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Bipolar Fuzzy n-Fold Positive Implicative and Fantastic Filters in Hoop Algebras with Applications to Fuzzy Logic

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Abstract

In this paper, we introduce and study the notion of *bipolar fuzzy n-fold positive implicative filters* within the framework of hoop algebras, and examine their fundamental properties. We also define and investigate *bipolar fuzzy n-fold fantastic filters*, thereby extending bipolar fuzzy set theory into a multi-valued logical setting. In addition, we explore the relationships between bipolar fuzzy n-fold positive implicative filters and other related classes of bipolar fuzzy filters, including fuzzy n-fold implicative filters and bipolar fuzzy n-fold fantastic filters. This work offers a detailed analysis of the structural characteristics and interrelations among these filters, contributing to the broader development of non-classical algebraic logic and fuzzy systems. Several illustrative examples are provided to support and clarify the theoretical results.

Key Words: Hoop algebra, bipolar fuzzy n-fold positive implicative filter, bipolar fuzzy n-fold fantastic filter.
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1 Introduction

Hoop algebras, a class of commutative residuated structures, have emerged as a significant algebraic framework for modeling substructural logics, particularly those lacking structural rules such as contraction and weakening. Originally introduced by Bosbach in [1, 2] as naturally ordered, integral, commutative, and residuated monoids, hoop algebras provide a fertile ground for the algebraic study of many-valued and intuitionistic logics. Their structural resemblance to MV-algebras, BL-algebras, and other generalizations of Łukasiewicz logic has drawn substantial interest from researchers in both logic and algebra.

A fundamental aspect of hoop algebras is the study of filters, which correspond to deductively closed sets in logical systems and play a crucial role in analyzing inference rules such as modus ponens. Over time, various fuzzy extensions of filters have been developed to accommodate the vagueness and uncertainty present in real-world applications. Notably, the incorporation of fuzzy set theory has led to the introduction of several types of fuzzy filters, including implicative, positive implicative, and fantastic filters [3, 4, 5]. These concepts were later extended to their n-fold counterparts to capture iterative logical behavior, as explored by Luo, Xin, and He. Subsequent research, such as [6, 7], further broadened this framework to include fuzzy fantastic and fuzzy Boolean filters, offering deeper insights into their mutual relationships.

Among the more recent contributions, Borumand Saeid and Namdar introduced and analyzed fuzzy obstinate filters, fuzzy n-fold obstinate filters, and fuzzy n-fold implicative filters, establishing important connections with other filter types such as fuzzy prime and fuzzy positive implicative filters [8]. Their work also demonstrated that certain quotient structures induced by these filters form Boolean algebras, enriching the structural understanding of hoop algebras.

In recent years, the theory of fuzzy logic has increasingly emphasized bipolar fuzzy sets, which allow for the simultaneous representation of degrees of acceptance and rejection. This dual nature makes them particularly effective in modeling situations involving ambivalence or conflicting information. Motivated by this perspective, we introduce and study *bipolar fuzzy n-fold fantastic filters* on hoop algebras, extending the classical notion of fantastic filters into a more expressive bipolar and iterative framework.

This paper is organized as follows. In the first part, we define the concept of *bipolar fuzzy n-fold positive implicative filters* in hoop algebras and examine their fundamental properties. Building on this foundation, we introduce *bipolar fuzzy n-fold fantastic filters*, a new class designed to capture uncertainty by incorporating both positive and negative membership values. We then explore the structural properties of these filters and investigate their connections with other bipolar fuzzy filters, including bipolar fuzzy n-fold implicative filters and bipolar fuzzy positive implicative filters.

Our results show that bipolar fuzzy n-fold fantastic filters generalize their classical analogs and contribute to the ongoing effort to model logical reasoning under uncertainty. The proposed theoretical framework holds potential applications in areas such as logical inference, decision-making systems, and knowledge representation in bipolar fuzzy environments.

2 Preliminaries

Zhang [16] introduced bipolar fuzzy sets as an extension of traditional fuzzy sets, wherein each element can simultaneously possess degrees of positive and negative membership, enabling a more nuanced representation of information involving dual perspectives.

Definition 1. *[16] Let $\mathcal{W}$ be a non-empty set. A bipolar fuzzy set (abbreviated as BFS) $\mathcal{B}_{\mathcal{FS}}$ on $\mathcal{W}$ is defined as*

$$\mathcal{B}_{\mathcal{FS}} = \left\{(\mathfrak{w}, \varphi^-(\mathfrak{w}), \varphi^+(\mathfrak{w})) \mid \mathfrak{w} \in \mathcal{W}\right\},$$

where the mappings $\varphi^+ : \mathcal{W} \to [0,1]$ and $\varphi^- : \mathcal{W} \to [-1,0]$ denote the positive and negative membership functions, respectively.

The value $\varphi^+(\mathfrak{w})$ quantifies the extent to which the element $\mathfrak{w}$ satisfies a given property associated with the BFS $\mathcal{B}_{\mathcal{FS}}$, whereas $\varphi^-(\mathfrak{w})$ represents the degree to which $\mathfrak{w}$ satisfies a contrary or counter-property. In particular, if $\varphi^+(\mathfrak{w}) \neq 0$ and $\varphi^-(\mathfrak{w}) = 0$, then $\mathfrak{w}$ is considered to fully support the positive aspect of the property in question. On the other hand, if $\varphi^+(\mathfrak{w}) = 0$ and $\varphi^-(\mathfrak{w}) \neq 0$, then $\mathfrak{w}$ does not support the primary property but exhibits partial membership in the associated counter-property.

It may also happen that both functions vanish at a point, i.e., $\varphi^+(\mathfrak{w}) = \varphi^-(\mathfrak{w}) = 0$, indicating that $\mathfrak{w}$ does not contribute meaningfully to either the property or its opposite, suggesting neutrality or lack of information at that point.

For notational simplicity, we denote the bipolar fuzzy set $\mathcal{B}_{\mathcal{FS}}$ by the pair $\varphi = (\varphi^+, \varphi^-)$.

Definition 2. *[9] A hoop (or hoop algebra) is an algebraic structure denoted as $(\mathcal{H}_{\mathcal{A}}, \odot, \to, 1)$, where $(\mathcal{H}_{\mathcal{A}}, \odot, 1)$ forms a commutative monoid, and the following properties hold:*

(H1) $\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_x} = 1$,

(H2) $\hbar_{\mathfrak{a}_x} \odot (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) = \hbar_{\mathfrak{a}_y} \odot (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})$,

(H3) $\hbar_{\mathfrak{a}_x} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z}) = (\hbar_{\mathfrak{a}_x} \odot \hbar_{\mathfrak{a}_y}) \to z$ *for all* $\hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y}, \hbar_{\mathfrak{a}_z} \in \mathcal{H}_{\mathcal{A}}$.

Next, we define a relation $\leq_{\mathcal{H}_{\mathcal{A}}}$ on a hoop $\mathcal{H}_{\mathcal{A}}$ as follows:

$$(\forall \hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y} \in \mathcal{H}_{\mathcal{A}}) \left(\hbar_{\mathfrak{a}_x} \leq_{\mathcal{H}_{\mathcal{A}}} \hbar_{\mathfrak{a}_y} \Leftrightarrow \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y} = 1 \right). \tag{1}$$

It is straightforward to verify that $(\mathcal{H}_{\mathcal{A}}, \leq_{\mathcal{H}_{\mathcal{A}}})$ forms a partially ordered set.

Proposition 1. *[11] Let $(\mathcal{H}_{\mathcal{A}}, \odot, \to, 1)$ be a hoop algebra. For any $\hbar_{\mathfrak{a}}, \hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y}, \hbar_{\mathfrak{a}_z} \in \mathcal{H}_{\mathcal{A}}$, the following conditions hold:*

