DOI: 10.2478/auom-2026-0005
An. Şt. Univ. Ovidius Constanţa **Vol. 34(1),2026, 101–115**

Gevrey Regularity and Local Well-Posedness for the HirotaSatsuma System

Feriel Boudersa, Abdelaziz Mennouni and Ravi P. Agarwal

Abstract

In this paper, we examine the local well-posedness of the initial value problem for the HirotaSatsuma system within Gevrey spaces. This system, which consists of a coupled nonlinear dispersive partial differential equation, models the interactions between long and short waves and is known for its integrable structure. We demonstrate that the problem is locally well-posed in the Gevrey spaces $G_{\eta,\delta,k}(\mathbb{R}) \times G_{\eta,\delta,k+1}(\mathbb{R})$ for $k > -\frac{1}{8}$, $\delta > 0$, and $\eta \geq 1$. This finding improves upon existing well-posedness results in Sobolev spaces $H^k(\mathbb{R}) \times H^{k+1}(\mathbb{R})$. Our approach involves a meticulous analysis of linear and bilinear estimates within Gevrey classes. By utilizing Fourier analytical techniques and reformulating the system into an integral equation through Duhamels principle, we establish the necessary bounds for applying a fixed-point argument. This process yields results regarding existence, uniqueness, and continuous dependence on initial data. Furthermore, we show that the solution demonstrates Gevrey-3η regularity in time, capturing the smoothing properties of the system. These results deepen our understanding of analytic-type regularity in nonlinear dispersive systems.

1 Introduction

The HirotaSatsuma system is a coupled, nonlinear, dispersive partial differential equation that arises in the study of shallow water waves and soliton

Key Words: Gevrey spaces, well-posedness, coupled system, HirotaSatsuma System, regularity.
2010 Mathematics Subject Classification: Primary 35K15, 35K55; Secondary 35K65, 35B40.
Received:24.06.2025.
Accepted: 30.09.2025.

interactions. Initially introduced by Hirota and Satsuma in 1976, this system models the interactions between long and short waves and has attracted significant mathematical interest due to its rich integrable structure and soliton solutions. Analyzing this system contributes to a deeper understanding of nonlinear wave phenomena and integrable systems.

A central question in the theory of nonlinear PDEs is the well-posedness of the associated initial value problem (IVP). This encompasses establishing the existence, uniqueness, and continuous dependence on initial data of solutions within an appropriate functional framework. While well-posedness results for the HirotaSatsuma system have been explored in Sobolev and analytic spaces, much less is known in intermediate function spaces such as Gevrey classes. Gevrey spaces serve as a bridge between real-analytic and Sobolev spaces, offering a refined scale for investigating the regularity of solutions.

In this paper, we consider the initial value problem of Hirota-Satsuma system

$$\begin{cases} \partial_t u - \alpha(u_{xxx} + 6uu_x) = 2\beta w w_x, \\ \partial_t w + w_{xxx} + 3uw_x = 0, \\ u(x,0) = u_0(x) \ \ , \ \ w(x,0) = w_0(x), \end{cases} \tag{1}$$

where $u = u(x,t)$ and $w = w(x,t)$ are the unknown function, $x \in \mathbb{R}$ and $t \in \mathbb{R}$, while u_0 and w_0 are a given functions. Moreover, α and β are real constants.

In [1] the author proved that (1) is locally well posed in the spaces $H^k(\mathbb{R}) \times H^{k+1}(\mathbb{R})$ with $k > -\frac{1}{8}$. This works establishs the local well-posedness of the HirotaSatsuma initial value problem (IVP) in Gevrey spaces. Our analysis employs techniques from Fourier analysis, energy estimates, and the abstract theory of evolution equations in Banach spaces. The results provide insights into the regularizing effects of the nonlinearities and the persistence of Gevrey regularity over a short time period.

The work is structured as follows: In Section 2, we introduce the fundamental functional spaces and preliminary lemmas that form the foundation of our analysis. In Section 3, we derive the necessary linear and bilinear estimates. By applying Duhamel's principletaking the Fourier transform in the spatial variable x for system (1), solving the resulting ODE in t, and then applying the inverse Fourier transform we reduce the initial value problem to an equivalent integral formulation. In Section 4, we address the local well-posedness of the system. We present key lemmas and establish the main result concerning the existence, uniqueness, and continuity of solutions in Gevrey spaces. In Section 5, we further analyze the regularity properties of the solution and show that it retains Gevrey regularity with respect to the time variable t.

2 Notation and function spaces

In this section, we will present the elementary spaces and lemmas used in this paper.

