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Karush-Kuhn-Tucker Necessary Optimality Conditions for $(h, \varphi)_\varepsilon$-Multiobjective Optimization Problems Based on Pseudo-Avriel-Ben-Tal Algebraic Operations

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Abstract

In this paper it is introduced a new generalized pseudo-operation with one parameter of the following form: $x \oplus_\varepsilon y = h^{-1}(h(x) + \varepsilon h(y))$, where h is an n vector-valued continuous function, defined on a subset H of $\mathbb{R}^n$ and possessing an inverse function h^{-1}, ε is an arbitrary but fixed positive real number. Five kinds of cones are introduced, which are used to establish the constraint qualifications. The generalized Karush-Kuhn-Tucker necessary optimality conditions are developed for a class of generalized $(h, \varphi)_\varepsilon$-differentiable single-objective programming problems and then for multiobjective programming problems, by using this generalized pseudo-operations, an extension of Avriel-Ben-Tal algebraic operations.The results obtained in this paper generalize and extend previous results obtained in this field. At the same time, in the final chapter, a crypto- graphic application using Ben-Tal type operators is presented.

INTRODUCTION

In mathematical programming involving differentiable functions, the Kuhn-Tucker conditions provide necessary conditions for an optimum, given certain

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qualifications on the constraints. A problem that continues to evoke very substantial interest is that of finding sufficient conditions for an optimum.

Many authors studied optimality conditions for vector optimization problems involving constraints are defined by single-valued mappings and obtained optimality conditions in terms of Lagrange-Kuhn-Tucker multipliers [3,4,8,11,12,18-23]. Some pseudo algebraic operations with applications can be found in [11].

Optimality conditions for various optimization problems are ever more, in particular, optimality sufficient conditions for a class (h, φ)-differentiable optimization problems [1,11, 26,27,28].

In [23-25], there have also been obtained significant contributions in the field of optimization problems, particularly regarding necessary and sufficient optimality conditions for various types of multiobjective programs involving locally Lipschitz functions, generalized invex functions, and generalized convex functions.

Preda has a important contribution in optimization problems, obtaining numerous results regarding necessary optimality conditions, sufficient optimality conditions and duality relations in different types of multitiobjective programs. In [12-14], Preda introduced some classes of Vunivex type-I functions, called (ρ, ρ' ')-V-univex type-I, (ρ, ρ')-quasi V-univex type-I, (ρ, ρ')pseudo V-univex type-I, (ρ, ρ')-quasi pseudo V-univex type-I, and (ρ, ρ')-pseudo quasi V-univex type-I. In [15] Preda introduced the class of locally Lipschitz (B, ρ, d) -preinvex functions and extended many results of B-vexity type stated in literature. Preda introduced (F, ρ)-convex function as extension of F-convex function and ρ-convex function[16,17]. The sufficient optimality conditions and duality results are obtained for nonlinear programming, generalized fractional programming, multi-objective programming and minmax programming problems, which involve these functions.

In [7], by using (h, φ) - generalized directional derivative and (h, φ) - generalized gradient, the authors directly derive the Karush-Kuhn-Tucker conditions by applying a corollary of Farkas's lemma under the MangasarianFromovitz constraint qualification and show the boundedness of Lagrange multipliers.

The results obtained by E. Pap in the domain of pseudo-analysis, and especially in solving of nonlinear equations (ODE, PDE, difference equations, etc.) using the pseudo linear principle[10], boosted us in the study of optimality necessary conditions for optimization problems based on pseudo-Avriel-Ben-Tal algebraic operations, case in which it is introduced a new generalized pseudo-operation with one parameter of the following form: $x_\varepsilon \oplus y = h^{-1}(\varepsilon h(x) + h(y))$, where h is an n vector-valued continuous function, defined on a subset H of R_n and possessing an inverse function h^{-1}, ε is a arbitrary but fixed positive real number and where there are obtained some Karush-Kuhn-Tucker optimality necessary conditions for a class of generalized $_{\varepsilon}(h, \varphi)$-differentiable single-objective programming problems [6].

The main aim of this paper is to study some optimality necessary conditions for optimization problems based on pseudo-Avriel-Ben-Tal algebraic operations. It is introduced a new generalized pseudo-operation with one parameter of the following form: $x \oplus_{\varepsilon} y = h^{-1}(h(x) + \varepsilon h(y))$, where h is an n vector-valued continuous function, defined on a subset H of R_n and possessing an inverse function h^{-1}, ε is an arbitrary but fixed positive real number. It is given the notion of $(h, \varphi)_{\varepsilon}$-differentiability of a function and the relation between $(h, \varphi)_{\varepsilon}$-generalized gradient and classic gradient . In order to establish the constraints qualifications, five kinds of cones are introduced. The generalized Karush-Kuhn-Tucker
optimality necessary conditions are derived for a class of generalized $(h, \varphi)_{\varepsilon}$-differentiable single and multi-objective programming problems by using these generalized pseudo-operations, an extension of Avriel-Ben-Tal algebraic operations. The results obtained in this paper generalize and extend the previously known results in this area [2,5,28].

Using Ben-Tal type operators and the structure of RANROT-type algorithms, it is developed a pseudo-random number generator with significant cryptographic properties that passed many of the NIST tests.

The paper is organized as follows. Section 1 contains preliminaries and related results that will be used to obtain the main results of the paper. In Section 2 it is introduced a new pseudo-operator on R_n and it is defined the concept of differentiable function, relative to the introduced operators. Constraint qualifications for single-objective problem are obtained in Section 3. KuhnTucker necessary conditions for $(h, \varphi)_{\varepsilon}$-differentiable single-objective programming optimization problems are derived in Section 4. Section 5 is dedicated to Kuhn-Tucker necessary conditions for $(h, \varphi)_{\varepsilon}$-differentiable multi-objective programming optimization problems. Section 6 presents a pseudo-random number generator that uses Ben-Tal's operators, along with the results of the tests performed on this generator.

1. PRELIMINARIES

Throughout the paper, R denotes the set of real numbers and R_k denotes the collection of k-dimensional real vectors, and it is written

$$R_k^+ = \left\{ (x_1, x_2, \ldots, x_k)^T \mid x_i \geq 0, \quad i = 1, 2, \ldots, k \right\};$$

$$R_k^{+0} = \left\{ (x_1, x_2, \ldots, x_k)^T \mid x_i \geq 0, \quad i = 1, 2, \ldots, k \right\}, \text{ there exists a least an } x_{i0} > 0 \Big\};$$

$$R_k^{++} = \left\{ (x_1, x_2, \ldots, x_k)^T \mid x_i \geq 0, \quad i = 1, 2, \ldots, k \right\};$$

Ben-Tal [5] introduced certain generalized operations of addition and multiplication:

1. Let h be an n vector-valued continuous function, defined on a subset H of R_n and possessing an inverse function h^{-1}. Define the h-vector addition of $x \in H$ and $y \in H$ as
$x \oplus y = h^{-1} \left(h\left(x\right) + h\left(y\right) \right)$,
and the h-scalar multiplication of $x \in H$and $\lambda \in R$ as
$\lambda \otimes y = h^{-1} \left(\lambda h\left(x\right) \right)$.

2. Let φ be a real-valued continuous functions, defined on $\Phi \subseteq R$ and possessing an inverse functions φ^{-1}.Then the φ-addition of two numbers, $\alpha \in \Phi$ and $\beta \in \Phi$, is given by $\alpha[+]\beta = \varphi^{-1}(\varphi(\alpha) + \varphi(\beta))$,
and the φ-scalar multiplication of $\alpha \in \Phi$ and $\lambda \in R$ by
$\lambda[\cdot]\alpha = \varphi^{-1}(\lambda\varphi(\alpha))$.