1. *$(\mathcal{H}_{\mathcal{A}}, \leq_{\mathcal{H}_{\mathcal{A}}})$ is a meet-semilattice with $\hbar_{\mathfrak{a}_x} \wedge \hbar_{\mathfrak{a}_y} = \hbar_{\mathfrak{a}_x} \odot (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})$,*
2. *$\hbar_{\mathfrak{a}_x} \leq_{\mathcal{H}_{\mathcal{A}}} (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}$, $\hbar_{\mathfrak{a}_x} \leq_{\mathcal{H}_{\mathcal{A}}} \hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}$, $\hbar_{\mathfrak{a}_x} \leq_{\mathcal{H}_{\mathcal{A}}} (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_x}$,*
3. *$1 \to \hbar_{\mathfrak{a}_x} = \hbar_{\mathfrak{a}_x}$, $\hbar_{\mathfrak{a}_x} \odot \hbar_{\mathfrak{a}_y} \leq_{\mathcal{H}_{\mathcal{A}}} \hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y}$, $\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_x} = 1$,*
4. *if $\hbar_{\mathfrak{a}_x} \leq \hbar_{\mathfrak{a}_y}$, then $\hbar_{\mathfrak{a}_z} \to \hbar_{\mathfrak{a}_x} \leq_{\mathcal{H}_{\mathcal{A}}} \hbar_{\mathfrak{a}_z} \to \hbar_{\mathfrak{a}_y}$, $\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z} \leq_{\mathcal{H}_{\mathcal{A}}} \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_z}$ and $\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y} = 1$,*
5. *$\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y} \leq_{\mathcal{H}_{\mathcal{A}}} (\hbar_{\mathfrak{a}_z} \to \hbar_{\mathfrak{a}_x}) \to (\hbar_{\mathfrak{a}_z} \to \hbar_{\mathfrak{a}_y})$, $\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y} \leq_{\mathcal{H}_{\mathcal{A}}} (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z}) \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_z})$,*
6. *$\hbar_{\mathfrak{a}_x} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z}) = \hbar_{\mathfrak{a}_y} \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_z})$*
7. *$\hbar_{\mathfrak{a}_x} \odot \hbar_{\mathfrak{a}_y} \leq \hbar_{\mathfrak{a}_z}$ if and only if $\hbar_{\mathfrak{a}_y} \leq_{\mathcal{H}_{\mathcal{A}}} \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_z}$,*
8. *$(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) \wedge (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z}) \leq_{\mathcal{H}_{\mathcal{A}}} (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_z})$.*

Definition 3. *[10, 13] Let $\mathcal{F}$ be a non-empty subset of $\mathcal{H}_{\mathcal{A}}$ such that $1 \in F$. Then for any $\hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y}, \hbar_{\mathfrak{a}_z} \in \mathcal{H}_{\mathcal{A}}$:*

1. *$\mathcal{F}$ is called an $\mathfrak{n}$-fold positive implicative filter of $\mathcal{H}_{\mathcal{A}}$, if ${\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z}) \in \mathcal{F}$ and ${\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y} \in \mathcal{F}$, then ${\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z} \in \mathcal{F}$.*
2. *$\mathcal{F}$ is called an $\mathfrak{n}$-fold fantastic filter of $\mathcal{H}_{\mathcal{A}}$, if $\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y} \in \mathcal{F}$, then $(({\hbar_{\mathfrak{a}_y}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_y} \in \mathcal{F}$.*

If $\mathcal{H}_{\mathcal{A}}$ is a bounded hoop, then a filter is proper if and only if it does not contain 0.

Definition 4. *[14] A bipolar fuzzy set $\varphi = (\mathcal{H}_{\mathcal{A}}, \varphi^+, \varphi^-)$ in a hoop $\mathcal{H}_{\mathcal{A}}$ is said to be a bipolar fuzzy filter of $\mathcal{H}_{\mathcal{A}}$ if the following conditions are hold for any $\bullet \in \{\odot, \to\}$:*

$$(\forall \hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y} \in \mathcal{H}_{\mathcal{A}}) \left(\begin{array}{c} \varphi^+(\hbar_{\mathfrak{a}_x} \bullet \hbar_{\mathfrak{a}_y}) \geq \min\{\varphi^+(\hbar_{\mathfrak{a}_x}), \varphi^+(\hbar_{\mathfrak{a}_y})\} \\ \varphi^-(\hbar_{\mathfrak{a}_x} \bullet \hbar_{\mathfrak{a}_y}) \leq \max\{\varphi^-(\hbar_{\mathfrak{a}_x}), \varphi^-(\hbar_{\mathfrak{a}_y})\} \end{array} \right), \tag{2}$$

$$(\forall\ \hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y} \in \mathcal{H}_{\mathcal{A}}) \left(\quad \hbar_{\mathfrak{a}_x} \leq \hbar_{\mathfrak{a}_y} \ \Rightarrow \left\{ \begin{array}{l} \varphi^+(\hbar_{\mathfrak{a}_x}) \leq \varphi^+(\hbar_{\mathfrak{a}_y}) \\ \varphi^-(\hbar_{\mathfrak{a}_x}) \geq \varphi^-(\hbar_{\mathfrak{a}_y}) \end{array} \right. \right). \tag{3}$$

Theorem 1. *[14] A bipolar fuzzy set $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy filter of $\mathcal{H}_\mathcal{A}$ if and only if*

$$(\forall \hbar_{\mathfrak{a}_x} \in \mathcal{H}_\mathcal{A}) \left(\begin{array}{c} \varphi^+(1) \geq \varphi^+(\hbar_{\mathfrak{a}_x}) \\ \varphi^-(1) \leq \varphi^-(\hbar_{\mathfrak{a}_x}) \end{array} \right) \tag{4}$$

$$(\forall \hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y} \in \mathcal{H}_\mathcal{A}) \left(\begin{array}{c} \varphi^+(\hbar_{\mathfrak{a}_y}) \geq \min\{\varphi^+(\hbar_{\mathfrak{a}_x}), \varphi^+(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})\} \\ \varphi^-(\hbar_{\mathfrak{a}_y}) \leq \max\{\varphi^-(\hbar_{\mathfrak{a}_x}), \varphi^-(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})\} \end{array} \right). \tag{5}$$

Definition 5. *[14] A bipolar fuzzy set $\varphi = (H, \varphi^+, \varphi^-)$ in a hoop $\mathcal{H}_\mathcal{A}$ is said to be a bipolar fuzzy $\mathfrak{n}$-fold implicative filter of $\mathcal{H}_\mathcal{A}$ if it satisfies* (4) *and*

$$(\forall \hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y} \in \mathcal{H}_\mathcal{A}) \left(\begin{array}{c} \varphi^+(({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \leq \varphi^+(\hbar_{\mathfrak{a}_x}) \\ \varphi^-(({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \geq \varphi^-(\hbar_{\mathfrak{a}_x}) \end{array} \right). \tag{6}$$

3 Bipolar Fuzzy $\mathfrak{n}$-fold positive implicative filters

In this section, we introduce the concept of bipolar fuzzy $\mathfrak{n}$-fold positive implicative filters within hoop algebras. This notion generalizes classical fuzzy filter structures by incorporating the bipolar framework and extending them to the $\mathfrak{n}$-fold setting, allowing for a more refined analysis of logical properties in a fuzzy environment.

Definition 6. *A bipolar fuzzy set $\varphi = (\varphi^+, \varphi^-)$ of $\mathcal{H}_\mathcal{A}$ is called a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter if*

$$(\forall \hbar_{\mathfrak{a}_x} \in \mathcal{H}_\mathcal{A}) \left(\begin{array}{c} \varphi^+(\hbar_{\mathfrak{a}_x}) \leq \varphi^+(1) \\ \varphi^-(\hbar_{\mathfrak{a}_x}) \geq \varphi^-(1) \end{array} \right) \tag{7}$$

$$(\forall \hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y}, \hbar_{\mathfrak{a}_z} \in \mathcal{H}_\mathcal{A}) \begin{pmatrix} \varphi^+({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \wedge \varphi^+({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \leq \varphi^+({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}) \\ \varphi^-({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \vee \varphi^-({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \geq \varphi^-({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}) \end{pmatrix} \tag{8}$$

Example 1. *Let $\mathcal{H}_\mathcal{A} = \{0, 1, 2, 3, 4, 5, 6\}$ be a set equipped with two binary operations, $\cdot$ and $\to$, whose values are provided in the following tables [1 - 2].*

Table 1: Cayley table of the binary operation $\to$

$\to$	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6
2	1	1	1	1	1	1	1
3	0	1	2	1	4	1	1
4	0	1	2	1	1	1	1
5	0	1	2	3	4	1	6
6	0	1	2	6	4	1	1

Table 2: Cayley table of the binary operation $\odot$

$\odot$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	0	2	2	2	2
3	0	3	2	3	4	3	3
4	0	4	2	4	4	4	4
5	0	5	2	3	4	5	6
6	0	6	2	3	4	6	3

Then $(\mathcal{H}_{\mathcal{A}}, \odot, \to, 1)$ is a hoop algebra. Let $\varphi = (\mathcal{H}_{\mathcal{A}}, \varphi^-, \varphi^+)$ be a bipolar fuzzy set in $\mathcal{H}_{\mathcal{A}}$ given by the Table 3:

Table 3: Tabular representation of of $\varphi = (\mathcal{H}_{\mathcal{A}}, \varphi^-, \varphi^+)$

$\mathcal{H}_{\mathcal{A}}$	0	1	2	3	4	5	6
$\varphi^-(\hbar_{\mathfrak{a}})$	-0.72	-0.83	-0.72	-0.83	-0.72	-0.83	-0.83
$\varphi^+(\hbar_{\mathfrak{a}})$	0.6	0.8	0.6	0.7	0.65	0.8	0.7

It is straightforward to verify that the bipolar fuzzy set $\varphi = (\mathcal{H}_{\mathcal{A}}, \varphi^-, \varphi^+)$ within the hoop $\mathcal{H}_{\mathcal{A}}$ qualifies as a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter on $\mathcal{H}_{\mathcal{A}}$.