It is now necessary to recall a definition of the needed spaces, where the analytic Gevrey spaces are given by $G^{\eta,\delta,k}(\mathbb{R}) = G^{\eta,\delta,k}$ for $k \in \mathbb{R}$, $\delta > 0$ and $\eta \geqslant 1$, defined by.

$$G^{\eta,\delta,k}(\mathbb{R}) = \{f \in L^2(\mathbb{R}); \|f\|^2_{G^{\eta,\delta,k}} = \int\limits_{\xi\in\mathbb{R}} e^{2\delta|\xi|^{1/\eta}} \langle\xi\rangle^{2k} |\widehat{f}(\xi)|^2 < \infty\}.$$

For $\eta = 1$, the space $G^{\eta,\delta,k}(\mathbb{R})$ reduce to the space $G^{\delta,k}(\mathbb{R})$, defined by

$$G^{\delta,k}(\mathbb{R}) = \{f \in L^2(\mathbb{R}); \|f\|^2_{G^{\delta,k}} = \int\limits_{\xi\in\mathbb{R}} e^{2\delta|\xi|} \langle\xi\rangle^{2k} |\widehat{f}(\xi)|^2 d\xi < \infty\}.$$

In the particular case where $\delta = 0$, the space $G^{0,k}(\mathbb{R})$ is reduced to the Sobolev space $H^k(\mathbb{R})$, defined by

$$H^k(\mathbb{R}) = \{f \in L^2(\mathbb{R}); \|f\|^2_{H^k}, = \int\limits_{\xi\in\mathbb{R}} \langle\xi\rangle^{2k} |\widehat{f}(\xi)|^2 d\xi < \infty\}.$$

With $\langle.\rangle = (1 + |.|)$. At a time, for $k, a \in \mathbb{R}$, $\delta > 0$ and $\eta \geq 1$ the analytic Gevrey-Bourgain space $X_{\eta,\delta,k,a}(\mathbb{R}^2) = X_{\eta,\delta,k,a}$ associated with $\partial_t \pm \alpha\partial_x^3$ is defined to be the closure of the Schwartz space $\mathscr{S}(\mathbb{R}^2)$ under the norm:

$$\|u\|_{X_{\eta,\delta,k,a}} = \Big(\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} e^{2\delta|\xi|^{1/\eta}} \langle\xi\rangle^{2k} \langle\tau \mp \alpha\xi^3\rangle^{2a} |\widehat{u}(\xi,\tau)|^2 d\tau d\xi\Big)^{1/2}.$$

The definition of the analytic Bourgain space $X_{k,a}$ is given in [1], where $\widehat{u}(\xi,\tau)$ denote as Fourier transformation of u with respect to x and t.

For any interval $I \subset \mathbb{R}$, we define the localized spaces $X^I_{\eta,\delta,k,a} = X_{\eta,\delta,k,a}(\mathbb{R} \times I)$ with the norms

$$\|u\|_{X^I_{\eta,\delta,k,a}} = \inf\Big\{\|U\|_{X_{\eta,\delta,k,a}};\ \ U\,|_{(\mathbb{R}\times I)} = u\Big\}$$

Lemma 2.1. *[2] Let $a > \frac{1}{2}$, $k \in \mathbb{R}$, $\delta > 0$ and $\eta \geq 1$ then we have $X_{\eta,\delta,k,a} \hookrightarrow C([-T,T], G^{\eta,\delta,k}(\mathbb{R}))$ where T is any posite constant, i.e. for all $u \in X_{\eta,\delta,k,a}$ and some canstant $c_0 > 0$ we have*

$$\sup_{t\in[-T,T]} \|u(.,t)\|_{G^{\eta,\delta,k}(\mathbb{R})} \leqslant c_0 \|u\|_{X_{\eta,\delta,k,a}}.$$

3 Linear and bilinear estimates

By using Duhamels formula (Taking Fourier transform with respect to x in (1), solving the resulting differential equation in t and using inverse Fourier transform reduces the initial value problem of Hirota-Satsuma system (1) to the following integral system, thus, we may write the solution as

$$\begin{cases} u(x,t) = S(t)u_0(x) - \displaystyle\int_0^t S(t-t')F(u,w,u_x,w_x)dt', \\ w(x,t) = S(t)w_0(x) - \displaystyle\int_0^t S(t-t')G(u,w,u_x,w_x)dt', \end{cases} \tag{2}$$

where $S(t)$ is the unitary group on $G^{\eta,\delta,k}(\mathbb{R})$ associated on the linear part of (1) given by

$$S(t)u_0 = \int_{-\infty}^{+\infty} e^{i(x\xi \mp t\xi^3)}\widehat{u_0}(\xi)d\xi,$$

and

$$\begin{cases} F(u,w,u_x,w_x) = 6\alpha uu_x - 2\beta ww_x, \\ G(u,w,u_x,w_x) = 3uw_x. \end{cases} \tag{3}$$

To prove the local well-posedness, let us consider the equivalent system of integral equations. Using a cut-off function satisfying $\theta \in C_0^\infty(\mathbb{R})$, with $\theta = 1$ in $[-\frac{1}{2}, \frac{1}{2}]$, supp$\theta \subset [-1,1]$. Denote $\theta_\lambda(t) = \theta(\frac{t}{\lambda})$ for $0 < \lambda \leqslant 1$.
Let us now define $(\Theta_1(u,w), \Theta_2(u,w))$ where the maps Θ_1 and Θ_2 are given by

$$\begin{cases} \Theta_1(u,w)(t) = \theta_1(t)S(t)u_0(x) - \theta_1 \displaystyle\int_0^t S(t-t')\theta_\lambda(t)F(t')dt', \\ \Theta_2(u,w)(t) = \theta_1(t)S(t)w_0(x) - \theta_1 \displaystyle\int_0^t S(t-t')\theta_\lambda(t)G(t')dt'. \end{cases} \tag{4}$$

Next, we estimate the first part of $(\Theta_1(u,w), \Theta_2(u,w))$ in the right-hand side of (4) and the integral part.