3. The (h, φ)-inner product of vectors $x, y \in H$ is defined as
$\left(x^T y\right)_{h,\varphi} = \varphi^{-1} \left(h(x)^T h(y) \right)$.
Denote

$$\left[\sum_{i=1}^{m} \right] \alpha_i = \alpha_1[+]\alpha_2[+] \ldots [+]\alpha_m, \quad \alpha_i \in \Phi, \quad i = 1, 2, \ldots, m;$$

$$\alpha[-]\beta = \alpha[+]((-1)[\cdot]\beta).$$

By Ben-Tal generalized algebraic operation, it is easy to obtain the following conclusions:

$$\left[\sum_{i=1}^{m} \right] \alpha_i = \varphi^{-1} \left(\sum_{i=1}^{m} \varphi\left(\alpha_i\right) \right) \tag{1.1}$$

$$\varphi(\lambda[\cdot]\alpha) = \lambda\varphi(\alpha)$$

$$h(\lambda \otimes x) = \lambda h(x) \tag{1.2}$$

Lemma 1.1 [28] Suppose $\varphi : R \to R$ is a continuous one-to-one strictly monotone and onto function, and $\alpha, \beta \in \Phi$. Then
$\alpha < \beta$ if and only if $\alpha[-]\beta < 0_\varphi$, where $0_\varphi = \varphi^{-1}(0)$.

2. A GENERALIZED PSEUDO-OPERATION. SOME LEMAS

It is introduced a new pseudo-operation of addition.
Let ε be arbitrary but fixed positive real number. Let h be an n vector-valued continuous function, defined on a subset H of R_n and possessing an inverse function h^{-1}. Define the left ε -h-vector addition of $x \in H$ and $y \in H$ as $x \oplus_\varepsilon y = h^{-1}(h(x) + \varepsilon h(y))$

Denote $\underset{i=1}{\oplus_{i=1}^m} x^i = x^1 \oplus_\varepsilon x^2 \oplus_\varepsilon \ldots \oplus_\varepsilon x^m, \quad x^i \in H, \quad i = 1, 2, \ldots, m.$
It is easy to obtain the following conclusion:

$$\bigoplus_{i=1}^{m} x^i = h^{-1}\left(h(x^1) + \varepsilon \sum_{i=2}^{m} h(x^i)\right) \tag{2.1}$$

Lemma 2.1 The following statements hold:
(i) $\lambda_1 \otimes x^1 \oplus_\varepsilon \frac{\lambda_2}{\varepsilon} \otimes x^2 \oplus_\varepsilon \ldots \oplus_\varepsilon \frac{\lambda_m}{\varepsilon} \otimes x^m = h^{-1}\left(\sum_{i=1}^m \lambda_i h\left(x^i\right)\right)$,
$x^i \in H, \lambda_i \in R$ for $i = 1, 2, \ldots, m$
(ii) $\lambda \otimes x \oplus_\varepsilon \mu \otimes x = (\lambda + \varepsilon\mu) \otimes x, \quad \lambda, \mu \in R, \quad x \in H$,
particularly, $x \oplus_\varepsilon \lambda \otimes x = (1 + \varepsilon\lambda) \otimes x$,
(iii) $x \oplus_\varepsilon \lambda \otimes d = h^{-1}(h(x) + \varepsilon\lambda h(d))$,
(iv) $\left[\sum_{i=1}^m\right] \mu_i [\cdot] \alpha_i = \varphi^{-1}\left(\sum_{i=1}^m \mu_i \varphi\left(\alpha_i\right)\right), \mu_i \in R, \alpha_i \in \Phi$, for $i = 1, 2, \ldots, m$,
(v) $\lambda \otimes (\mu \otimes x) = (\lambda\mu) \otimes x, \lambda, \mu \in R, \quad x \in H$,
(vi) $\lambda \otimes (x \otimes_\varepsilon y) = (\lambda \otimes x) \oplus_\varepsilon (\lambda \otimes y)$
Proof. It is only proved (i). One can similarly obtain (ii) - (vii).
Proof of (i)
$\lambda_1 \otimes x^1 \oplus_\varepsilon \frac{\lambda_2}{\varepsilon} \otimes x^2 \oplus_\varepsilon \ldots \oplus \frac{\lambda_m}{\varepsilon} \otimes x^m =$
$= h^{-1}\left(\lambda_1 h\left(x^1\right)\right) \oplus_\varepsilon h^{-1}\left(\frac{\lambda_2}{\varepsilon} h\left(x^2\right)\right) \oplus_\varepsilon \ldots \oplus_\varepsilon h^{-1}\left(\frac{\lambda_m}{\varepsilon} h\left(x^m\right)\right) =$
$= h^{-1}\left(\lambda_1 h\left(x^1\right) + \lambda_2 h\left(x^2\right) + \ldots + \lambda_m h\left(x^m\right)\right) = h^{-1}\left(\sum_{i=1}^m \lambda_i h\left(x^i\right)\right)$.
The relation relation between $(h, \varphi)_\varepsilon$-generalized directional derivative and Clarke directional derivative can be given by the following theorem.
Theorem 2.1 Let f be a real valued function, $\varphi(t)$ be strictly increasing and continuous on R, and let $\hat{f}(t) = \varphi\left(f\left(h^{-1}(t)\right)\right)$.
Then $f_\varepsilon^*(x; d) = \varphi^{-1}\left(\varepsilon \hat{f}^0(h(x), h(d))\right)$.

Proof: Note that

$$
\begin{aligned}
f_\varepsilon^*(x;d) &= \lim_{\substack{y\to x\\ \mu\setminus 0}} \sup \frac{1}{\mu}[\cdot]\left(f\left(y\oplus_\varepsilon \mu\otimes d\right)[-]f(y)\right)\\
&= \lim_{\substack{y\to x\\ \mu\nmid 0}} \sup \frac{1}{\mu}[\cdot]\varphi^{-1}\left(\varphi h^{-1}(h(y)+\varepsilon\mu h(d))-\varphi f(y)\right)\\
&= \lim_{\substack{y\to x\\ \mu\setminus 0}} \overset{-1}{\sup}\left(\frac{\varphi h^{-1}(h(y)+\varepsilon\mu h(d))-\varphi f(y)}{\mu}\right)\\
&= \lim_{\substack{y\to x\\ \mu\setminus 0}} \overset{-1}{\sup}\left(\frac{\hat f(h(y)+\varepsilon\mu h(d))-\hat f(h(y))}{\mu}\right)\\
&= \varphi^{-1}\left(\lim_{\substack{y\to x\\ \mu\setminus 0}} \sup \frac{\hat f(h(y)+\varepsilon\mu h(d))-\hat f(h(y))}{\varepsilon\mu}\cdot\varepsilon\right)\\
&= \varphi^{-1}\left(\varepsilon\hat f^0(h(x),h(d))\right).
\end{aligned}
$$

Therefore, it can be given a similar theorem as Theorem 2.1 about the relation between the generalized gradients.

