1}{2}$ is provided. It is clear that vertex $\frac{1}{2}$ covers vertex 0. Consequently, $\frac{1}{2}$ is unique atom. Moreover, since vertex $\frac{1}{2}$ is an element in $V(\mathbf{F}_2) - \{0, 1\}$, $\frac{1}{2}$ is a proper element. To essential element, if $\frac{1}{2} \wedge 1 \neq 0$ for $\frac{1}{2} \neq 0 \in \mathbf{F}_2$, 1 is an essential element in Farey lattice $\mathbf{F}_2$. □

Theorem 3.12. *Farey graph $\boldsymbol{F}_2$ is a complete graph whenever Farey lattice $\boldsymbol{F}_2$ has an unique atom.*

Proof. In Theorem 3.11, we show that the Farey lattice $\mathbf{F}_2$ has a single atom. Assume that $E(\mathbf{F}_2)$ is a set of vertices of Farey graph $\mathbf{F}_2$ for $\forall v_i, v_j \in \mathbf{F}_2$. Since $v_i v_j \in E(\mathbf{F}_2)$ for $\forall v_i, v_j \in \mathbf{F}_2$, Farey lattice $\mathbf{F}_2$ is a complete graph. □

Theorem 3.13. *Assume that Farey lattice $\boldsymbol{F}_2$ has a unique a coatom. Consequently, $\frac{1}{2}$ is a proper element.*

Proof. Let Farey lattice $\mathbf{F}_2$ is an bounded lattice by 0 from the bottom and 1 from the top. For $v = \frac{1}{2}$, since $\frac{1}{2} \leqslant 1$ is provided, vertex $\frac{1}{2}$ is covered by vertex 1. Vertex $\frac{1}{2}$ is an unique coatom as well. Moreover, $1 \wedge \frac{1}{2} \neq 0$ is satisfied for $1 \neq 0 \in \mathbf{F}_2$. Thus, $\frac{1}{2}$ is a proper element in Farey lattice $\mathbf{F}_2$. □

3.3 Investigation of Hasse Diagram of Farey Graph

Hasse diagram orders vertices in a graph according to the relevant partial ordered relation. At that time, Farey graph $\mathbf{F}$ and Farey graph Hasse diagrams $\mathfrak{F}_n$ are demonstrated as follows:
Step 1: Remove all self-loops. Here, since group action is transitivity, $(v_i, v_j) \in \mathbf{F}$ is a directed edge from v_i to v_j, where vertices v_i and v_j on $\widehat{\mathbb{Q}}$.
Step 2: Remove all the edges that provide the transitivity. The number of hyperbolic geodesics increases whenever number of vertex n increases in interval $[0, 1]$. The hyperbolic geodesics with vertices n are provided transitive for hyperbolic geodesics with vertices $n + 1$.
Step 3: Arrange so that all edges points upwards. Since the hyperbolic geodesics are obtained on hyperbolic plane, action is accepted from 0 to 1 to up.
Step 4: Remove redirects on edges.
The edges of the Farey graph $\mathbf{F}$ is taken as undirected. The edges representing the transition between Figures 9 and 10 are shown in red.
For $n = 1$,

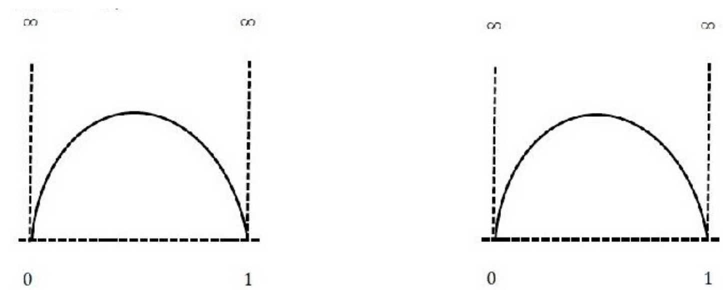

Figure 8: Farey graph $\mathbf{F}_1$ and Farey graph Hasse diagram $\mathfrak{F}_1$.

For $n = 2$,

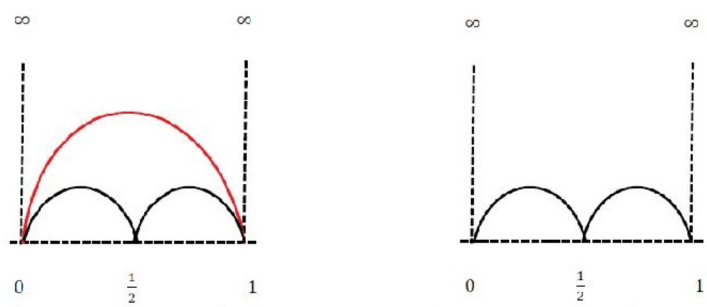

Figure 9: Farey graph $\mathbf{F}_2$ and Farey graph Hasse diagram $\mathfrak{F}_2$.

For $n = 3$,

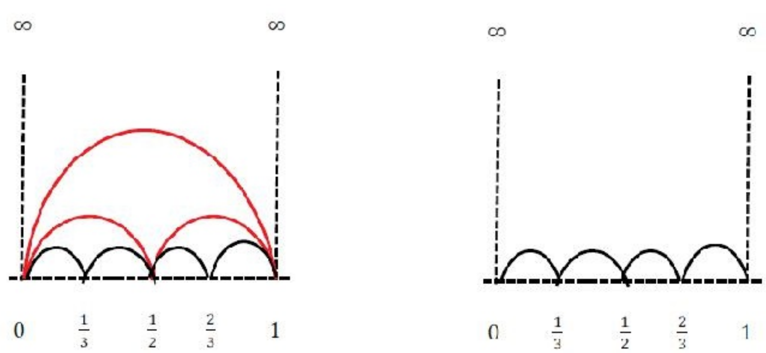

Figure 10: Farey graph $\mathbf{F}_3$ and Farey graph Hasse diagram $\mathfrak{F}_3$.