Lemma 3.1. *Let $k \in \mathbb{R}$, if $\frac{1}{2} < a < a' \leq 1$, there is a constant $c > 0$, such that for any $\lambda \in (0,1]$ we have the estimates (see [1])*

$$\|\theta_1(t)S(t)u_0(x)\|_{X_{k,a}} \leq c\|u_0\|_{H^k},$$

and

$$\|\theta_1(t)S(t)w_0(x)\|_{X_{k+1,a}} \leq c\|w_0\|_{H^{k+1}}$$

for $(u_0, w_0) \in H^k \times H^{k+1}$. *Furthermore, for* $(F, G) \in X_{k,a'-1} \times X_{k+1,a'-1}$ *there exists* $c > 0$ *such that (see [1])*

$$\|\theta_1 \int_0^t S(t-t')\theta_\lambda(t)F(t')dt'\|_{X_{k,a}} \leq c\|\theta_\lambda(t)F\|_{X_{k,a-1}},$$
$$\|\theta_\lambda(t)F\|_{X_{k,a-1}} \leq c\lambda^{a'-a}\|F\|_{X_{k,a'-1}},$$

and

$$\|\theta_1 \int_0^t S(t-t')\theta_\lambda(t)G(t')dt'\|_{X_{k+1,a}} \leq c\|\theta_\lambda(t)G\|_{X_{k+1,a-1}},$$
$$\|\theta_\lambda(t)G\|_{X_{k+1,a-1}} \leq c\lambda^{a'-a}\|G\|_{X_{k+1,a'-1}}.$$

Lemma 3.2. *Let* $k \in \mathbb{R}$, $\delta > 0$, $\eta \geq 1$ *if* $\frac{1}{2} < a < a' \leq 1$, *there is a constant* $c > 0$, *such that for any* $\lambda \in (0, 1]$ *we have the estimates*

$$\|\theta_1(t)S(t)u_0(x)\|_{X_{\eta,\delta,k,a}} \leq c\|u_0\|_{G^{\eta,\delta,k}},$$

and

$$\|\theta_1(t)S(t)w_0(x)\|_{X_{\eta,\delta,k+1,a}} \leq c\|w_0\|_{G^{\eta,\delta,k+1}}$$

for $(u_0, w_0) \in G^{\eta,\delta,k} \times G^{\eta,\delta,k+1}$.

Furthermore, for $(F, G) \in X_{\eta,\delta,k,a'-1} \times X_{\eta,\delta,k+1,a'-1}$ *there exists* $c > 0$ *such that*

$$\|\theta_1 \int_0^t S(t-t')\theta_\lambda(t)F(t')dt'\|_{X_{\eta,\delta,k,a}} \leq c\|\theta_\lambda(t)F\|_{X_{\eta,\delta,k,a-1}},$$
$$\|\theta_\lambda(t)F\|_{X_{\eta,\delta,k,a-1}} \leq c\lambda^{a'-a}\|F\|_{X_{\eta,\delta,k,a'-1}},$$

and

$$\|\theta_1 \int_0^t S(t-t')\theta_\lambda(t)G(t')dt'\|_{X_{\eta,\delta,k+1,a}} \leq c\|\theta_\lambda(t)G\|_{X_{\eta,\delta,k+1,a-1}},$$
$$\|\theta_\lambda(t)G\|_{X_{\eta,\delta,k+1,a-1}} \leq c\lambda^{a'-a}\|G\|_{X_{\eta,\delta,k+1,a'-1}}.$$

Proof. As in [4, 3, 2], the desired result is achieved by using Lemma (3.1) and employing the operator $A^{\delta,\eta}$, which is defined by

$$\widehat{A^{\delta,\eta}u}(\xi) = e^{\delta|\xi|^{\frac{1}{\eta}}}\widehat{u}(\xi)$$

satisfies

$$\|A^{\delta,\eta}u\|_{X_{k,a}} = \|u\|_{X_{\eta,\delta,k,a}},$$

and

$$\|A^{\delta,\eta}u\|_{H^k} = \|u\|_{G^{\eta,\delta,k}}.$$

□

Now, we studing the nonlinear part of the system (1). More specifically, we obtain the following result.