Theorem 2.2 Let f be a real valued function, $\varphi(t)$ be strictly increasing and continuous on R, and let $\hat f(t)=\varphi\left(f\left(h^{-1}(t)\right)\right)$. Then

$$
\partial_\varepsilon^* f(x)=h^{-1}(\varepsilon\partial\hat f(h(x)))\triangleq\left\{h^{-1}(\xi)/\xi\in\varepsilon\partial(\hat f(t))_{|l=h(x)}\right\}.
$$

Proof:It can be proved only that $\partial_\varepsilon^* f(x)\subseteq h^{-1}(\varepsilon\partial\hat f(h(x)))$ since $\partial_\varepsilon^* f(x)\supseteq h^{-1}(\varepsilon\partial\hat f(h(x)))$ can be proved in the similar way. Let $\xi^*\in\partial_\varepsilon^* f(x)$, then $f_\varepsilon^*(x;d)\geq\left(\xi^{*T}d\right)_{(h,\varphi)_\varepsilon},\forall d\in R^n$.

According to Theorem 2.1, it follows that

$$
\varphi^{-1}\left(\varepsilon\hat f^0(h(x),h(d))\right)\geq\varphi^{-1}\left(h\left(\xi^*\right)^T h(d)\right),\forall d\in R^n
$$

or

$\varepsilon\hat f^0(h(x),h(d))\geq\left(h\left(\xi^*\right)^T h(d)\right),\forall d\in R^n.$

Taking $y=h(d)$ and noting that h is one to one mapping, it can be concluded that $\varepsilon\hat f^0(h(x),y)\geq\left(h\left(\xi^*\right)^T y\right),\forall d\in R^n.$

The last inequality shows that $h\left(\xi^*\right)\in\varepsilon\partial\hat f(h(x))$ or $\xi^*\in h^{-1}(\varepsilon\partial\hat f(h(x)))$ which complets the proof.

It is introduced the following concept, which plays an important role in this article.

Definition 2.1 Let f be a real-valued function defined on R_n, denote $\hat f(t)=\varphi\left(f\left(h^{-1}(t)\right)\right)$. For simplicity, write $\hat f(t)=\varphi f h^{-1}(t)$. The function

f is said to be $(h,\varphi)_\varepsilon$-differentiable at x, if $\hat{f}(t)$ is differentiable at $t=h(x)$. Denote $\nabla^*_\varepsilon f(x)=h^{-1}\left(\varepsilon\nabla\hat{f}(t)_{t=h(x)}\right)$.

In addition, f is differentiable on R_n, if and only if it is $(h,\varphi)_\varepsilon$-differentiable at x, where $h(t)=t$.

Lemma 2.2 The following assertions hold.

(i) Suppose f is $(h,\varphi)_\varepsilon$-differentiable at $\mathrm{x}^0, k\in R$. Then

$$\nabla^*_\varepsilon\left(k[\cdot]f\left(x^0\right)\right)=k\otimes\nabla^*_\varepsilon f\left(x^0\right)$$

(ii) Let $\mathrm{f_i}$ for $\mathrm{i}=1,2,\ldots,\mathrm{p}$ be $(h,\varphi)_\varepsilon-$ differentiable at x^0. Then

$$\nabla^*_\varepsilon\left(\left[\sum_{i=1}^p\right]f_i\left(x^0\right)\right)=\frac{1}{\varepsilon}\otimes\left(\varepsilon\otimes\nabla^*_\varepsilon f_1\left(x^0\right)\oplus_\varepsilon\nabla^*_\varepsilon f_2\left(x^0\right)\oplus_\varepsilon\ldots\oplus_\varepsilon\nabla^*_\varepsilon f_p\left(x^0\right)\right).$$

(iii) Assume f is $(h,\varphi)_\varepsilon-$ differentiable at $\mathrm{x}^0, c(x)=f(x)[-]f\left(x^0\right)$.Then $\nabla_\varepsilon{}^*c\left(x^0\right)=\nabla^*_\varepsilon f\left(x^0\right)$.

Proof:

(i) $\nabla^*_\varepsilon\left(k[\cdot]f\left(x^0\right)\right)=\nabla^*_\varepsilon\left(\varphi^{-1}\left(k\cdot\varphi f\left(x^0\right)\right)\right)=$
$=h^{-1}\left(\varepsilon\nabla\varphi\varphi^{-1}(k\cdot\varphi f)h^{-1}\left(t_0\right)\big|_{t_0=h(x^0)}\right)=$
$=h^{-1}\left(\varepsilon\nabla k\cdot\hat{f}\left(t_0\right)\Big|_{t_0=h(x^0)}\right)=h^{-1}\left(\varepsilon k\nabla\hat{f}\left(t_0\right)\Big|_{t_0=h(x^0)}\right)$.

(ii). Let $g(x)=\left[\sum_{i=1}^p\right]f_i(x)$.

Then $\varphi g(x)=\varphi\varphi^{-1}\left(\sum_{i=1}^p\varphi f_i(x)\right)=\sum_{i=1}^p\varphi f_i(x)$.

Writing $x=h^{-1}(t)$, it is obtained $\varphi gh^{-1}(t)=\sum_{i=1}^p\varphi f_ih^{-1}(t)$.

By hypotheses, it is concluded that φf_ih^{-1} for $i=1,2,\ldots,p$ are differentiable at $t_0=h\left(x^0\right)$, hence

$\nabla\varphi gh^{-1}\left(t_0\right)=\sum_{i=1}^p\nabla\varphi f_ih^{-1}\left(t_0\right)$.Thus, $\nabla\hat{g}\left(t_0\right)=\sum_{i=1}^p\nabla\hat{f}_i\left(t_0\right)$.

Multiplying the relation with ε it is obtained: $\varepsilon\nabla\hat{g}\left(t_0\right)=\sum_{i=1}^p\varepsilon\nabla\hat{f}_i\left(t_0\right)$. Therefore

$$hh^{-1}\left(\varepsilon\nabla\hat{g}\left(t_0\right)\right)=\sum_{i=1}^p hh^{-1}\left(\varepsilon\nabla\hat{f}_i\left(t_0\right)\right)\Rightarrow h\left(\nabla_\varepsilon{}^*g\left(x_0\right)\right)=\sum_{i=1}^p h\left(\nabla_\varepsilon{}^*f_i\left(x_0\right)\right).$$

Thus, $\nabla^*_\varepsilon g\left(x_0\right)=h^{-1}\left(\sum_{i=1}^p h\left(\nabla^*_\varepsilon f_i\left(x_0\right)\right)\right)$, which, together with 3.1 leads to: $\nabla_\varepsilon{}^*g\left(x_0\right)=\frac{1}{\varepsilon}\otimes\left(\varepsilon\otimes\nabla^*_\varepsilon f_1\left(x^0\right)\oplus_\varepsilon\nabla_\varepsilon{}^*f\left(x^2\right)\oplus_\varepsilon\ldots\oplus_\varepsilon\nabla^*_\varepsilon f\left(x^p\right)\right)$.

By Lemma 2.2 (i) and (ii), it is easy to obtain the following theorem, which characterizes the generalized linearity of $(h,\varphi)_\varepsilon$-differentiable operations.

Theorem 2.3 Suppose f_i for $i=1,2,\ldots,\mathrm{p}$ are $(h,\varphi)_\varepsilon$-differentiable at $x,\lambda\in R_p$ and $g(x)=\left[\sum_{i=1}^p\right]\lambda_i[\cdot]f_i(x)$.Then

$\nabla_\varepsilon{}^*g(x) = \frac{1}{\varepsilon} \otimes (\varepsilon\lambda_1 \otimes \nabla_\varepsilon{}^* f_1(x) \oplus_\varepsilon \lambda_2 \otimes \nabla_\varepsilon^* f_2(x) \oplus_\varepsilon \ldots \oplus_\varepsilon \lambda_p \otimes \nabla_\varepsilon^* f_p(x))$.