Proposition 2. *Each bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter on $\mathcal{H}_{\mathcal{A}}$ is, in particular, a bipolar fuzzy filter.*

Proof. Assume that $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter on the hoop algebra $\mathcal{H}_{\mathcal{A}}$. By definition, it satisfies the conditions $\varphi^+(\hbar_{\mathfrak{a}_x}) \leq \varphi^+(1)$ and $\varphi^-(\hbar_{\mathfrak{a}_x}) \geq \varphi^-(1)$ for all $\hbar_{\mathfrak{a}_x} \in \mathcal{H}_{\mathcal{A}}$. Moreover, we have

$$\varphi^+(1^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})) \wedge \varphi^+(1^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_x}) \leq \varphi^+(1^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y})$$

and

$$\varphi^-(1^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})) \vee \varphi^-(1^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_x}) \geq \varphi^-(1^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}).$$

According to Proposition 1 (3), it follows that

$\varphi^+(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) \wedge \varphi^+(\hbar_{\mathfrak{a}_x}) \leq \varphi^+(\hbar_{\mathfrak{a}_y})$ and $\varphi^-(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) \vee \varphi^-(\hbar_{\mathfrak{a}_x}) \geq \varphi^-(\hbar_{\mathfrak{a}_y})$.

These inequalities confirm that $\varphi = (\varphi^+, \varphi^-)$ satisfies the conditions required for a bipolar fuzzy filter on $\mathcal{H}_{\mathcal{A}}$. □

The converse of Proposition 2 does not necessarily hold, as demonstrated by the following example.

Example 2. *Let $\mathcal{H}_{\mathcal{A}} = \{0, 1, 2, 3, 4, 5, 6\}$ be a set equipped with two binary operations, $\cdot$ and $\rightarrow$, whose values are provided in the following tables [4 - 5].*

Table 4: Cayley table of the binary operation $\rightarrow$

$\rightarrow$	0	1	2	3	4	5	6
0	1	1	2	1	1	1	1
1	0	1	2	3	4	5	6
2	1	1	1	1	1	1	1
3	0	1	2	1	1	5	6
4	0	1	2	3	1	5	6
5	0	1	2	1	1	1	6
6	1	2	1	1	1	1	1

Table 5: Cayley table of the binary operation $\odot$

$\odot$	0	1	2	3	4	5	6
0	0	0	2	0	0	0	0
1	0	1	2	3	4	5	6
2	2	2	2	2	2	2	2
3	0	3	2	3	3	5	6
4	0	4	2	3	4	5	6
5	0	5	2	5	5	5	6
6	0	6	2	6	6	6	0

Then $(\mathcal{H}_{\mathcal{A}}, \odot, \rightarrow, 1)$ is a hoop algebra. Let $\varphi = (\mathcal{H}_{\mathcal{A}}, \varphi^-, \varphi^+)$ be a bipolar fuzzy set in $\mathcal{H}_{\mathcal{A}}$ given by the Table 6:

Table 6: Tabular representation of of $\varphi = (\mathcal{H}_{\mathcal{A}}, \varphi^-, \varphi^+)$

$\mathcal{H}_{\mathcal{A}}$	0	1	2	3	4	5	6
$\varphi^-(\hbar_{\mathfrak{a}})$	-0.6	-0.8	-0.6	-0.8	-0.8	-0.7	-0.6
$\varphi^+(\hbar_{\mathfrak{a}})$	0.7	0.8	0.7	0.7	0.85	0.7	0.7

It is straightforward to verify that the bipolar fuzzy set $\varphi = (\mathcal{H}_{\mathcal{A}}, \varphi^-, \varphi^+)$ within the hoop $\mathcal{H}_{\mathcal{A}}$ qualifies as a bipolar fuzzy filter of $\mathcal{H}_{\mathcal{A}}$. But it is not an $\mathfrak{n}$-fold positive implicative filter of $\mathcal{H}_{\mathcal{A}}$ since

$$\varphi^+(6^3 \rightarrow (6 \rightarrow 2)) \wedge \varphi^+(6^3 \rightarrow 6) = \varphi^+(0 \rightarrow 1) \wedge \varphi^+(0 \rightarrow 6) = \varphi^+(1) \wedge \varphi^+(1) = 0.8$$

and

$$\varphi^+(6^3 \rightarrow 2) = \varphi^+(0 \rightarrow 2) = \varphi^+(2) = 0.7.$$

Therefore $\varphi^+(6^3 \rightarrow (6 \rightarrow 2)) \wedge \varphi^+(6^3 \rightarrow 6) \nleq \varphi^+(6^3 \rightarrow 2)$.

Definition 7. *Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy set on $\mathcal{H}_{\mathcal{A}}$, where $(\mathfrak{s}, \mathfrak{t}) \in [-1, 0] \times [0, 1]$. The sets $\mathcal{U}(\varphi^+, \mathfrak{t}) = \{\hbar_{\mathfrak{a}} \in \mathcal{H}_{\mathcal{A}} : \varphi^+(\hbar_{\mathfrak{a}}) \geq \mathfrak{t}\}$ and $\mathcal{L}(\varphi^-, \mathfrak{s}) = \{\hbar_{\mathfrak{a}} \in \mathcal{H}_{\mathcal{A}} : \varphi^-(\hbar_{\mathfrak{a}}) \leq \mathfrak{s}\}$ are called the level subsets of φ.*

Theorem 2. *A bipolar fuzzy set $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter of $\mathcal{H}_{\mathcal{A}}$ if and only if, for all $(\mathfrak{t}, \mathfrak{s}) \in [-1, 0] \times [0, 1]$, the nonempty level subsets $\mathcal{U}(\varphi^+, \mathfrak{t})$ and $\mathcal{L}(\varphi^-, \mathfrak{s})$ are $\mathfrak{n}$-fold positive implicative filters of $\mathcal{H}_{\mathcal{A}}$.*

Proof. Assume that $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter of $\mathcal{H}_{\mathcal{A}}$ and that $U(\varphi^+, \mathfrak{t}) \neq \emptyset$ and $L(\varphi^-, \mathfrak{s}) \neq \emptyset$ for all $(\mathfrak{t}, \mathfrak{s}) \in [-1, 0] \times [0, 1]$. It is evident that $1 \in \mathcal{U}(\varphi^+, \mathfrak{t}) \cap \mathcal{L}(\varphi^-, \mathfrak{s})$.

Let $\hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y}, \hbar_{\mathfrak{a}_z}, \hbar_{\mathfrak{a}_p}, \hbar_{\mathfrak{a}_q}, \hbar_{\mathfrak{a}_r} \in \mathcal{H}_{\mathcal{A}}$ such that

$$(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z}), \hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_q} \to \hbar_{\mathfrak{a}_r})) \in \mathcal{L}(\varphi^-, \mathfrak{s}) \times \mathcal{U}(\varphi^+, \mathfrak{t}),$$

and

$$(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}, \hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_q}) \in \mathcal{L}(\varphi^-, \mathfrak{s}) \times \mathcal{U}(\varphi^+, \mathfrak{t}).$$

Then we have

$$\varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \leq \mathfrak{s}, \quad \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \leq \mathfrak{s},$$

$$\varphi^+(\hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_q} \to \hbar_{\mathfrak{a}_r})) \geq \mathfrak{t}, \quad \varphi^+(\hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_q}) \geq \mathfrak{t}.$$

It follows that

$$\varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}) \leq \max\{\varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})), \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y})\} \leq \mathfrak{s},$$

$$\varphi^+(\hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_r}) \geq \min\{\varphi^+(\hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_q} \to \hbar_{\mathfrak{a}_r})), \varphi^+(\hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_q})\} \geq \mathfrak{t}.$$

Hence,

$$(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}, \hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_r}) \in \mathcal{L}(\varphi^-, \mathfrak{s}) \times \mathcal{U}(\varphi^+, \mathfrak{t}),$$

which proves that $\mathcal{L}(\varphi^-, \mathfrak{s})$ and $\mathcal{U}(\varphi^+, \mathfrak{t})$ are $\mathfrak{n}$-fold positive implicative filters.