From here, following sequences are achieved by number of vertices and edges of Hasse diagram of Farey graph.

No. of $\mathfrak{F}_n$	$V(\mathfrak{F}_n)$	$E(\mathfrak{F}_n)$
$\mathfrak{F}_1$	2	1
$\mathfrak{F}_2$	3	2
$\mathfrak{F}_3$	5	4
$\mathfrak{F}_4$	9	8
$\mathfrak{F}_5$	17	16
$\vdots$	$\vdots$	$\vdots$
$\mathfrak{F}_n$	$2^{n-1}+1$	2^{n-1}

Table 2: Sequences of constitued of the vertices and edges of the Farey graph Hasse diagram $\mathfrak{F}_n$.

It follows that, with respect to the variable n, the general terms of the sequences $V(\mathfrak{F}_n)$ and $E(\mathfrak{F}_n)$ are given by $2^{n-1}+1$ and 2^{n-1}, respectively.

3.4 On the Lattice Property of the Farey Graph Under the Minimal Length Condition

In this section, we present the lemma establishing that the Farey graph satisfies the minimal length condition, followed by the theorem demonstrating that the associated graph forms a lattice structure.

Lemma 3.14. *Let $\frac{r}{s}$ and $\frac{x}{y}$ be two reduced rational numbers that are adjacent in the Farey graph $\boldsymbol{F}$ satisfying $|ry-sx|=1$. Accordingly, $\frac{x}{y}$ is unique neighbour of $\frac{r}{s}$ in the direction corresponding to the edge connecting to the edge connecting them in the Farey graph. In particular, there exist no other neighbour of $\frac{r}{s}$ that lies along the same geodesic arc in the hyperbolic plane as the one connecting $\frac{r}{s}$ and $\frac{x}{y}$.*

Proof. The Farey graph $\mathbf{F}$ is the graph between two rational numbers whose vertices are the set of reduced rational numbers in the interval $[0,1]$ and the edges only satisfy the Farey neighbourhood. In a Farey graph, the neighbourhood between two vertices is given in Lemma 2.3. The Farey graph forms an acyclic and connected structure, therefore, a tree. The basic characteristic of trees in graphs is not that only one path is formed in between two vertices. Since this path is structurally unique, this path provides the shortest path condition between two vertices. □

Theorem 3.15. *Assume that $\boldsymbol{F}$ is a Farey graph satisfying the minimal length condition, that is each edge corresponds to a unique geodesic arc in the hyperbolic plane, and each pair of adjacent vertices $\frac{r}{s}$ and $\frac{x}{y}$ satisfies $|ry - sx|= 1$. Then the graph $\boldsymbol{F}$ admits a lattice structure.*

Proof. In Lemma 3.14, the minimal length condition is demonstrated for the edges in a Farey graph. Therefore, Farey graph **F** satisfies the minimal length condition, namely that each edge corresponds to a unique geodesic in the hyperbolic plane and adjacent vertices $\frac{r}{s}, \frac{x}{y} \in \mathbf{F}$ satisfy $|ry - sx|= 1$. By Lemma 3.5, it follows that the set of vertices of Farey graph's is a poset according to relation $\triangle$. We aim to establish that **F**, equipped with the partial order $\triangle$, admits a lattice structure. The poset structure arises naturally from the Farey tree, a rooted binary tree in which each vertex corresponds to a reduced rational number and each vertex is obtained via mediant $\frac{r+x}{s+y}$, which lies strictly between its parent vertices on the real line. This recursive generation ensures that every pair of vertices in **F** has a unique least common ancestor (LCA) in a tree, which we identify as their greatest lower upper bound (join) is given explicitly by their Farey sum, which, by construction, lies in **F** and serves as a minimal common successor in the ordering. For non-adjacent pairs, successive applications of the mediant along minimal paths within the graph yield a common upper bound that also respects the combinatorial and geometric structure of **F**. As both meet and joint operations are closed in **F** and defined for all vertex pairs, it follows that the Farey graph, under this ordering, constitutes a lattice. □

4 Conclusion

This paper strengthens the theoretical foundation by expanding proof, consolidating definitions, and adding illustrative examples. We demonstrated that Farey graphs admit lattice structure under minimal length conditions and analysed their essential element graphs. Applications in number theory, hyperbolic geometry, and network science are suggested as follows. In the context of number theory, Fuchsian groups and lattices derived from these groups allow the reduced rational numbers in the Farey graph to be arranged more regularly. Thus, the geometric and topological studies of modular forms and automorphic forms can be more understandable. The Farey graph is related to ideal triangular tilings in the hyperbolic plane. Hence, in the framework of hyperbolic geometry and tilings, the lattice structure defined on a Farey graph can make these tilings appear more symmetric, repetitive, and locally regular. Moreover, by extending the Farey lattice to more general geometric structures, it can contribute to the construction of Riemann surfaces and the modular parametrization of small-type surfaces such as spheres, tori, and surfaces with two holes. From a network science perspective, since many networks can be embedded within a hyperbolic space, defining a lattice structure for a Farey graph in hyperbolic space can provide stronger scaling and navigation for these types of networks. Further work may include computational algorithms for detecting lattice properties in large graphs and exploring connections with algebraic graph theory.

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