Proposition 1. *[1] If $k > -\frac{1}{8}$ and $\frac{1}{2} < a' < \frac{9}{16}$, then $\forall a > \frac{1}{2}$ we have*

$$\|(\partial_x u_1)u_2\|_{X_{k+1,a'-1}} \leq \|u_1\|_{X_{k+1,a}}\|u_2\|_{X_{k,a}}$$

and

$$\|(\partial_x u_1 u_2)\|_{X_{k,a'-1}} \leq \|u_1\|_{X_{k,a}}\|u_2\|_{X_{k,a}}$$

Proposition 2. *If $k > -\frac{1}{8}$, $\frac{1}{2} < a' < \frac{9}{16}$, $\eta \geq 1$ and $\delta > 0$, then $\forall a > \frac{1}{2}$ we have*

$$\|(\partial_x u_1)u_2\|_{X_{\eta,\delta,k+1,a'-1}} \leq \|u_1\|_{X_{\eta,\delta,k+1,a}}\|u_2\|_{X_{\eta,\delta,k,a}}$$

and

$$\|(\partial_x u_1 u_2)\|_{X_{\eta,\delta,k,a'-1}} \leq \|u_1\|_{X_{\eta,\delta,k,a}}\|u_2\|_{X_{\eta,\delta,k,a}}$$

Proof. In a similar manner to [3, 2], we apply proposition 1 and the operator $A^{\delta,\eta}$ to demonstrate proposition 2.

□

We are now ready to present the theorem regarding the well-posedness of System (1).

Theorem 3.3. *Let $k > -\frac{1}{8}$, $\delta > 0$ and $\eta \geq 1$. Then for any initial data $(u_0, w_0) \in G^{\eta,\delta,k}(\mathbb{R})\times G^{\eta,\delta,k+1}(\mathbb{R})$, there exists $T = T(\|(u_0, w_0)\|_{G^{\eta,\delta,k}(\mathbb{R})\times G^{\eta,\delta,k+1}(\mathbb{R})})$, and a unique solution (u, w) of (1) on the time interval, as follows:*

$$(u, w) \in C([0, T], G^{\eta,\delta,k}(\mathbb{R})) \times C([0, T], G^{\eta,\delta,k+1}(\mathbb{R})).$$

4 Local Well-Posedness and Proof

The following lemmas will be used to prove our main Theorem. In the subsequent lemma, we will define the ball B_ω and show that the map $\Theta_1 \times \Theta_2$ is a contraction from B_ω to B_ω

Lemma 4.1. *Let $k > -\frac{1}{8}$, $\delta > 0$, $\eta \geq 1$, $\frac{1}{2} < a < a' \leq \frac{9}{16}$. Then, for all $(u_0, w_0) \in G^{\eta,\delta,k} \times G^{\eta,\delta,k+1}$ and $\|(u_0, w_0)\|_{G^{\eta,\delta,k} \times G^{\eta,\delta,k+1}} \equiv \|u_0\|_{G^{\eta,\delta,k}} + \|w_0\|_{G^{\eta,\delta,k+1}} = \omega$, the map $\Theta_1 \times \Theta_2 : B_\omega \to B_\omega$ is a contraction, where B_ω is given by*

$$B_\omega := \{(u, w) \in X_{\eta,\delta,k,a} \times X_{\eta,\delta,k+1,a} \quad : \quad \|(u, w)\|_{X_{\eta,\delta,k,a} \times X_{\eta,\delta,k+1,a}} \leq 2c\omega\}.$$

We note that B_ω is Banach space, whose norm is

$$\|(u, w)\|_{X_{\eta,\delta,k,a} \times X_{\eta,\delta,k+1,a}} = \|u\|_{X_{\eta,\delta,k,a}} + \|w\|_{X_{\eta,\delta,k+1,a}}.$$

Proof. By using linear estimates in Lemma 3.2 and bilinear estimates in Proposition 2 we obtain

$$\begin{aligned}
\|\Theta_1(u, w)\|_{X_{\eta,\delta,k,a}} &\leq \|\theta_1(t)S(t)u_0(x)\|_{X_{\eta,\delta,k,a}} \\
&\quad + \|\theta_1 \int_0^t S(t-t')\theta_\lambda(t)[6\alpha u u_x - 2\beta w w_x](t')dt'\|_{X_{\eta,\delta,k,a}} \\
&\leq c\|u_0\|_{G^{\eta,\delta,k}} + c\|\theta_\lambda(t) u u_x\|_{X_{\eta,\delta,k,a-1}} + c\|\theta_\lambda(t) w w_x\|_{X_{\eta,\delta,k,a-1}} \\
&\leq c\|w_0\|_{G^{\eta,\delta,k}} + c\lambda^{a'-a}\|u u_x\|_{X_{\eta,\delta,k,a'-1}} + c\lambda^{a'-a}\|w w_x\|_{X_{\eta,\delta,k,a'-1}} \\
&\leq c\|u_0\|_{G^{\eta,\delta,k}} + c\lambda^{a'-a}\|u\|^2_{X_{\eta,\delta,k,a}} + c\lambda^{a'-a}\|w\|^2_{X_{\eta,\delta,k,a}} \\
&\leq c\|u_0\|_{G^{\eta,\delta,k}} + c\lambda^{a'-a}\|u\|^2_{X_{\eta,\delta,k,a}} + c\lambda^{a'-a}\|w\|^2_{X_{\eta,\delta,k+1,a}}, \qquad (5)
\end{aligned}$$