Conversely, assume that for every $(\mathfrak{t}, \mathfrak{s}) \in [-1, 0] \times [0, 1]$, the nonempty level sets $\mathcal{U}(\varphi^+, \mathfrak{t})$ and $\mathcal{L}(\varphi^-, \mathfrak{s})$ are $\mathfrak{n}$-fold positive implicative filters of $\mathcal{H}_{\mathcal{A}}$. For any $\hbar_{\mathfrak{a}_x} \in \mathcal{H}_{\mathcal{A}}$, define $\mathfrak{t}_{\hbar_{\mathfrak{a}_x}} = \varphi^+(\hbar_{\mathfrak{a}_x})$ and $\mathfrak{s}_{\hbar_{\mathfrak{a}_x}} = \varphi^-(\hbar_{\mathfrak{a}_x})$. Then clearly $\hbar_{\mathfrak{a}_x} \in \mathcal{U}(\varphi^+, \mathfrak{t}_{\hbar_{\mathfrak{a}_x}}) \cap L(\varphi^-, \mathfrak{s}_{\hbar_{\mathfrak{a}_x}})$, and since $1 \in \mathcal{H}_{\mathcal{A}}$, we get

$$\varphi^+(1) \geq \mathfrak{t}_{\hbar_{\mathfrak{a}_x}} = \varphi^+(\hbar_{\mathfrak{a}_x}), \quad \varphi^-(1) \leq \mathfrak{s}_{\hbar_{\mathfrak{a}_x}} = \varphi^-(\hbar_{\mathfrak{a}_x}).$$

Suppose that, for some elements $\hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y}, \hbar_{\mathfrak{a}_z}, \hbar_{\mathfrak{a}_p}, \hbar_{\mathfrak{a}_q}, \hbar_{\mathfrak{a}_r} \in \mathcal{H}_{\mathcal{A}}$, the following inequality fails:

$$\varphi^-(\hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_r}) > \varphi^-(\hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_q} \to \hbar_{\mathfrak{a}_r})) \vee \varphi^-(\hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_q}),$$

or

$$\varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}) < \varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \wedge \varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}).$$

Let $\mathfrak{s} = \varphi^{-}(\hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_q} \to \hbar_{\mathfrak{a}_r})) \vee \varphi^{-}(\hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_q})$ and $\mathfrak{t} = \varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \wedge \varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y})$. Then $\hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_q} \to \hbar_{\mathfrak{a}_r}), \hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_q} \in \mathcal{L}(\varphi^{-}, \mathfrak{s})$, and $\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z}), \hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y} \in \mathcal{U}(\varphi^{+}, \mathfrak{t})$. But this implies either $\hbar_{\mathfrak{a}_p}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_r} \notin \mathcal{L}(\varphi^{-}, \mathfrak{s})$ or $\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z} \notin \mathcal{U}(\varphi^{+}, \mathfrak{t})$, a contradiction.

Therefore, for all $\hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y}, \hbar_{\mathfrak{a}_z} \in \mathcal{H}_{\mathcal{A}}$,

$$\varphi^{-}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}) \leq \varphi^{-}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to z)) \vee \varphi^{-}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}),$$

and

$$\varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}) \geq \varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \wedge \varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to y).$$

Hence, $\varphi = (\varphi^{+}, \varphi^{-})$ satisfies the conditions for a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter of $\mathcal{H}_{\mathcal{A}}$. □

Proposition 3. *Let $\varphi = (\varphi^{+}, \varphi^{-})$ be a bipolar fuzzy filter of $\mathcal{H}_{\mathcal{A}}$. The following statements are equivalent for any $\hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y}, \hbar_{\mathfrak{a}_z} \in \mathcal{H}_{\mathcal{A}}$:*

1. *φ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter of $\mathcal{H}_{\mathcal{A}}$.*
2. *$\varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \leq \varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y})$ and $\varphi^{-}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \geq \varphi^{-}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y})$.*
3. *$\varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \leq \varphi^{+}((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}))$ and $\varphi^{-}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \geq \varphi^{-}((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}))$.*

Proof. (1) $\Rightarrow$ (2): Let $\hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y} \in \mathcal{H}_{\mathcal{A}}$. Since $\varphi = (\varphi^{+}, \varphi^{-})$ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter of $\mathcal{H}_{\mathcal{A}}$, we have

$$\varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})) \wedge \varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_x}) \leq \varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}),$$

$$\varphi^{-}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})) \vee \varphi^{-}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_x}) \geq \varphi^{-}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}).$$

Since $\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) = \hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to y$ and $\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_x} = 1$, the inequalities simplify to

$$\varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \leq \varphi^{+}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}), \quad \varphi^{-}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \geq \varphi^{-}(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}).$$

(2) $\Rightarrow$ (3): From Proposition 1(5), we have $\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z} \leq (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z})$, so

$$\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z}) \leq \hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z})).$$

Using associativity and substitution, we have

$$\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z})) = \hbar_{\mathfrak{a}_x}{}^{2\mathfrak{n}} \to ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_z}).$$

By the assumption in (2), we iteratively apply

$$\varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{k}+1} \to \eta) \leq \varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{k}} \to \ \zeta), \quad \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{k}+1} \to \eta) \geq \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{k}} \to \eta),$$

to conclude

$$\varphi^+(\hbar_{\mathfrak{a}_x}{}^{2\mathfrak{n}} \to ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_z})) \leq \varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_z})),$$

$$\varphi^-(\hbar_{\mathfrak{a}_x}{}^{2\mathfrak{n}} \to ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_z})) \geq \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_z})).$$

Combining the above, we obtain

$$\varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \leq \varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_z})),$$

$$\varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \geq \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_z})).$$

Since $\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_z}) = (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z})$, we obtain (3).

(3) $\Rightarrow$ (1): Assume (3) holds. As φ is a bipolar fuzzy filter, we have

$$\varphi^+((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z})) \wedge \varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \leq \varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}),$$

$$\varphi^-((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z})) \vee \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \geq \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}).$$

Combining these with the assumption, we get

$$\varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \wedge \varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \leq \varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}),$$

$$\varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \vee \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \geq \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}),$$

thus showing that φ satisfies the condition for being a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter of $\mathcal{H}_{\mathcal{A}}$. $\square$

Corollary 1. *Let $\mathcal{H}_{\mathcal{A}}$ be a bounded hoop, and let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter of $\mathcal{H}_{\mathcal{A}}$. Then for any $\hbar_{\mathfrak{a}} \in \mathcal{H}_{\mathcal{A}}$,*

$$\varphi^+((\hbar_{\mathfrak{a}}{}^{\mathfrak{n}+1})') = \varphi^+((\hbar_{\mathfrak{a}}{}^{\mathfrak{n}})') \quad \textit{and} \quad \varphi^-((\hbar_{\mathfrak{a}}{}^{\mathfrak{n}+1})') = \varphi^-((\hbar_{\mathfrak{a}}{}^{\mathfrak{n}})').$$

Proposition 4. *A bipolar fuzzy set $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter of $\mathcal{H}_{\mathcal{A}}$ if and only if*

$$(\forall \hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y}, \hbar_{\mathfrak{a}_z} \in \mathcal{H}_{\mathcal{A}}) \begin{pmatrix} \varphi^+(\hbar_{\mathfrak{a}_x}) \wedge \varphi^+(\hbar_{\mathfrak{a}_x} \to (\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_z})) \leq \varphi^+(\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}), \\ \varphi^-(\hbar_{\mathfrak{a}_x}) \vee \varphi^-(\hbar_{\mathfrak{a}_x} \to (\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_z})) \geq \varphi^-(\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}) \end{pmatrix}. \tag{9}$$

Proof. Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter. Then φ is a bipolar fuzzy filter and satisfies

$$\varphi^+(\hbar_{\mathfrak{a}_x}) \wedge \varphi^+\big(\hbar_{\mathfrak{a}_x} \to ({\hbar_{\mathfrak{a}_y}}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_z})\big) \leq \varphi^+({\hbar_{\mathfrak{a}_y}}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_z}),$$

and

$$\varphi^-(\hbar_{\mathfrak{a}_x}) \vee \varphi^-\big(\hbar_{\mathfrak{a}_x} \to ({\hbar_{\mathfrak{a}_y}}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_z})\big) \geq \varphi^-({\hbar_{\mathfrak{a}_y}}^{\mathfrak{n}+1} \to z).$$

By Proposition 3 (2), we have

$$\varphi^+({\hbar_{\mathfrak{a}_y}}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_z}) \leq \varphi^+({\hbar_{\mathfrak{a}_y}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}), \quad \varphi^-({\hbar_{\mathfrak{a}_y}}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_z}) \geq \varphi^-({\hbar_{\mathfrak{a}_y}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}).$$

Hence, we have

$$\varphi^+(\hbar_{\mathfrak{a}_x}) \wedge \varphi^+\big(\hbar_{\mathfrak{a}_x} \to ({\hbar_{\mathfrak{a}_y}}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_z})\big) \leq \varphi^+({\hbar_{\mathfrak{a}_y}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}),$$

and

$$\varphi^-(\hbar_{\mathfrak{a}_x}) \vee \varphi^-\big(\hbar_{\mathfrak{a}_x} \to ({\hbar_{\mathfrak{a}_y}}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_z})\big) \geq \varphi^-({\hbar_{\mathfrak{a}_y}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_z}).$$

Conversely, assume that the conditions in (12) hold. Let $\hbar_{\mathfrak{a}_y} = \hbar_{\mathfrak{a}_z} = 1$. Then

$$\varphi^+(\hbar_{\mathfrak{a}_x}) \wedge \varphi^+(\hbar_{\mathfrak{a}_x} \to (1^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y})) \leq \varphi^+(1^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}),$$

and

$$\varphi^-(\hbar_{\mathfrak{a}_x}) \vee \varphi^-(\hbar_{\mathfrak{a}_x} \to (1^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y})) \geq \varphi^-(1^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}).$$

But since $1^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y} = \hbar_{\mathfrak{a}_y}$ and $\hbar_{\mathfrak{a}_x} \to (1^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) = \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}$, it follows that

$$\varphi^+(\hbar_{\mathfrak{a}_x}) \wedge \varphi^+(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) \leq \varphi^+(\hbar_{\mathfrak{a}_y}), \quad \varphi^-(\hbar_{\mathfrak{a}_x}) \vee \varphi^-(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) \geq \varphi^-(\hbar_{\mathfrak{a}_y}),$$

which shows that φ is a bipolar fuzzy filter.