Proof:

Since $g(x) = [\sum_{i=1}^{p}] \lambda_1[\cdot] f_i(x)$, applying Lema 2.2. (ii) it is obtained:

$$\nabla_\varepsilon{}^*g(x) = \frac{1}{\varepsilon} \otimes \left(\varepsilon \otimes \nabla_\varepsilon{}^* \left(\lambda_1[\cdot]f_1(x)\right) \oplus_\varepsilon \nabla_\varepsilon{}^* \left(\lambda_2[\cdot]f_2(x)\right) \oplus_\varepsilon \ldots \oplus_\varepsilon \nabla_\varepsilon{}^* \left(\lambda_p[\cdot]f_p(x)\right)\right).$$

And from Lema 2.2 (i) it is obtained:

$$\nabla_\varepsilon^* g(x) = \frac{1}{\varepsilon} \otimes \left(\varepsilon \otimes \left(\lambda_1 \otimes \nabla_\varepsilon^* f_1(x)\right) \oplus_\varepsilon \left(\lambda_2 \otimes \nabla_\varepsilon^* f_2(x)\right) \oplus_\varepsilon \ldots \oplus_\varepsilon \left(\lambda_p \otimes \nabla_\varepsilon^* f_p(x)\right)\right) =$$
$$= \frac{1}{\varepsilon} \otimes \left(\varepsilon\lambda_1 \otimes \nabla_\varepsilon^* f_1(x) \oplus_\varepsilon \lambda_2 \otimes \nabla_\varepsilon^* f_2(x) \oplus_\varepsilon \ldots \oplus_\varepsilon \lambda_p \otimes \nabla_\varepsilon^* f_p(x)\right).$$

In the rest of the paper, it is further assumed that $h : R_n \to R_n$ there are continuous one-to-one and onto function. Similarly, suppose $\varphi : R \to R$ is a continuous one-to-one strictly monotone and onto function.

In the next section, we consider the constraint qualifications for single-objective programming problems with both inequality and equality constraints.

3. CONSTRAINT QUALIFICATIONS

Consider the following program:

(HFP) $\min f(x)$

$$\begin{aligned} \text{s.t. } & g_i(x) \leq 0_\varphi \text{ for } i = 1, 2, \ldots, m, \\ & h_j(x) = 0_\varphi \text{ for } j = 1, 2, \ldots, l, \end{aligned}$$

where $0_\varphi = \varphi^{-1}(0)$.

Throughout the remainder of this article let

$$X = \{x \in R_n \mid g_i(x) \leq 0_\varphi \text{ for } i = 1, 2, \ldots, m, h_j(x) = 0_\varphi \text{ for } j = 1, 2, \ldots, l\}$$

denote the feasible region of problem (HFP), and let

$$I(\bar{x}) = \{i \mid g_i(\bar{x}) = 0_\varphi, \quad i = 1, 2, \ldots, m\}$$

denote the set of generalized binding constraints, where $\bar{x} \in X$.

Definition 3.1 Let g_i for $i = 1, 2, \ldots, m$ and h_j for $j = 1, 2, \ldots, l$ be $(h, \varphi)_\varepsilon$-differentiable on $R_n, \bar{x} \in X$. The $(h, \varphi)_\varepsilon$ - cone of local constraint directions of X at $\bar{x}$ is defined by $Z^1_{h,\varphi}(X, \bar{x}) = \Big\{d \in R_n \mid \left(d^T \nabla_\varepsilon^* g_i(\bar{x})\right)_{h.\varphi} \leq 0$ for $i \in I(\bar{x})$ and $\left(d^T \nabla_\varepsilon{}^* h_i(\bar{x})\right)_{h.\varphi} \leq 0_\varphi$ for $j = 1, 2, \ldots, l\}$.

Each nonzero vector $d \in Z^1_{h,\varphi}(X, \bar{x})$ is called an $(h, \varphi)_\varepsilon$ - local constraint direction.
Similarly to the definition of the cone of feasible direction of S at x^0 ([3]), is introduced the following concept.
Definition 3.2 Let S be a nonempty set in R_n, and $x^0 \in \mathrm{cl}S$. The $(h, \varphi)_\varepsilon$ - cone of feasible directions of S ar x^0, denoted by $D_{h,\varphi}\left(S, x^0\right)$, is given by

$$D_{h,\varphi}\left(S, x^0\right) = \left\{d \mid x^0 \oplus_\varepsilon \lambda \otimes d \in S \text{ for all } \lambda \in (0, \delta) \text{ for some } \delta > 0\right\}.$$

Every nonzero vector $d \in D_{h,\varphi}\left(S, x^0\right)$ is said to be an $((h, \varphi)_\varepsilon$-feasible direction.
Remark 3.1 Each feasible direction of S at x_0 is an $(h, \varphi)_\varepsilon$-feasible direction of S at x_0, where $h(x) = x, x \in R$. However, the converse is not true, as is shown in the following.

Example 3.1 Let $S = \left\{(x_1, x_2)^T \mid x_2 \geq -x_1^3, x_1, x_2 \in R\right\}, x^0 = (0, 0)^T$, $h\left((x_1, x_2)^T\right) = \left(x_1, \sqrt[3]{x_2}\right)^T$, $d = (1, -1)^T$.
Then, $h(S) = \left\{(x_1, x_2)^T \mid x_2 \geq -x_1, x_1, x_2 \in R\right\}, h\left(x^0\right) = (0, 0)^T, h(d) = d$.
We can verify $d = (1, -1)^T$ is an $(h, \varphi)_\varepsilon$ - feasible direction of S as x_0, but it is not a feasible direction of S as x^0.

The relationship between the two cones defined above is characterized in the form of the following lemma:

Lemma 3.1 Let $\mathrm{g_i}$ for $i = 1, 2, \ldots, m$ and $\mathrm{h_j}$ for $j = 1, 2, \ldots, \mathrm{l}$ be $(h, \varphi)_\varepsilon$-differentiable on R_n and $\bar{x} \in X$. Then
$D_{h,\varphi}(X, \bar{x}) \subseteq Z^1_{h,\varphi}(X, \bar{x})$.
Proof: It is supposed that φ is strictly monotone decreasing on R. By this assumption the generality is not lost.
Let $d \in D_{(h,\varphi)_e}(X, \bar{x})$. It is necessary to show $d \in Z^1_{(h,\varphi\varphi_e}(X, \bar{x})$.
Assume to the contrary that $d \notin Z^1_{(h,\varphi)_e}(X, \bar{x})$.
From the definition of $Z^1_{(h,\varphi)_s}(X, \bar{x})$ one deduces that at least one of the two cases holds:
Case 1. There exists a $k \in I(\bar{x})$ such that $\left(d^T \nabla^*_\varepsilon g_k(\bar{x})\right)_{h,\varphi} > 0_\varphi$.
Case 2. There exists a $j \in (1, 2, \ldots, l)$ such that $\left(d^T \nabla_\varepsilon{}^* h_j(\bar{x})\right)_{h,\varphi} \neq 0_\varphi$.
For Case 1, the inequality $\left(d^T \nabla_{\varepsilon\varepsilon} g_k(\bar{x})\right)_{h,\varphi} > 0_\varphi$ gives
$\varphi^{-1}\left(h(d)^T h\left(\nabla^*_\varepsilon k(\bar{x})\right)\right) > \varphi^{-1}(0)$, which along with the strictly monotone decrease of φ leads to

$$h(d)^T h\left(\nabla^*_\varepsilon g_k(\bar{x})\right) > 0 \tag{3.1}$$

Since g_k is $(h, \varphi)_\varepsilon$-differentiable at $\bar{x}$, hence $\hat{g}_k(t) = \varphi g_k h^{-1}(t)$ is differentiable at $\bar{t} = h(\bar{x})$, thus

$$\hat{g}_k(\bar{t} + \varepsilon\theta h(d)) = \hat{g}_k(\bar{t}) + \varepsilon\theta h(d)^T \nabla \hat{g}_k(\bar{t}) + \varepsilon\theta\varepsilon_k(\theta) \tag{3.2}$$