and

$$\begin{aligned}
\|\Theta_2(u, w)\|_{X_{\eta,\delta,k+1,a}} &\leq \|\theta_1(t)S(t)w_0(x)\|_{X_{\eta,\delta,k+1,a}} \\
&\quad + \|\theta_1(t) \int_0^t S(t-t')\theta_\lambda(t)[3u w_x](t')dt'\|_{X_{\eta,\delta,k+1,a}} \\
&\leq c\|w_0\|_{G^{\eta,\delta,k+1}} + c\|\theta_\lambda(t) u w_x\|_{X_{\eta,\delta,k+1,a-1}} \\
&\leq c\|w_0\|_{G^{\eta,\delta,k+1}} + c\lambda^{a'-a}\|u w_x\|_{X_{\eta,\delta,k+1,a'-1}} \\
&\leq c\|w_0\|_{G^{\eta,\delta,k+1}} + c\lambda^{a'-a}\|u\|_{X_{\eta,\delta,k,a}}\|w\|_{X_{\eta,\delta,k+1,a}} \\
&\leq c\|w_0\|_{G^{\eta,\delta,k+1}} + c\lambda^{a'-a}\|u\|^2_{X_{\eta,\delta,k,a}} + c\lambda^{a'-a}\|w\|^2_{X_{\eta,\delta,k+1,a}}. \qquad (6)
\end{aligned}$$

Thus, by the estimates (5) and (6), we have

$$\begin{aligned}\|[\Theta_1(u,w),\Theta_2(u,w)]\|_{X_{\eta,\delta,k,a}\times X_{\eta,\delta,k+1,a}} &\le c\|u_0\|_{G^{\eta,\delta,k}}+c\|w_0\|_{G^{\eta,\delta,k+1}}\\ &+c\lambda^{a'-a}\|u\|^2_{X_{\eta,\delta,k,a}}\\ &+c\lambda^{a'-a}\|w\|^2_{X_{\eta,\delta,k+1,a}}\\ &\le c\|(u,w)\|_{G^{\eta,\delta,k}\times G^{\eta,\delta,k+1}}\\ &+c\lambda^{a'-a}\|(u,w)\|^2_{X_{\eta,\delta,k,a}\times X_{\eta,\delta,k+1,a}}.\end{aligned}$$

Thus, when taking $\lambda < [(2c)^2\omega]^{\frac{1}{a-a'}}$, $\Theta_1\times\Theta_2$ mapping B_ω into B_ω.

We assume that $(u,w),(u^*,w^*)\in B_\omega$. Similar to (5) and (6), for λ determined above, we have

$$\begin{aligned}&\|(\Theta_1(u,w)-\Theta_1(u^*,w^*),\Theta_2(u,w)-\Theta_2(u^*,w^*))\|_{X_{\eta,\delta,k,a}\times X_{\eta,\delta,k+1,a}}\\ &\qquad< \frac{1}{2}\|(u-u^*,w-w^*\|_{X_{\eta,\delta,k,a}\times X_{\eta,\delta,k+1,a}}.\end{aligned}$$

Thus, $\Theta_1\times\Theta_2$ is contract mapping. Using the Banach fixed point theorem, there is exactly one fixed point because of $\Theta_1\times\Theta_2\in B_\omega$. □

5 Gevrey's Regularity

In this section, we will demonstrate in the following theorem that the solution to the initial value problem of Hirota-Satsuma system (1) exhibits Gevrey regularity with respect to the time variable t. Specifically, we aim to establish the estimates given in Equations (7) and (8), as outlined in Proposition 3. To prove these estimates, we will employ mathematical induction on l. Additionally, when estimating nonlinear terms, we will utilize Leibniz's formula.

Theorem 5.1. *Let $k>-\frac{1}{8}$, $\delta>0$, $\eta\ge 1$ and $(u,w)\in C([0,T],G^{\eta,\delta,k}(\mathbb{R})\times C([0,T],G^{\eta,\delta,k+1}(\mathbb{R}))$ be the solution of (1). Then, $(u,w)\in G^{3\eta}([0,T])\times G^{3\eta}([0,T])$ in the time variable t.*

For the proof of the Theorem 5.1, it is enough to prove the following results:

Proposition 3. *Let $l,m\in\{0,1,2,\cdots\}$, for some $L>0$. Thus, we have*

$$|\partial_t^l\partial_x^m u(x,t)|\le L^{l+m+1}((m+3l)!)^\eta M^l, \tag{7}$$

and

$$|\partial_t^l\partial_x^m w(x,t)|\le L^{l+m+1}((m+3l)!)^\eta M^l, \tag{8}$$

where

$$M:=|\alpha|L^2+\frac{3L}{2^\eta}+\frac{|\beta|L}{2^\eta}.$$

Proof. To prove Proposition 3, we will use induction on l. For the case when $l=0$, Inequalities (7) and (8) follow from the result below (see, [2, 5, 6, 7]).

$$|\partial_x^m u(x,t)| \leq L^{m+1}(m!)^\eta, \tag{9}$$

and

$$|\partial_x^m w(x,t)| \leq L^{m+1}(m!)^\eta, \tag{10}$$

Thus, (u,w) is a $G^\eta \times G^\eta$ in x for for all $k > -\frac{1}{8}$. For proof of this result, see [2].