To show that φ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter, we must prove

$$\varphi^+({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \leq \varphi^+({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}), \quad \varphi^-({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \geq \varphi^-({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}).$$

By hypothesis, taking $\hbar_{\mathfrak{a}_x} = 1$, $\hbar_{\mathfrak{a}_y} = {\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}+1}$, and $\hbar_{\mathfrak{a}_z} = \hbar_{\mathfrak{a}_y}$, we get

$$\varphi^+(1) \wedge \varphi^+(1 \to ({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y})) \leq \varphi^+({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}),$$

and

$$\varphi^-(1) \vee \varphi^-(1 \to ({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y})) \geq \varphi^-({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}).$$

Since $1 \to \hbar_{\mathfrak{a}_p} = \hbar_{\mathfrak{a}_p}$ in a bounded hoop, this yields

$$\varphi^+({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \leq \varphi^+({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}), \quad \varphi^-({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \geq \varphi^-({\hbar_{\mathfrak{a}_x}}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}).$$

Therefore, by Proposition 3, φ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter. □

Proposition 5. *1. If $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter, then it is also a bipolar fuzzy $(\mathfrak{n}+1)$-fold positive implicative filter.*

2. Moreover, such a filter is necessarily a bipolar fuzzy $(\mathfrak{n}+\mathfrak{k})$-fold positive implicative filter for any integer $\mathfrak{k} \geq 0$.

Proof. (1) Assume that $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter. According to Proposition 3 (2), the following inequalities hold:

$$\varphi^+\big(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})\big) \leq \varphi^+\big(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})\big),$$

$$\varphi^-\big(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})\big) \geq \varphi^-\big(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})\big).$$

Utilizing the identities

$$\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) = \hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}$$

and

$$\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) = \hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+2} \to \hbar_{\mathfrak{a}_y},$$

we derive

$$\varphi^+\big(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+2} \to \hbar_{\mathfrak{a}_y}\big) \leq \varphi^+\big(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}\big),$$

and

$$\varphi^-\big(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+2} \to \hbar_{\mathfrak{a}_y}\big) \geq \varphi^-\big(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}\big).$$

Consequently, φ satisfies the condition for being a bipolar fuzzy $(\mathfrak{n}+1)$-fold positive implicative filter.

(2) The result directly follows by induction from part (1), as the property holds for each increment in $\mathfrak{k}$. □

Proposition 6. *1. If $\mathcal{H}_{\mathcal{A}}$ is locally finite for some $\mathfrak{n} \in \mathbb{N}$, then each bipolar fuzzy filter on $\mathcal{H}_{\mathcal{A}}$ qualifies as a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter.*

2. If every element $\hbar_{\mathfrak{a}}$ in H satisfies $\hbar_{\mathfrak{a}}{}^2 = \hbar_{\mathfrak{a}}$, then any bipolar fuzzy filter $\varphi = (\varphi^+, \varphi^-)$ on $\mathcal{H}_{\mathcal{A}}$ is inherently a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter.

Proof. (1). Suppose $\mathfrak{m}$ denotes the maximum of $\mathrm{ord}(x)$ for elements $\hbar_{\mathfrak{a}_x}$ in $\mathcal{H}_{\mathcal{A}}$. For every $\mathfrak{n} \geq \mathfrak{m}$, we observe that $\varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \leq \varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y})$ and $\varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \geq \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y})$. Consequently, $\varphi = (\varphi^+, \varphi^-)$ meets the conditions of a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter.

(2). Given the idempotency condition $\hbar_{\mathfrak{a}}{}^2 = \hbar_{\mathfrak{a}}$ for all $\hbar_{\mathfrak{a}} \in \mathcal{H}_{\mathcal{A}}$, it directly follows that $\varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \leq \varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y})$ and $\varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \geq \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y})$. From Proposition 1 (2), we conclude that φ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter. □

Theorem 3. *1. Every bipolar fuzzy n-fold implicative filter in H is also a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter.*

2. Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy n-fold positive implicative filter of H satisfying

$$\varphi^+((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y})^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \leq \varphi^+((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}} \to x)^{\mathfrak{n}} \to x),$$

and

$$\varphi^-((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to y)^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \geq \varphi^-((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}} \to x)^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_x}),$$

and assume $\hbar_{\mathfrak{a}_x} \leq (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}$ for all $\hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y} \in \mathcal{H}_{\mathcal{A}}$. Then, φ qualifies as a bipolar fuzzy $\mathfrak{n}$-fold implicative filter.

Proof. (1). From Proposition 1 (5), (6), and using algebraic properties of the implication operator $\to$, one can derive

$$(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y})^{\mathfrak{n}} \to (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \geq (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y})^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}.$$

Hence, through repeated derivations and inequalities involving fuzzy membership evaluations φ^+, φ^-, and leveraging the $\mathfrak{n}$-fold structure, we infer that

$$\varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \leq \varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}), \quad \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \geq \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}).$$

Thus, based on Proposition 3 (2), φ must be a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter.

(2). Assuming φ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter and the additional ordering $\hbar_{\mathfrak{a}_x} \leq (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}$, we apply Proposition 1 to deduce:

$$(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to y)^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_x} \leq (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y})^{\mathfrak{n}} \to ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}).$$

Then, evaluating via φ^+ and φ^-, and applying monotonicity and level-set arguments, we construct the nested inclusions:

$$\varphi^+((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y})^{n} \to \hbar_{\mathfrak{a}_x}) \leq \varphi^+(x), \quad \varphi^-((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y})^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_x}) \geq \varphi^-(\hbar_{\mathfrak{a}_x}).$$

Hence, the conditions are satisfied for φ to be a bipolar fuzzy $\mathfrak{n}$-fold implicative filter. □

4 Bipolar fuzzy $\mathfrak{n}$-fold fantastic filters

In this section, we introduce the concept of *bipolar fuzzy $\mathfrak{n}$-fold fantastic filters* in the context of hoop algebras. We examine essential properties of these filters and explore their connections with other types of fuzzy $\mathfrak{n}$-fold filters, particularly *bipolar fuzzy $\mathfrak{n}$-fold positive implicative filters* and *bipolar fuzzy $\mathfrak{n}$-fold implicative filters*. These relationships provide deeper insight into the structure and hierarchy of fuzzy filters defined on hoop algebras.

Definition 8. *Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy set on a hoop algebra $\mathcal{H}_{\mathcal{A}}$. Then φ is said to be a* bipolar fuzzy $\mathfrak{n}$-fold fantastic filter *of $\mathcal{H}_{\mathcal{A}}$ if it satisfies* (7) *and the following condition:*

$$(\forall \hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y}, \hbar_{\mathfrak{a}_z} \in \mathcal{H}_{\mathcal{A}}) \begin{pmatrix} \varphi^+(\hbar_{\mathfrak{a}_z} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})) \wedge \varphi^+(\hbar_{\mathfrak{a}_z}) \leq \varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \\ \varphi^-(\hbar_{\mathfrak{a}_z} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})) \vee \varphi^-(\hbar_{\mathfrak{a}_z}) \geq \varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \end{pmatrix} \tag{10}$$

Example 3. *Let $(\mathcal{H}_{\mathcal{A}}, \odot, \to, 1)$ be a hoop algebra given in Example 1. Let $\varphi = (\mathcal{H}_{\mathcal{A}}, \varphi^-, \varphi^+)$ be a bipolar fuzzy set in $\mathcal{H}_{\mathcal{A}}$ given by the Table 8:*

Table 7: Tabular representation of of $\varphi = (\mathcal{H}_{\mathcal{A}}, \varphi^-, \varphi^+)$

$\mathcal{H}_{\mathcal{A}}$	0	1	2	3	4	5	6
$\varphi^-(\hbar_{\mathfrak{a}})$	-0.72	-0.83	-0.72	-0.83	-0.83	-0.83	-0.83
$\varphi^+(\hbar_{\mathfrak{a}})$	0.6	0.8	0.6	0.8	0.8	0.8	0.8

It is straightforward to verify that the bipolar fuzzy set $\varphi = (\mathcal{H}_{\mathcal{A}}, \varphi^-, \varphi^+)$ within the hoop $\mathcal{H}_{\mathcal{A}}$ qualifies as a bipolar fuzzy $\mathfrak{n}$-fold fantastic filter of $\mathcal{H}_{\mathcal{A}}$.