It follows from (3.2) that

$$\varphi g_k h^{-1}(h(\bar{x}) + \varepsilon\theta h(d)) = \varphi g_k h^{-1}(h(\bar{x})) + \theta h(d)^T \varepsilon \nabla \hat{g}_k(\bar{t}) + \varepsilon\theta\varepsilon_k(\theta)$$
$$\varphi g_k h^{-1}(h(\bar{x}) + \varepsilon\theta h(d)) = \varphi g_k h^{-1}(h(\bar{x})) + \theta h(d)^T h h^{-1} (\varepsilon \nabla \hat{g}_k(\bar{t})) + \varepsilon\theta\varepsilon_k(\theta)$$

$$\varphi g_k h^{-1}(h(\bar{x}) + \varepsilon\theta h(d)) = \varphi g_k h^{-1}(h(\bar{x})) + \theta h(d)^T h (\nabla_\varepsilon^* g_k(\bar{x})) + \varepsilon\theta\varepsilon_k(\theta) \tag{3.3}$$

On the other hand, by $\bar{x} \in X$, it is obtained

$$\varphi g_k h^{-1}(h(\bar{x})) = \varphi g_k(\bar{x}) = \varphi(0_\varphi) = 0 \tag{3.4}$$

Substituting (3.4) into (3.3), one deduces that

$$\varphi g_k h^{-1}(h(\bar{x}) + \varepsilon\theta h(d)) = \theta h(d)^T h (\nabla_\varepsilon^* g_k(\bar{x})) + \varepsilon\theta\varepsilon_k(\theta)$$

Hence

$$\frac{\varphi g_k h^{-1}(h(\bar{x}) + \varepsilon\theta h(d))}{\theta} = h(d)^T h (\nabla_\varepsilon^* g_k(\bar{x})) + \varepsilon \cdot \varepsilon_k(\theta) \tag{3.5}$$

Since $\varepsilon_k(\theta) \to 0$ as $\theta \to 0^+$, it follows from (3.1) and (3.5) that
$\varphi g_k h^{-1}(h(\bar{x}) + \varepsilon\theta h(d)) < 0$ for a sufficiently small positive scalar θ.
By Lemma 2.1 (iii), it is obtained
$\varphi g_k (\bar{x} \oplus_\varepsilon \theta \otimes d) < 0$ for above θ.
Since φ is strictly monotone decreasing, one derives that
$g_k (\bar{x} \oplus_\varepsilon \theta \otimes d) > \varphi^{-1}(0) = 0_\varphi$ for a sufficiently small positive scalar θ.
This contradicts $d \in D_{h,\varphi}(X, \bar{x})$.
Therefore, Case 1 does not hold.
For Case 2, without loss of generality, , it is assumed that $\left(d^T \nabla_\varepsilon{}^* h_j(\bar{x})\right) > 0_\varphi$.
Similarly to the foregoing discussion, it is concluded that Case 2 doesnt hold either.
A summary of the above discussions leads to the validity of the lemma.
Analogously to the contingent cone defined in [2],a definition is given as follows:
Definition 3.3 Let K be a subset of R_n and x^0 belong to the closure of K. The $(h,\varphi)_\varepsilon-$ contingent cone $T_{h,\varphi}\left(K, x^0\right)$ is defined by $d \in T_{h,\varphi}\left(K, x^0\right)$ if and only if $\exists h_n \to 0^+$and $\exists d_n \to d$ such that $\forall n, x^0 \oplus_\varepsilon h_n \otimes d_n \in K$.

Remark 3.2 Let $h(x) = x, x \in R_n$. Then $T_{h,\varphi}\left(S, x^0\right) = T\left(S, x^0\right)$. However, $T_{h,\varphi}\left(S, x^0\right)$ is not necessarily contained in $T\left(S, x^0\right)$. Continuing the consideration of Example 3.1, it can be verified that $d \in T_{h,\varphi}\left(S, x^0\right)$, but $d \notin T\left(S, x^0\right)$.

Definition 3.4 [32] Let K be a nonempty subset of R_n. The $(h,\varphi)_\varepsilon$-positive polar cone $K^+_{h,\varphi}$of

K is defined by $K^+_{h,\varphi} = \left\{ d \in R_n \mid \left(d^T y\right)_{h,\varphi} \geq 0_\varphi, \forall y \in K \right\}$

Remark 3.3 The positive polar cone of K is the $(h,\varphi)_\varepsilon$ - positive polar cone $K^+_{h,\varphi}$of K with respect to $h(x) = x$. But the converse does not hold. In order to show this point, let us continue to consider Example 2.1, it can be verified that $S^+ = \varnothing$, but $(1,1)^T \in S^+_{h,\varphi}$.

Definition 3.5 Suppose f is $(h,\varphi)_\varepsilon$-differentiable on $R_n, \bar{x} \in X$. We shall say that

$$Z^2_{h,\varphi}(X, \bar{x}) = \left\{ d \in R_n \mid \left(d^T \nabla^*_\varepsilon f(\bar{x})\right)_{h,\varphi} < 0_\varphi \right\}$$

is the $(h,\varphi)_\varepsilon$ - cone of descent directions of f at $\bar{x}$.

We now present the Kuhn-Tucker constraint qualification of X at $\bar{X}$ that will be used to validate the Kuhn-Tucker necessary condition in the next section. Kuhn-Tucker constraint qualification: let g_i for $i = 1, 2, \ldots, m$ and h_j for $j = 1, 2, \ldots, l$ be $(h,\varphi)_\varepsilon-$ differentiable on R_n

and $\left[Z^1_{h,\varphi}(X, \bar{x})\right]^+_{h,\varphi} = \left[T_{h,\varphi}(X, \bar{x})\right]^+_{h,\varphi}$, where $\bar{x} \in X$.

4. NECESSARY CONDITIONS FOR $(h,\varphi)_\varepsilon$-SINGLE-OBJECTIVE PROGRAMMING

The programming problem (HFP) described in Section 3 is taken into consideration.

The $(h,\varphi)_\varepsilon$-Lagrangian function associated with (HFP) is given by

$$L_{h,\varphi}(x, u, v) = f(x)[+]\left[\sum_{i=1}^{m}\right] u_i[\cdot]g_i(x)[+]\left[\sum_{j=1}^{1}\right] v_j[\cdot]h_j(x) \qquad (4.1)$$

where $u \in R_n, v \in R_l$, whose components u_i for $i = 1, 2, \ldots, m$ and v_j for $j = 1, 2, \ldots, l$ are called the $(h,\varphi)_\varepsilon$ - Lagrangian multipliers.

The following lemma is needed later.