For $l=1$ and $m \in \{0,1,2,\cdots\}$, we have

$$\Big|\partial_t \partial_x^m u\Big| \leqslant |\alpha|\Big|\partial_x^{m+3}u\Big| + 3\Big|\partial_x^{m+1}u^2\Big| + |\beta|\Big|\partial_x^{m+1}w^2\Big|. \tag{11}$$

The terms of (11) can be estimated as

$$\begin{aligned} |\alpha|\Big|\partial_x^{m+3}u\Big| &\leqslant |\alpha| L^{m+3+1}\Big((m+3)!\Big)^\eta \\ &\leqslant L^{m+1+1}\Big((m+3.1)!\Big)^\eta |\alpha| L^2. \end{aligned} \tag{12}$$

For the second term in (11), we have

$$\begin{aligned} 3\Big|\partial_x^{m+1}u^2\Big| &\leqslant 3\sum_{p=0}^{m+1}\binom{m+1}{p}\Big|\partial_x^{m+1-p}u\Big|\Big|\partial_x^p u\Big| \\ &\leqslant 3\sum_{p=0}^{m+1}\frac{(m+1)!}{((m+1-p)!)(p!)}L^{m+1-p+1}((m+1-p)!)^\eta L^{p+1}(p!)^\eta \\ &\leqslant 3\sum_{p=0}^{m+1}\frac{((m+1)!)^\eta}{((m+1-p)!)^\eta(p!)^\eta}L^{m+1-p+1}((m+1-p)!)^\eta L^{p+1}((p+1)!)^\eta \\ &\leqslant 3((m+1)!)^\eta L^{m+3}\sum_{p=0}^{m+1}(p+1)^\eta. \end{aligned}$$

The fact

$$\sum_{p=0}^{m+1}(p+1) = \frac{(m+2)(m+3)}{2}$$

gives

$$\begin{aligned} 3\Big|\partial_x^{m+1}u^2\Big| &\leqslant 3((m+1)!)^\eta L^{m+3}\frac{(m+2)^\eta(m+3)^\eta}{2^\eta} \\ &\leqslant L^{m+1+1}\Big((m+3.1)!\Big)^\eta \frac{3L}{2^\eta}. \end{aligned} \tag{13}$$

A similar calculation for the last term in (11)

$$|\beta|\Big|\partial_x^{m+1}w^2\Big| \leqslant L^{m+1+1}\Big((m+3.1)!\Big)^{\eta}\frac{|\beta|L}{2^{\eta}}. \tag{14}$$

From (12), (13) and (14) we obtain

$$\Big|\partial_t\partial_x^m u\Big| \leqslant L^{m+1+1}\Big((m+3.1)!\Big)^{\eta}M^1$$

where

$$M = |\alpha|L^2 + \frac{3L}{2^{\eta}} + \frac{|\beta|L}{2^{\eta}}$$

We assume that (7) is correct for l and then we prove for $(l+1)$ we obtain

$$\Big|\partial_t^{l+1}\partial_x^m u\Big| \leqslant |\alpha|\Big|\partial_t^l\partial_x^{m+3}u\Big| + 3\Big|\partial_t^l\partial_x^{m+1}u^2\Big| + |\beta|\Big|\partial_t^l\partial_x^{m+1}w^2\Big|. \tag{15}$$

These terms are estimated as follows:

$$\begin{aligned}|\alpha|\Big|\partial_t^l\partial_x^{m+3}u\Big| &\leqslant |\alpha|L^{l+m+2+1}((m+3+3l)!)^{\eta}M^l\\ &\leqslant L^{l+m+1}((m+3(l+1))!)^{\eta}M^l|\alpha|L^2. \end{aligned}\tag{16}$$

For the second term of (15), we have

$$\begin{aligned}3\Big|\partial_t^l\partial_x^{m+1}u^2\Big| &\leqslant 3\sum_{j=0}^{l}\sum_{p=0}^{m+1}\binom{l}{j}\binom{m+1}{p}\Big|\partial_t^{l-j}\partial_x^{m+1-p}u\Big|\Big|\partial_t^j\partial_x^p u\Big|\\ &\leqslant 3\sum_{j=0}^{l}\sum_{p=0}^{m+1}\binom{l+m+1}{j+p}\\ &\quad L^{l-j+m+1-p+1}\Big((m+1-p+3.(l-j))!\Big)^{\eta}M^{l-j}L^{j+p+1}\Big((p+3.j)!\Big)^{\eta}M^j\\ &\leqslant L^{m+(l+1)+1}\Big((m+3.(l+1))!\Big)^{\eta}M^l\frac{3L}{2^{\eta}}. \end{aligned}\tag{17}$$

Using the same calculus for the third term of equation (15), we find that

$$|\beta|\Big|\partial_t^l\partial_x^{m+1}w^2 \leqslant L^{m+(l+1)+1}\Big((m+3.(l+1))!\Big)^{\eta}M^l\frac{|\beta|L}{2^{\eta}}. \tag{18}$$

Finally, by combining (16), (17) and (18), we deduce that

$$\Big|\partial_t^{l+1}\partial_x^m u\Big| \leqslant L^{m+(l+1)+1}\Big((m+3.(l+1))!\Big)^{\eta}M^{l+1}.$$