Proposition 7. *Every bipolar fuzzy $\mathfrak{n}$-fold fantastic filter $\varphi = (\varphi^+, \varphi^-)$ of a hoop algebra $\mathcal{H}_{\mathcal{A}}$ is a bipolar fuzzy filter.*

Proof. Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy n-fold fantastic filter of $\mathcal{H}_{\mathcal{A}}$. Then, by definition, we have

$$\varphi^+(\hbar_{\mathfrak{a}}) \leq \varphi^+(1) \quad \text{and} \quad \varphi^-(\hbar_{\mathfrak{a}}) \geq \varphi^-(1), \quad \text{for all } \hbar_{\mathfrak{a}} \in \mathcal{H}_{\mathcal{A}}.$$

By Proposition 1 (3), we know that for all $\hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y} \in \mathcal{H}_{\mathcal{A}}$,

$$\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y} = \hbar_{\mathfrak{a}_x} \to (1 \to \hbar_{\mathfrak{a}_y}).$$

Therefore,

$$\varphi^+(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) = \varphi^+(\hbar_{\mathfrak{a}_x} \to (1 \to \hbar_{\mathfrak{a}_y})),$$
$$\varphi^-(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) = \varphi^-(\hbar_{\mathfrak{a}_x} \to (1 \to \hbar_{\mathfrak{a}_y})).$$

It follows that

$$\varphi^+(\hbar_{\mathfrak{a}_x}) \wedge \varphi^+(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) = \varphi^+(\hbar_{\mathfrak{a}_x}) \wedge \varphi^+(\hbar_{\mathfrak{a}_x} \to (1 \to \hbar_{\mathfrak{a}_y})),$$

$$\varphi^-(\hbar_{\mathfrak{a}_x}) \vee \varphi^-(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) = \varphi^-(\hbar_{\mathfrak{a}_x}) \vee \varphi^-(\hbar_{\mathfrak{a}_x} \to (1 \to \hbar_{\mathfrak{a}_y})).$$

Since φ is a bipolar fuzzy $\mathfrak{n}$-fold fantastic filter, it satisfies

$$\varphi^+(\hbar_{\mathfrak{a}_x}) \wedge \varphi^+(\hbar_{\mathfrak{a}_x} \to (1 \to \hbar_{\mathfrak{a}_y})) \leq \varphi^+\left(((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}} \to 1) \to 1) \to \hbar_{\mathfrak{a}_y}\right),$$

$$\varphi^-(\hbar_{\mathfrak{a}_x}) \vee \varphi^-(\hbar_{\mathfrak{a}_x} \to (1 \to \hbar_{\mathfrak{a}_y})) \geq \varphi^-\left(((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}} \to 1) \to 1) \to \hbar_{\mathfrak{a}_y}\right).$$

By Proposition 1 (3) again, we have

$$\left(((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}} \to 1) \to 1) \to \hbar_{\mathfrak{a}_y}\right) = \hbar_{\mathfrak{a}_y}.$$

Hence,

$$\varphi^+(\hbar_{\mathfrak{a}_x}) \wedge \varphi^+(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) \leq \varphi^+(\hbar_{\mathfrak{a}_y}),$$

and

$$\varphi^-(\hbar_{\mathfrak{a}_x}) \vee \varphi^-(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) \geq \varphi^-(\hbar_{\mathfrak{a}_y}).$$

Therefore, $\varphi = (\varphi^+, \varphi^-)$ satisfies the condition of a bipolar fuzzy filter. □

The converse of Proposition 7 may not be true as shown in the following example.

Example 4. *Let $(\mathcal{H}_{\mathcal{A}}, \odot, \to, 1)$ be a hoop algebra given in Example 1. Let $\varphi = (\mathcal{H}_{\mathcal{A}}, \varphi^-, \varphi^+)$ be a bipolar fuzzy set in $\mathcal{H}_{\mathcal{A}}$ given by the Table 8:*

Table 8: Tabular representation of of $\varphi = (\mathcal{H}_{\mathcal{A}}, \varphi^-, \varphi^+)$

$\mathcal{H}_{\mathcal{A}}$	0	1	2	3	4	5	6
$\varphi^-(\hbar_{\mathfrak{a}})$	-0.72	-0.83	-0.72	-0.83	-0.72	-0.83	-0.83
$\varphi^+(\hbar_{\mathfrak{a}})$	0.1	0.2	0.1	0.1	0.1	0.1	0.1

It is straightforward to verify that the bipolar fuzzy set $\varphi = (\mathcal{H}_{\mathcal{A}}, \varphi^-, \varphi^+)$ within the hoop $\mathcal{H}_{\mathcal{A}}$ qualifies as a bipolar fuzzy filter of $\mathcal{H}_{\mathcal{A}}$. But it is not an $\mathfrak{n}$-fold fantastic filter of $\mathcal{H}_{\mathcal{A}}$ since

$$\varphi^+(1 \to (4 \to 6)) \wedge \varphi^+(1) = \varphi^+(1 \to 1) \wedge \varphi^+(1) = \varphi^+(1) \wedge \varphi^+(1) = \varphi^+(1) = 0.2$$

and

$$\begin{aligned} \varphi^+\left((6^3 \to 4) \to 4) \to 6\right) &= \varphi^+\left(((3 \to 4) \to 4) \to 6\right) \\ &= \varphi^+\left((4 \to 4) \to 6\right) \\ &= \varphi^+\left(1 \to 6\right) \\ &= \varphi^+(6) \\ &= 0.1. \end{aligned}$$

Therefore $\varphi^+(1 \to (4 \to 6)) \wedge \varphi^+(1) \nleq \varphi^+(((6^3 \to 4) \to 4) \to 6)$.

Theorem 4. *A bipolar fuzzy set $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold fantastic filter if and only if it satisfies the following condition:*

$$(\forall \hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y} \in \mathcal{H}_{\mathcal{A}}) \begin{pmatrix} \varphi^+(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})) \leq \varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \\ \varphi^-(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})) \geq \varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \end{pmatrix}. \tag{11}$$

Proof. If $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy n-fold fantastic filter and $\hbar_{\mathfrak{a}_z} = 1$, then

$$\begin{array}{rcl}\varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) & \geq & \varphi^+(1 \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})) \wedge \varphi^+(1) \\ & = & \varphi^+(1 \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})),\end{array}$$

and

$$\begin{array}{rcl}\varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) & \leq & \varphi^-(1 \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})) \vee \varphi^-(1) \\ & = & \varphi^-(1 \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})).\end{array}$$

Hence, we obtain

$$\varphi^+(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}) \leq \varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}),$$

$$\varphi^-(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}) \geq \varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}).$$

Conversely, if $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy filter, then $\varphi^+(\hbar_{\mathfrak{a}_x}) \leq \varphi^+(1)$, $\varphi^-(\hbar_{\mathfrak{a}_x}) \geq \varphi^-(1)$, $\varphi^+(\hbar_{\mathfrak{a}_z}) \wedge \varphi^+(\hbar_{\mathfrak{a}_z} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})) \leq \varphi^+(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})$ and $\varphi^-(\hbar_{\mathfrak{a}_z}) \vee \varphi^-(\hbar_{\mathfrak{a}_z} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})) \geq \varphi^-(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})$. By assumption, we obtain

$$\varphi^+(\hbar_{\mathfrak{a}_z}) \wedge \varphi^+(\hbar_{\mathfrak{a}_z} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})) \leq \varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to y) \to y) \to \hbar_{\mathfrak{a}_x}),$$

$$\varphi^-(\hbar_{\mathfrak{a}_z}) \vee \varphi^-(\hbar_{\mathfrak{a}_z} \to (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})) \geq \varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}).$$

Therefore, $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold fantastic filter. □

Proposition 8. *Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy filter of $\mathcal{H}_{\mathcal{A}}$. Then $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold fantastic filter if and only if*

$$(\forall \hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y} \in \mathcal{H}_{\mathcal{A}}) \begin{pmatrix} \varphi^+(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}) = \varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \\ \varphi^-(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}) = \varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \end{pmatrix}. \tag{12}$$

Proof. By Proposition 1 (2), $\hbar_{\mathfrak{a}_y} \leq (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}$ and by Proposition 1 (4), $(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \leq \hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}$. Then

$$\varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \leq \varphi^+(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})$$

and

$$\varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \geq \varphi^-(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}).$$

Therefore, by Theorem 4, $\varphi^+(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}) = \varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to x)$ and $\varphi^-(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}) = \varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x})$. □

Theorem 5. *Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy prefilter of $\mathcal{H}_{\mathcal{A}}$. Then $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative prefilter if and only if its nonempty level sets $\mathcal{U}(\varphi^+, \mathfrak{t})$ and $\mathcal{L}(\varphi^-, \mathfrak{s})$ are $\mathfrak{n}$-fold positive implicative prefilter of $\mathcal{H}_{\mathcal{A}}$ for all $(\mathfrak{t}, \mathfrak{s}) \in [-1, 0] \times [0, 1]$.*