Lemma 4.1 Let f, g_i for $i = 1, 2, \ldots, m$ and h_j for $j = 1, 2, \ldots, l$ be $((h,\varphi)_\varepsilon$ - differentiable on $\mathrm{R_n}$, $\bar{x} \in X$, namely, $\bar{x}$ is a feasible solution for (HFP). Then $Z^1_{h,\varphi}(X, \bar{x}) \cap Z^2_{h,\varphi}(X, \bar{x}) = \varnothing$ implies that there exist vectors $\bar{u} \in R^+_m$and $\bar{v} \in R_l$ such that:

$\nabla_\varepsilon{}^* L_{h,\varphi}(\bar{x},\bar{u},\bar{v}) =$
$= \nabla_\varepsilon{}^* f(\bar{x}) \oplus_\varepsilon \frac{1}{\varepsilon^2} \otimes (\varepsilon\bar{u}_1 \otimes \nabla_\varepsilon{}^* g_1(\bar{x}) \oplus_\varepsilon \bar{u}_2 \otimes \nabla_\varepsilon{}^* g_2(\bar{x}) \oplus_\varepsilon \ldots \oplus_\varepsilon \bar{u}_m \otimes \nabla_{\varepsilon\varepsilon} g_m(\bar{x})) \oplus_\varepsilon$
$\oplus_\varepsilon \frac{1}{\varepsilon^2} \otimes \left(\varepsilon\bar{v}_1 \otimes \nabla_\varepsilon{}^* h_1(\bar{x}) \oplus_\varepsilon \bar{V}_2 \otimes \nabla_\varepsilon{}^* h_2(\bar{x}) \oplus_\varepsilon \ldots \oplus_\varepsilon \bar{v}_l \otimes \nabla_\varepsilon{}^* h_l(\bar{x})\right) = 0_h$
$u_i[\cdot]g_i(\bar{x}) = 0_\varphi$ for $i = 1, 2, \ldots, m$, where $0_h = h^{-1}(0)$.

Before proving the lemma, a Motzkin's alternative theorem is given:
Theorem 4.1[27] Let A be a nonzero $m \times n$ matrix, B be an $r \times n$ matrix and C be an $s \times n$ matrix. Then exactly one of the following two systems has a solution:

System 1: $Ax \in R_m^{++}, Bx \in R_r^{++}, Cx = 0$, for some $x \in R_n$.
System 2: $A^T u_1 + B^T u_2 + C^T u_3 = 0$, for some $u_1 \in R_m^{+0}, u_2 \in R_r^{+}.u_3 \in R_s$.
Lemma 4.1. is going to be proved.
Proof of Lemma 4.1: Without loss of generality, it is assumed that φ is strictly monotone decreasing on R. From $Z^1_{h,\varphi}(X,\bar{x}) \cap Z^2_{h,\varphi}(X,\bar{x}) = \varnothing$, it is concluded that the following system:
$\left(d^T \nabla^*_\varepsilon f(\bar{x})\right)_{h,\varphi} < 0_\varphi$,
$\left(d^T \nabla^*_\varepsilon g_i(\bar{x})\right)_{h,\varphi} \leq 0_\varphi$, for $i \in I(\bar{x})$
$\left(d^T \nabla^*_\varepsilon h_j(\bar{x})\right)_{h,\varphi} = 0_\varphi$, for $j = 1, 2, \ldots, l$
has no solution.
This, by the strictly monotone decrease of φ, is equivalent to the fact that the following system:
$h(d)^T h\left(\nabla^*_\varepsilon f(\bar{x})\right) > 0$
$h(d)^T h\left(\nabla^*_\varepsilon g_i(\bar{x})\right) \geq 0$, for $i \in I(\bar{x})$
$h(d)^T h\left(\nabla_\varepsilon{}^* h_j(\bar{x})\right) = 0$, for $j = 1, 2, \ldots, l$
is inconsistent.
Because $h : R_n \to R_n$ is a one-to-one and onto function, it does not exist a z satisfying

$$\begin{aligned} &z^T h\left(\nabla^*_\varepsilon f(\bar{x})\right) > 0 \\ &z^T h\left(\nabla^*_\varepsilon g_i(\bar{x})\right) \geq 0, \text{ for } i \in I(\bar{x}) \\ &z^T h\left(\nabla^*_\varepsilon h_j(\bar{x})\right) = 0, \text{ for } j = 1, 2, \ldots, l \end{aligned}$$

Denote $A = h\left(\nabla^*_\varepsilon f(\bar{x})\right)^T, B = \left(h\left(\nabla^*_\varepsilon g_i(\bar{x})\right)\right)^T_{i \in I(\bar{x})}$, in other words, B is a matrix whose rows are $h\left(\nabla_\varepsilon{}^* g_i(\bar{x})\right)^T$ for $i \in I(\bar{x})$, and C is a matrix whose rows are $h\left(\nabla^*_\varepsilon h_i(\bar{x})\right)^T$ for $j = 1, 2, \ldots, l$, namely,
$C = \left(h\left(\nabla^*_\varepsilon h_1(\bar{x})\right), h\left(\nabla_\varepsilon{}^* h_2(\bar{x})\right), \ldots, h\left(\nabla_\varepsilon{}^* h_l(\bar{x})\right)\right)^T$,

It follows from the above discussion that the system Az $\in R_1^{++}$, Bz $\in R^+_{I(\bar{x})}$, Cz $= 0$
is inconsistent, where $|I(\bar{x})|$ denotes the number of elements in $I(\bar{x})$. By Theorem 4.1, there exist $\alpha \in R_1^{+0}, u \in R^+_{I(\bar{x}),}, v \in R_l$ so that

$$\alpha h\left(\nabla_{\varepsilon}^{*} f(\bar{x})\right)+\sum_{i \in I(\bar{x})} u_{i} h\left(\nabla_{\varepsilon}^{*} g_{i}(\bar{x})\right)+\sum_{j=1}^{l} v_{j} h\left(\nabla_{\varepsilon}{}^{*} h_{j}(\bar{x})\right)=0 \tag{4.2}$$

According to the definition of R_1^{+0}, one has $\alpha > 0$. Division of (4.2) by α leads to

$$h\left(\nabla_{\varepsilon}^{*} f(\bar{x})\right)+\sum_{i \in I(\bar{x})} \bar{u}_{i} h\left(\nabla_{\varepsilon}^{*} g_{i}(\bar{x})\right)+\sum_{j=1}^{l} \bar{v}_{j} h\left(\nabla_{\varepsilon}^{*} h_{j}(\bar{x})\right)=0 \tag{4.3}$$

where $\bar{u}_i = \frac{u_i}{\alpha}$, $\bar{v}_i = \frac{v_i}{\alpha}$.