Now, we prove (8), for $l = 1$ and $m \in \{0, 1, 2, \cdots\}$, we have

$$\Big|\partial_t \partial_x^m w(x,t)\Big| \leqslant \Big|\partial_x^{m+3} w\Big| + 3\Big|\partial_x^m (u\partial_x w)\Big| \tag{19}$$

For the first term in (19) we have

$$\begin{aligned}\Big|\partial_x^{m+3} w\Big| &\leqslant L^{m+3+1}\Big((m+3)!\Big)^{\eta} \\ &\leqslant L^{m+1+1}\Big((m+3.1)!\Big)^{\eta} L^2.\end{aligned} \tag{20}$$

for the second term we have

$$\begin{aligned}3\Big|\partial_x^m (u\partial_x w)\Big| &\leqslant 3\sum_{p=0}^{m}\binom{m}{p}\Big|\partial_x^{m-p} u\Big|\Big|\partial_x^{p+1} w\Big| \\ &\leqslant 3\sum_{p=0}^{m}\frac{(m)!}{((m-p)!)(p!)} L^{m-p+1}((m-p)!)^{\eta} L^{p+2}((p+1)!)^{\eta} \\ &\leqslant 3\sum_{p=0}^{m}\frac{((m)!)^{\eta}}{((m-p)!)^{\eta}(p!)^{\eta}} L^{m-p+1}((m-p)!)^{\eta} L^{p+2}((p+1)!)^{\eta} \\ &\leqslant 3((m!)^{\eta} L^{m+3}\sum_{p=0}^{m}(p+1)^{\eta}.\end{aligned}$$

Now, we use the fact that

$$\sum_{p=0}^{m}(p+1) = \frac{(m+1)(m+2)}{2}$$

we obtain

$$\begin{aligned}3\Big|\partial_x^m (u\partial_x w)\Big| &\leqslant 3L^{m+3}((m!)^{\eta}\frac{(m+1)^{\eta}(m+2)^{\eta}}{2^{\eta}} \\ &\leqslant L^{m+1+1}((m+3.1)!)^{\eta}\frac{3L}{2^{\eta}(m+3)^{\eta}} \\ &\leqslant L^{m+1+1}((m+3.1)!)^{\eta}\frac{3L}{6^{\eta}}.\end{aligned} \tag{21}$$

Then from (20) and (21) we obtain

$$\Big|\partial_t \partial_x^m w(x,t)\Big| \leqslant L^{m+1+1}((m+3.1)!)^{\eta} N^1,$$

where

$$N = L^2 + \frac{3L}{6^\eta}$$

We assume that 8 is correct for l and then we prove it for $(l+1)$ we have

$$\left|\partial_t^{l+1}\partial_x^m w(x,t)\right| \leqslant \left|\partial_t^l\partial_x^{m+3} w\right| + 3\left|\partial_t^l\partial_x^m (u\partial_x w)\right|. \tag{22}$$

These terms are estimated as follows:

$$\begin{aligned}\left|\partial_t^l\partial_x^{m+3} w\right| &\leqslant L^{m+3+l+1}((m+3+3.l)!)^\eta N^l \\ &\leqslant L^{m+(l+1)+1}((m+3.(l+1))!)^\eta N^l L^2. \end{aligned} \tag{23}$$

and

$$\begin{aligned}3\left|\partial_t^l\partial_x^m (u\partial_x w)\right| &\leqslant 3\sum_{j=0}^{l}\sum_{p=0}^{m}\binom{l}{j}\binom{m}{p}\left|\partial_t^{l-j}\partial_x^{m-p}u\right|\left|\partial_t^{j}\partial_x^{p+1}w\right| \\ &\leqslant 3\sum_{j=0}^{l}\sum_{p=0}^{m+1}\binom{l+m}{j+p} \\ &L^{l-j+m-p+1}\Big((m-p+3.(l-j))!\Big)^\eta N^{l-j}L^{j+p+2}\Big((p+3.j)!\Big)^\eta N^j \\ &\leqslant L^{m+(l+1)+1}\Big((m+3.(l+1))!\Big)^\eta N^l \frac{3L}{6^\eta}. \end{aligned} \tag{24}$$

Finally, by combining (23) and (24), we deduce that

$$\left|\partial_t^{l+1}\partial_x^m w(\right| \leqslant L^{m+(l+1)+1}\Big((m+3.(l+1))!\Big)^\eta N^{l+1}$$

In the end, we choose

$$\max(M,N) = M$$

then we obtain

$$\left|\partial_t^l\partial_x^m u\right| \leqslant L^{m+l+1}\Big((m+3.l)!\Big)^\eta M^l,$$

and

$$\left|\partial_t^l\partial_x^m w\right| \leqslant L^{m+l+1}\Big((m+3.l)!\Big)^\eta M^l,$$

Consequently,

$$(u,w) \in G^{3\eta}([0,T]) \times G^{3\eta}([0,T])$$

. $\square$

6 Conclusion

In this work, we investigate the initial value problem for the HirotaSatsuma system within the framework of Gevrey spaces. Specifically, we proved local well-posedness in $G_{\eta,\delta,k}(\mathbb{R}) \times G_{\eta,\delta,k+1}(\mathbb{R})$ with $k > -\frac{1}{8}$, $\delta > 0$ and $\eta \geq 1$, thus refining and extending previous results obtained in the Sobolev setting. This establishes the persistence of Gevrey's regularity for solutions over short time intervals, highlighting the system's smoothing behavior.