Proof. Similar to the proof of Theorem 2. □

Proposition 9. *Every fuzzy $\mathfrak{n}$-fold fantastic filter $\varphi = (\varphi^+, \varphi^-)$ of $\mathcal{H}_{\mathcal{A}}$ is a bipolar fuzzy $(\mathfrak{n}+1)$-fold fantastic filter.*

Proof. By Proposition 1 (3),(4), $((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x} \leq ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to x$. So $\varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \leq \varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x})$ and $\varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \geq \varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x})$. Therefore, $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $(\mathfrak{n}+1)$-fold fantastic filter. □

Proposition 10.
1. *If $\mathcal{H}_{\mathcal{A}}$ is a chain, then every bipolar fuzzy filter of $\mathcal{H}_{\mathcal{A}}$ is a bipolar fuzzy $\mathfrak{n}$-fold fantastic filter.*
2. *If $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold fantastic filter of bounded hoop H, then $\varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})'') \to \hbar_{\mathfrak{a}_x}) = \varphi^+(1)$ and $\varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})'') \to x) = \varphi^-(1)$.*
3. *If every element of bounded hoop $\mathcal{H}_{\mathcal{A}}$ is dense, then any bipolar fuzzy n-fold fantastic filter of $\mathcal{H}_{\mathcal{A}}$ is constant.*

Proof. (1). Let $\hbar_{\mathfrak{a}_x} \leq \hbar_{\mathfrak{a}_y}$. Then by Proposition 1 (3), $\varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) = \varphi^+(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})$ and $\varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) = \varphi^-(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})$. Therefore, by Theorem 5, $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold fantastic filter.

(2). Let $\hbar_{\mathfrak{a}_y} = 0$. Then $\varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) = \varphi^+(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})$ and $\varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) = \varphi^-(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x})$.

(3). Let $0 \neq x \in H$ be dense. Then by (2),

$$\varphi^+((((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})') \to 0) \to \hbar_{\mathfrak{a}_x}) = \varphi^+(1 \to \hbar_{\mathfrak{a}_x}) = \varphi^+(\hbar_{\mathfrak{a}_x}) = \varphi^+(0 \to \hbar_{\mathfrak{a}_x}) = \varphi^+(1),$$

$$\varphi^-((((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})') \to 0) \to \hbar_{\mathfrak{a}_x}) = \varphi^-(1 \to \hbar_{\mathfrak{a}_x}) = \varphi^-(\hbar_{\mathfrak{a}_x}) = \varphi^-(0 \to \hbar_{\mathfrak{a}_x}) = \varphi^-(1).$$

Therefore, $\varphi^+(\hbar_{\mathfrak{a}_x}) = \varphi^+(1)$ and $\varphi^-(\hbar_{\mathfrak{a}_x}) = \varphi^-(1)$. □

Proposition 11. *Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy $\mathfrak{n}$-fold fantastic filter of $\mathcal{H}_{\mathcal{A}}$. Then $\mathcal{G} = \{\hbar_{\mathfrak{a}} \in \mathcal{H}_{\mathcal{A}} \mid \varphi^+(\hbar_{\mathfrak{a}}) = \varphi^+(1), \varphi^-(\hbar_{\mathfrak{a}}) = \varphi^-(1)\}$ is an $\mathfrak{n}$-fold fantastic filter.*

Proof. Since $1 \in \mathcal{H}_{\mathcal{A}}$, $\varphi^+(1) = \varphi^+(1)$ and $\varphi^-(1) = \varphi^-(1)$, so $1 \in \mathcal{G}$. Let $\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x} \in \mathcal{G}$. Then $\varphi^+(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}) = \varphi^+(1)$ and $\varphi^-(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}) = \varphi^-(1)$. By Theorem 4, $\varphi^+(1) = \varphi^+(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}) \le \varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x})$ and $\varphi^-(1) = \varphi^-(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}) \ge \varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x})$. Hence, $\varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) = \varphi^+(1)$ and $\varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) = \varphi^-(1)$. Thus, $((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x} \in \mathcal{G}$. Therefore, $\mathcal{G}$ is an $\mathfrak{n}$-fold fantastic filter. □

Lemma 1. *Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy filter of a bounded hoop $\mathcal{H}_{\mathcal{A}}$. Then for any $\hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y}, \hbar_{\mathfrak{a}_z} \in \mathcal{H}_{\mathcal{A}}$, the following conditions are equivalent:*

1. *$\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold implicative filter of $\mathcal{H}_{\mathcal{A}}$,*
2. *$\varphi^+(((\hbar_{\mathfrak{a}_z}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_y}) \wedge \varphi^+(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z}) \le \varphi^+(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_z})$ and $\varphi^-(((\hbar_{\mathfrak{a}_z}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_y}) \vee \varphi^-(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z}) \ge \varphi^-(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_z})$,*
3. *$\varphi^+(((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_y}) \le \varphi^+(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})$ and $\varphi^-(((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_y}) \ge \varphi^-(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})$.*

Proof. (1) ⇒ (3): By Proposition 1 (2), $\hbar_{\mathfrak{a}_y} \le \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}$, then $\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}} \le (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})^{\mathfrak{n}}$. By Proposition 1 (4), $(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})^{\mathfrak{n}} \to 0 \le \hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}} \to 0 = (\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})'$. So, $(\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})' \to \hbar_{\mathfrak{a}_y} \le (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})' \to \hbar_{\mathfrak{a}_y}$. Thus, $\hbar_{\mathfrak{a}_x} \to ((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})' \to \hbar_{\mathfrak{a}_y}) \le \hbar_{\mathfrak{a}_x} \to (((\hbar_{\mathfrak{a}_x} \to y)^{n})' \to y)$. So, $(y^{n})' \odot \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y} \le (((\hbar_{\mathfrak{a}_x} \to y)^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_y}$. Hence, $\varphi^+((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) \le \varphi^+(((\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})^{\mathfrak{n}})' \odot x \to \hbar_{\mathfrak{a}_y})$.

Let $\eta_1 = \varphi^+((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})$, $\eta_2 = \varphi^-((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})$, $\rho_1 = \varphi^+(((\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})$, and $\rho_2 = \varphi^-(((\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})$. Then $(\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y} \in \mathcal{U}(\varphi^+, \eta_1) \cap \mathcal{L}(\varphi^-, \eta_2)$ and $((\hbar_{\mathfrak{a}_x} \to y)^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y} \in \mathcal{U}(\varphi^+, \rho_1) \cap \mathcal{L}(\varphi^-, \rho_2)$. Since $\eta_1 \le \rho_1$ and $\eta_2 \ge \rho_2$, $\mathcal{U}(\varphi^+, \rho_1) \subseteq \mathcal{U}(\varphi^+, \eta_1)$ and $L(\varphi^-, \eta_2) \supseteq \mathcal{L}(\varphi^-, \rho_2)$. Hence, $((\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y} \in \mathcal{U}(\varphi^+, \eta_1) \cap \mathcal{L}(\varphi^-, \eta_2)$ and $((\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})^{\mathfrak{n}})' \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) \in \mathcal{U}(\varphi^+, \eta_1) \cap \mathcal{L}(\varphi^-, \eta_2)$. Since $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold implicative filter, then $\mathcal{U}(\varphi^+, \eta_1)$ and $\mathcal{L}(\varphi^-, \eta_2)$ are $\mathfrak{n}$-fold implicative filters, thus $\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y} \in \mathcal{U}(\varphi^+, \eta_1) \cap \mathcal{L}(\varphi^-, \eta_2)$ and $\varphi^+((y^{n})' \odot \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) = \eta_1 \le \varphi^+(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})$ and $\varphi^-((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) = \eta_2 \ge \varphi^-(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})$.

(3) ⇒ (2): By Proposition 1 (8), $(((\hbar_{\mathfrak{a}_z}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_y}) \wedge (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z}) \le (((\hbar_{\mathfrak{a}_z}{}^{\mathfrak{n}})') \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_z}$. So, $\varphi^+((((\hbar_{\mathfrak{a}_z}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_y}) \wedge (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \le \varphi^+((((\hbar_{\mathfrak{a}_z}{}^{\mathfrak{n}})') \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_z})$ and $\varphi^-((((\hbar_{\mathfrak{a}_z}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_y}) \vee (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \ge \varphi^+((((\hbar_{\mathfrak{a}_z}{}^{\mathfrak{n}})') \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_z})$. By hypothesis, $\varphi^+((((\hbar_{\mathfrak{a}_z}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_y}) \wedge (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \le \varphi^+(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_z})$ and $\varphi^-((((\hbar_{\mathfrak{a}_z}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_y}) \vee (\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_z})) \ge \varphi^-(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_z})$.

(2) ⇒ (3): Let $\hbar_{\mathfrak{a}_z} = \hbar_{\mathfrak{a}_y}$ in part (2). Then $\varphi^+(((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_y}) \wedge \varphi^+(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_y}) \le \varphi^+(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})$ and $\varphi^-(((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_y}) \vee \varphi^-(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_y}) \ge \varphi^-(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})$. By Proposition 1 (3), $\varphi^+(((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x}) \to \hbar_{\mathfrak{a}_y}) \le \varphi^+(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})$ and $\varphi^-(((\hbar_{\mathfrak{a}_y}{}^{\mathfrak{n}})' \odot \hbar_{\mathfrak{a}_x}) \to y) \ge \varphi^-(\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})$.