Letting h^{-1} act on (4.3), it is obtained

$$h^{-1}\left(h\left(\nabla_{\varepsilon}^{*} f(\bar{x})\right)+\sum_{i \in I(\bar{x})} \bar{u}_{i} h\left(\nabla_{\varepsilon}^{*} g_{i}(\bar{x})\right)+\sum_{j=1}^{l} \bar{v}_{j} h\left(\nabla_{\varepsilon}^{*} h_{j}(\bar{x})\right)\right)=h^{-1}(0) \tag{4.4}$$

Let $\bar{u}_i = 0$ for $i \notin I(\bar{x})$. Then we arrive at

$$h^{-1}\left(h\left(\nabla_{\varepsilon}^{*} f(\bar{x})\right)+\sum_{i=1}^{m} \bar{u}_{i} h\left(\nabla_{\varepsilon}^{*} g_{i}(\bar{x})\right)+\sum_{j=1}^{l} \bar{v}_{j} h\left(\nabla_{\varepsilon}^{*} h_{j}(\bar{x})\right)\right)=h^{-1}(0)$$

which, together with Lemma 2.1 (i) and (v) and $h^{-1}(0) = 0_h$, leads to

$$\begin{aligned}
h^{-1}\Bigg(& h\left(\nabla_{\varepsilon}^{*} f(\bar{x})\right)+h\Bigg(\frac{1}{\varepsilon} \otimes \Big(\varepsilon \bar{u}_{1} \otimes \nabla_{\varepsilon}^{*} g_{1}(\bar{x}) \oplus_{\varepsilon} \bar{u}_{2} \otimes \nabla_{\varepsilon}^{*} g_{2}(\bar{x}) \\
& \oplus_{\varepsilon} \ldots \oplus_{\varepsilon} \bar{u}_{m} \otimes \nabla_{\varepsilon}^{*} g_{m}(\bar{x})\Big)\Bigg) \\
& + h\Bigg(\frac{1}{\varepsilon} \otimes \Big(\varepsilon \bar{v}_{1} \otimes \nabla_{\varepsilon}^{*} h_{1}(\bar{x}) \oplus_{\varepsilon} \bar{v}_{2} \otimes \nabla_{\varepsilon}^{*} h_{2}(\bar{x}) \\
& \oplus_{\varepsilon} \ldots \oplus_{\varepsilon} \bar{v}_{l} \otimes \nabla_{\varepsilon}^{*} h_{l}(\bar{x})\Big)\Bigg)\Bigg)=0_{h} .
\end{aligned}$$

Thus,

$$\frac{1}{\varepsilon} \otimes \Bigg[(\varepsilon \otimes \nabla_\varepsilon^* f(\bar{x})) \oplus_\varepsilon \frac{1}{\varepsilon} \otimes \Big(\varepsilon \bar{u}_1 \otimes \nabla_\varepsilon^* g_1(\bar{x})$$

$$\oplus_\varepsilon \bar{u}_2 \otimes \nabla_\varepsilon^* g_2(\bar{x}) \oplus_\varepsilon \ldots \oplus_\varepsilon \bar{u}_m \otimes \nabla_\varepsilon^* g_m(\bar{x}) \Big)$$

$$\oplus_\varepsilon \frac{1}{\varepsilon} \otimes \Big(\varepsilon \bar{v}_1 \otimes \nabla_\varepsilon^* h_1(\bar{x}) \oplus_\varepsilon \bar{v}_2 \otimes \nabla_\varepsilon^* h_2(\bar{x})$$

$$\oplus_\varepsilon \ldots \oplus_\varepsilon \bar{v}_l \otimes \nabla_\varepsilon^* h_l(\bar{x}) \Big) \Bigg] = 0_h$$

So,

$$\nabla_\varepsilon^* f(\bar{x}) \oplus_\varepsilon \frac{1}{\varepsilon^2} \otimes \Big(\varepsilon \bar{u}_1 \otimes \nabla_\varepsilon^* g_1(\bar{x}) \oplus_\varepsilon \bar{u}_2 \otimes \nabla_\varepsilon^* g_2(\bar{x})$$

$$\oplus_\varepsilon \ldots \oplus_\varepsilon \bar{u}_m \otimes \nabla_\varepsilon^* g_m(\bar{x}) \Big) \oplus_\varepsilon \frac{1}{\varepsilon^2} \otimes \Big(\varepsilon \bar{v}_1 \otimes \nabla_\varepsilon^* h_1(\bar{x})$$

$$\oplus_\varepsilon \bar{v}_2 \otimes \nabla_\varepsilon^* h_2(\bar{x}) \oplus_\varepsilon \ldots \oplus_\varepsilon \bar{v}_l \otimes \nabla_\varepsilon^* h_l(\bar{x}) \Big) = 0_h$$

On the other hand,

$$\nabla_\varepsilon^* L_{h,\varphi}(\bar{x}, \bar{u}, \bar{v}) = \nabla_\varepsilon^* \left(f(\bar{x})[+] \sum_{i=1}^{m} \bar{u}_i[\cdot] g_i(\bar{x}) [+] \sum_{j=1}^{l} \bar{v}_j [\cdot] h_j(\bar{x}) \right)$$

$$= \frac{1}{\varepsilon} \otimes \left(\varepsilon \otimes \nabla_\varepsilon^* f(\bar{x}) \oplus_\varepsilon \nabla_\varepsilon^* \left(\sum_{i=1}^{m} \bar{u}_i[\cdot] g_i(\bar{x}) \right) \oplus_\varepsilon \nabla_\varepsilon^* \left(\sum_{j=1}^{l} \bar{v}_j [\cdot] h_j(\bar{x}) \right) \right)$$

$$= \frac{1}{\varepsilon} \otimes \Bigg(\varepsilon \otimes \nabla_\varepsilon^* f(\bar{x}) \oplus_\varepsilon \frac{1}{\varepsilon} \otimes \Big(\varepsilon \bar{u}_1 \otimes \nabla_\varepsilon^* g_1(\bar{x}) \oplus_\varepsilon \bar{u}_2 \otimes \nabla_\varepsilon^* g_2(\bar{x})$$

$$\oplus_\varepsilon \ldots \oplus_\varepsilon \bar{u}_m \otimes \nabla_\varepsilon^* g_m(\bar{x}) \Big) \oplus_\varepsilon \frac{1}{\varepsilon} \otimes \Big(\varepsilon \bar{v}_1 \otimes \nabla_\varepsilon^* h_1(\bar{x}) \oplus_\varepsilon \bar{v}_2 \otimes \nabla_\varepsilon^* h_2(\bar{x})$$

$$\oplus_\varepsilon \ldots \oplus_\varepsilon \bar{v}_l \otimes \nabla_\varepsilon^* h_l(\bar{x}) \Big) \Bigg)$$

$$= \nabla_\varepsilon^* f(\bar{x}) \oplus_\varepsilon \frac{1}{\varepsilon^2} \otimes \Big(\varepsilon \bar{u}_1 \otimes \nabla_\varepsilon^* g_1(\bar{x}) \oplus_\varepsilon \bar{u}_2 \otimes \nabla_\varepsilon^* g_2(\bar{x})$$

$$\oplus_\varepsilon \ldots \oplus_\varepsilon \bar{u}_m \otimes \nabla_\varepsilon^* g_m(\bar{x}) \Big) \oplus_\varepsilon \frac{1}{\varepsilon^2} \otimes \Big(\varepsilon \bar{v}_1 \otimes \nabla_\varepsilon^* h_1(\bar{x}) \oplus_\varepsilon \bar{v}_2 \otimes \nabla_\varepsilon^* h_2(\bar{x})$$

$$\oplus_\varepsilon \ldots \oplus_\varepsilon \bar{v}_l \otimes \nabla_\varepsilon^* h_l(\bar{x}) \Big)$$

Combining the definition of $I(\bar{x})$ and $\bar{u}_i$, one observes that $\bar{u}_i = 0$ or

$g_i(\bar{x}) = 0$ for each $i \in \{1, 2, \ldots, m\}$, which means $\bar{u}_i\varphi(g_i(\bar{x})) = 0$. Therefore $\bar{u}_i[\cdot]g_i(\bar{x}) = \varphi^{-1}(\bar{u}_i\varphi(g_i(\bar{x}))) = 0_\varphi$. Thus the proof is completed.