The methodology involved applying Duhamels principle to convert the original PDE system into an equivalent integral formulation, followed by precise linear and bilinear estimates in the Gevrey class. These estimates were crucial for establishing a contraction mapping in an appropriate function space, which in turn led to the existence, uniqueness, and continuous dependence of solutions. Additionally, we demonstrated Gevrey-3η regularity in the time variable, which provides further insight into the regularizing nature of the system. This study suggests several interesting directions for future research. One natural extension is to investigate the global well-posedness in Gevrey spaces and determine whether the solutions retain Gevrey regularity for all time. Another promising direction is the study of the analytic continuation and decay properties of solutions in the complex domain. Moreover, exploring ill-posedness thresholds and sharp regularity results within the Gevrey scale could provide a complete picture of the systems behavior across different functional settings. It is important to further investigate the q-integro multipoint boundary value problems involving fractional q-difference equations with dual hybrid terms, as presented in [8]. Additionally, the study of integro-differential equations with non-instantaneous impulses, introduced in [9], offers valuable insight into complex dynamical behaviors. Continued exploration in these directions could significantly advance the theory and applications of fractional calculus and impulsive differential systems. Furthermore, we will investigate the Gevrey regularity of the problems presented in [10, 11, 12].

References

[1] X. Zhao & Z. Lv, Well-posedness of initial value problem of Hirota-Satsuma system in low regularity Sobolev space. AIMS Mathematics **(2022)**, Volume 7, Issue 4: 6702-6710. doi: 10.3934/math.2022374.

[2] J. Gorsky, A. A. Himonas, C. Holliman, G. Petronilho, The Cauchy problem of a periodic higher order KdV equation in analytic Gevrey spaces. J. Math. Anal. Appl., 40, 349361, **(2013)**.

[3] A. Boukarou & K. Guerbati, The Cauchy problem of a periodic Kawahara equation in analytic Gevrey spaces, **(2021)**, Issue: 2, 91 - 100, 30.10.2021. https://doi.org/10.47087/mjm.930045.

[4] A. A. Himonas, H. Kalisch & S. Selberg, On persistence of spatial analyticity for the dispersion-generalized periodic KdV equation. Nonlinear Analysis: Real World Applications, **2017**, *38*, 35-48.

[5] F. Boudersa, A. Mennouni & RP. Agarwal, Advancements in Gevrey Regularity for a Coupled KadomtsevPetviashvili II System: New Insights and Findings. Axioms. **2025**; 14(4):251. https://doi.org/10.3390/axioms14040251.

[6] F. Boudersa, A. Mennouni & RP. Agarwal, New Results on Gevrey Well-Posedness for the SchrdingerKortewegde Vries System. Math. Comput. Appl. **2025**, 30(3), 52; https://doi.org/10.3390/mca30030052.

[7] G. Menon, Gevrey class regularity for the attractor of the laser equations, Nonlinearity 12 **(1999)** 15051510. Printed in the UK

[8] B. Alharbi , A. Alsaedi, R.P. Agarwal, & Ahmad, B. Existence results for a nonlocal q-integro multipoint boundary value problem involving a fractional q-difference equation with dual hybrid terms. Analele Stiintifice ale Universitatii Ovidius Constanta: Seria Matematica, 32 (3) **2024**.

[9] A. Bensalem, A. Salim, M. Benchohra, & E. Karapinar, Existence and attractivity results on semi-infinite intervals for integrodifferential equations with non-instantaneous impulsions in Banach spaces. Analele Stiintifice ale Universitatii Ovidius Constanta: Seria Matematica, 32 (1) **(2024)**.

[10] T. Bentrcia & A. Mennouni, On the energy decay of a nonlinear time-fractional Euler-Bernoulli beam problem including time-delay: theoretical treatment and numerical solution techniques. J. Eng. Math. 145, (1), 21 **(2024)**.

[11] A. Mennouni & A. Youkana, Finite time blow-up of solutions for a nonlinear system of fractional differential equations. Electron. J. Differ. Equ. **2017**.

[12] S. Otsmane, A. Mennouni, New contributions to a complex system of quadratic heat equations with a generalized kernels: global solutions. Monatsh. Math., 1-20 **(2024)**.

Feriel Boudersa,
Department of Mathematics, LTM,
University of Batna 2, Mostefa Ben Boulaïd,
Fesdis, Batna 05078, Algeria.
Email: f.boudersa@univ-batna2.dz

Abdelaziz Mennouni,
Department of Mathematics, LTM,
University of Batna 2, Mostefa Ben Boulaïd,
Fesdis, Batna 05078, Algeria.
Email: a.mennouni@univ-batna2.dz

Ravi P. Agarwal,
Department of Mathematics and Systems Engineering,
Florida Institute of Technology,
Melbourne, FL 32901, USA,
Email: agarwalr@fit.edu