(3) $\Rightarrow$ (1): By Proposition 1 (3), $((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \odot ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \rightarrow \hbar_{\mathfrak{a}_x})) \rightarrow \hbar_{\mathfrak{a}_x} = 1$. Then $\varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \odot ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \rightarrow \hbar_{\mathfrak{a}_x})) \rightarrow \hbar_{\mathfrak{a}_x}) = \varphi^+(1)$ and $\varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \odot ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \rightarrow \hbar_{\mathfrak{a}_x})) \rightarrow \hbar_{\mathfrak{a}_x}) = \varphi^-(1)$. By part (3),

$$\varphi^+(1) = \varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \odot ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \rightarrow \hbar_{\mathfrak{a}_x})) \rightarrow \hbar_{\mathfrak{a}_x}) \leq \varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \rightarrow \hbar_{\mathfrak{a}_x}) \rightarrow \hbar_{\mathfrak{a}_x}),$$

$$\varphi^-(1) = \varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \odot ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \rightarrow \hbar_{\mathfrak{a}_x})) \rightarrow \hbar_{\mathfrak{a}_x}) \geq \varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \rightarrow \hbar_{\mathfrak{a}_x}) \rightarrow x).$$

Hence, $\varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \rightarrow \hbar_{\mathfrak{a}_x}) \rightarrow \hbar_{\mathfrak{a}_x}) = \varphi^+(1)$ and $\varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \rightarrow \hbar_{\mathfrak{a}_x}) \rightarrow \hbar_{\mathfrak{a}_x}) = \varphi^-(1)$. Therefore, $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy n-fold implicative filter of $\mathcal{H}_{\mathcal{A}}$. □

Theorem 6. *If $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold implicative filter of a bounded hoop $\mathcal{H}_{\mathcal{A}}$, then $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold fantastic filter.*

Proof. By Proposition 1 (4), $(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \leq \hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow \hbar_{\mathfrak{a}_y}$. Hence, $(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_y} \leq (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \rightarrow \hbar_{\mathfrak{a}_y}$ and $((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \rightarrow \hbar_{\mathfrak{a}_y}) = 1$. Then $((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \odot ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_y})) \rightarrow \hbar_{\mathfrak{a}_y} = 1$. So $\varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \odot ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_y})) \rightarrow \hbar_{\mathfrak{a}_y}) = \varphi^+(1)$ and $\varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \odot ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_y})) \rightarrow \hbar_{\mathfrak{a}_y}) = \varphi^-(1)$.

Since $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold implicative filter, by Lemma 1 (2),

$$\varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \odot ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_y})) \rightarrow \hbar_{\mathfrak{a}_y}) \wedge \varphi^+(\hbar_{\mathfrak{a}_y} \rightarrow \hbar_{\mathfrak{a}_x}) \leq \varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_x}),$$

$$\varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}})' \odot ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_y})) \rightarrow \hbar_{\mathfrak{a}_y}) \vee \varphi^-(\hbar_{\mathfrak{a}_y} \rightarrow \hbar_{\mathfrak{a}_x}) \geq \varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_x}).$$

Then $\varphi^+(\hbar_{\mathfrak{a}_y} \rightarrow \hbar_{\mathfrak{a}_x}) \leq \varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow x)$ and $\varphi^-(\hbar_{\mathfrak{a}_y} \rightarrow \hbar_{\mathfrak{a}_x}) \geq \varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_x})$.

By Theorem 4, $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold fantastic filter. □

Theorem 7. *Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy $\mathfrak{n}$-fold fantastic filter of $\mathcal{H}_{\mathcal{A}}$ such that $\hbar_{\mathfrak{a}}^2 = \hbar_{\mathfrak{a}}$ for all $\hbar_{\mathfrak{a}} \in \mathcal{H}_{\mathcal{A}}$. Then $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold implicative filter.*

Proof. Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy $\mathfrak{n}$-fold fantastic filter. Then

$$\varphi^+((\hbar_{\mathfrak{a}_x} \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_x}) \leq \varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow (\hbar_{\mathfrak{a}_x} \rightarrow \hbar_{\mathfrak{a}_y})) \rightarrow (\hbar_{\mathfrak{a}_x} \rightarrow \hbar_{\mathfrak{a}_y})) \rightarrow \hbar_{\mathfrak{a}_x})$$

and

$$\varphi^-((\hbar_{\mathfrak{a}_x} \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow \hbar_{\mathfrak{a}_x}) \geq \varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow (\hbar_{\mathfrak{a}_x} \rightarrow \hbar_{\mathfrak{a}_y})) \rightarrow (\hbar_{\mathfrak{a}_x} \rightarrow \hbar_{\mathfrak{a}_y})) \rightarrow \hbar_{\mathfrak{a}_x}).$$

Also,

$$((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \rightarrow (\hbar_{\mathfrak{a}_x} \rightarrow \hbar_{\mathfrak{a}_y})) \rightarrow (\hbar_{\mathfrak{a}_x} \rightarrow \hbar_{\mathfrak{a}_y})) \rightarrow \hbar_{\mathfrak{a}_x} = ((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \rightarrow \hbar_{\mathfrak{a}_y}) \rightarrow (x \rightarrow \hbar_{\mathfrak{a}_y})) \rightarrow \hbar_{\mathfrak{a}_x}.$$

Since $\hbar_{\mathfrak{a}_x}^2 = \hbar_{\mathfrak{a}_x}$, by Proposition 1 (3),

$$((\hbar_{\mathfrak{a}_x}^2 \to \hbar_{\mathfrak{a}_y}) \to (\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y})) \to \hbar_{\mathfrak{a}_x} = 1 \to \hbar_{\mathfrak{a}_x} = \hbar_{\mathfrak{a}_x}.$$

Hence,

$$\varphi^+((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) = \varphi^+((\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \leq \varphi^+(\hbar_{\mathfrak{a}_x})$$

and

$$\varphi^-((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) = \varphi^-((\hbar_{\mathfrak{a}_x} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \geq \varphi^-(\hbar_{\mathfrak{a}_x}).$$

Therefore, $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold implicative filter. □

Proposition 12. *If $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold fantastic filter and $y \odot (\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \leq \hbar_{\mathfrak{a}_x}$ and $\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \odot (((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \leq \hbar_{\mathfrak{a}_y}$ for all $\hbar_{\mathfrak{a}_x}, \hbar_{\mathfrak{a}_y} \in \mathcal{H}_{\mathcal{A}}$, then $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter.*

Proof. By hypothesis and Proposition 1 (7), $\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y} \leq \hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}$, then

$$\varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \leq \varphi^+(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}) \leq \varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}),$$

$$\varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}+1} \to \hbar_{\mathfrak{a}_y}) \geq \varphi^-(\hbar_{\mathfrak{a}_y} \to \hbar_{\mathfrak{a}_x}) \geq \varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}).$$

By Proposition 1 (7), $((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x} \leq \hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}$. Hence,

$$\varphi^+(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \leq \varphi^+(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to y)$$

and

$$\varphi^-(((\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_y}) \to \hbar_{\mathfrak{a}_x}) \geq \varphi^-(\hbar_{\mathfrak{a}_x}{}^{\mathfrak{n}} \to \hbar_{\mathfrak{a}_y}).$$

Therefore, by Proposition 3 (2), $\varphi = (\varphi^+, \varphi^-)$ is a bipolar fuzzy $\mathfrak{n}$-fold positive implicative filter. □

5 Conclusion and Future Work

In this study, we introduced the notion of *bipolar fuzzy n-fold fantastic filters* in hoop algebras, extending classical filter theory into a more expressive bipolar-valued setting. We provided formal definitions, established several characterizations, and examined the relationships between bipolar fuzzy n-fold fantastic filters and other known filter types. The results obtained underscore the robustness and generality of the proposed framework, demonstrating that classical fantastic filters emerge as special cases.

By incorporating bipolar fuzzy logic into the algebraic treatment of uncertainty, this work contributes to both the theoretical advancement of hoop algebras and their applicability in systems that accommodate both positive and negative information. The developed theory presents a more flexible algebraic tool for addressing dual-valued logical frameworks and complex fuzzy environments.

Several directions for future research are worth pursuing. First, extending the concept of bipolar fuzzy n-fold fantastic filters to other algebraic structuressuch as BL-algebras, residuated lattices, and pseudo-hoop algebrasmay yield fruitful generalizations. Second, exploring potential applications in approximate reasoning, bipolar decision-making models, and knowledge-based systems could further demonstrate the practical relevance of these filters. Lastly, the computational aspects, including the algorithmic detection and implementation of such filters in fuzzy control systems or logic programming environments, remain promising areas for further investigation.

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