Lemma 4.2 Let f, g_i for $i = 1, 2, \ldots, n$ and h_j for $j = 1, 2, \ldots, l$ be $(h, \varphi)_\varepsilon$-differentiable on R_n, and suppose $\bar{x}$ is an optimal solution for (HFP). Then $\nabla_\varepsilon{}^* f(\bar{x}) \in [T_{h,\varphi}(X, \bar{x})]^+_{h,\varphi}$

Proof: It suffices to show that for an arbitrary vector $d \in [T_{h,\varphi}(X, \bar{x})]^+_{h,\varphi}$, one has $\left(d^T \nabla^*_\varepsilon f(\bar{x})\right)_{h,\varphi} \geq 0_\varphi$ Let $d \in [T_{h,\varphi}(X, \bar{x})]^+_{h,\varphi}$. Then there exist sequences $d_n \to d$ and $t_n \to 0^+$such that

$$\bar{x} \oplus_\varepsilon t_n \otimes d_n \in X \tag{4.5}$$

By the $(h, \varphi)_\varepsilon$-differentiability of f at $\bar{x}$, it is concluded that $\hat{f}(t) = \varphi f h^{-1}(t)$ is differentiable at $\bar{t} = h(\bar{x})$, hence

$$\hat{f}\left(\bar{t} + \varepsilon t_n h\left(d_n\right)\right) = \hat{f}(\bar{t}) + \varepsilon t_n h\left(d_n\right)^T \nabla \hat{f}(\bar{t}) + \varepsilon t_n \left\| h\left(d_n\right)\right\| \varepsilon_n$$

where $\varepsilon_n \to 0$ as $n \to \infty$.

In other words,

$$\varphi f h^{-1}\left(h(\bar{x}) + \varepsilon t_n h\left(d_n\right)\right) = \varphi f h^{-1}(h(\bar{x})) + t_n h\left(\nabla^*_\varepsilon f(\bar{x})\right)^T h\left(d_n\right) + \varepsilon t_n \left\| h\left(d_n\right)\right\| \varepsilon_n$$

Consequently, it follows from Lemma 2.1 (iii) that

$$\varphi f\left(\bar{x} \oplus_\varepsilon t_n \otimes d_n\right) = \varphi f(\bar{x}) + t_n h\left(\nabla^*_\varepsilon f(\bar{x})\right)^T h\left(d_n\right) + \varepsilon t_n \left\| h\left(d_n\right)\right\| \varepsilon_n \tag{4.6}$$

Hence

$$\frac{\varphi f\left(\bar{x} \oplus_\varepsilon t_n \otimes d_n\right) - \varphi f(\bar{x})}{t_n} = h\left(\nabla^*_\varepsilon f(\bar{x})\right)^T h\left(d_n\right) + \varepsilon \left\| h\left(d_n\right)\right\| \varepsilon_n \tag{4.7}$$

Since $\bar{x}$ is optimal for (HFP), by (4.5) one obtains $f\left(\bar{x} \oplus_\varepsilon t_n \otimes d_n\right) \geq f(\bar{x})$ Without loss of generality, assume φ is strictly monotone decreasing on R_n,so that

$$\varphi f\left(\bar{x} \oplus_\varepsilon t_n \otimes d_n\right) \leq \varphi f(\bar{x}) \tag{4.8}$$

Taking the limit in (4.7), since $t_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$, it follows from (4.8) that $h\left(\nabla^*_\varepsilon f(\bar{x})\right)^T h(d) \leq 0$, which together with the strictly monotone decrease of φ leads to $\varphi^{-1}\left(h\left(\nabla^*_\varepsilon f(\bar{x})\right)^T h(d)\right) \geq \varphi^{-1}(0)$. Namely $\left(\nabla^*_\varepsilon f(\bar{x})^T d\right)_{h,\varphi} \geq 0_\varphi$. Thus
$\nabla_\varepsilon{}^* f(\bar{x}) \in [T_{h,\varphi}(X, \bar{x})]^+_{h,\varphi}$.

In the remainder of this section, it is presented the necessary condition for a

feasible $\bar{x}$ to be optimal for (HFP) in the form of the following theorem.
Theorem 4.2. Let the hypotheses of Lemma 4.2 be satisfied, and

$$\left[Z^1_{h,\varphi}(X,\bar{x})\right]^+_{h,\varphi} = \left[T_{h,\varphi}(X,\bar{x})\right]^+_{h,\varphi}.$$

Then there exist vectors $\bar{u} \in R^+_m$ and $\bar{v} \in R_l$ so that

$$\begin{aligned}\nabla^*_\varepsilon L_{h,\varphi}(\bar{x},\bar{u},\bar{v}) = \nabla^*_\varepsilon f(\bar{x}) \oplus_\varepsilon \frac{1}{\varepsilon^2} \otimes \Big(&\varepsilon \bar{u}_1 \otimes \nabla^*_\varepsilon g_1(\bar{x}) \oplus_\varepsilon \bar{u}_2 \otimes \nabla^*_\varepsilon g_2(\bar{x}) \\ &\oplus_\varepsilon \ldots \oplus_\varepsilon \bar{u}_m \otimes \nabla^*_\varepsilon g_m(\bar{x})\Big) \oplus_\varepsilon \frac{1}{\varepsilon^2} \otimes \Big(\varepsilon \bar{v}_1 \otimes \nabla^*_\varepsilon h_1(\bar{x}) \\ &\oplus_\varepsilon \bar{v}_2 \otimes \nabla^*_\varepsilon h_2(\bar{x}) \oplus_\varepsilon \ldots \oplus_\varepsilon \bar{v}_l \otimes \nabla^*_\varepsilon h_l(\bar{x})\Big) = 0_h,\end{aligned}$$

and

$$u_i[\cdot]g_i(\bar{x}) = 0_\varphi, \quad \text{for } i = 1, 2, \ldots, m,$$

where $0_h = h^1(0)$.

Proof. It follows from Theorem 2.1 that

$$\begin{aligned}\nabla^*_\varepsilon L_{h,\varphi}(\bar{x},\bar{u},\bar{v}) = \nabla^*_\varepsilon f(\bar{x}) \oplus_\varepsilon \frac{1}{\varepsilon^2} \otimes \Big(&\varepsilon \bar{u}_1 \otimes \nabla^*_\varepsilon g_1(\bar{x}) \oplus_\varepsilon \bar{u}_2 \otimes \nabla^*_\varepsilon g_2(\bar{x}) \\ &\oplus_\varepsilon \ldots \oplus_\varepsilon \bar{u}_m \otimes \nabla^*_\varepsilon g_m(\bar{x})\Big) \oplus_\varepsilon \frac{1}{\varepsilon^2} \otimes \Big(\varepsilon \bar{v}_1 \otimes \nabla^*_\varepsilon h_1(\bar{x}) \\ &\oplus_\varepsilon \bar{v}_2 \otimes \nabla^*_\varepsilon h_2(\bar{x}) \oplus_\varepsilon \ldots \oplus_\varepsilon \bar{v}_l \otimes \nabla^*_\varepsilon h_l(\bar{x})\Big).\end{aligned}$$

By Lemma 4.2 and the hypothesis

$$\left[Z^1_{h,\varphi}(X,\bar{x})\right]^+_{h,\varphi} = \left[T_{h,\varphi}(X,\bar{x})\right]^+_{h,\varphi},$$

it is derived that

$$\nabla^*_\varepsilon f(\bar{x}) \in \left[Z^1_{h,\varphi}(X,\bar{x})\right]^+_{h,\varphi}.$$

Hence, for any $d \in Z^1_{h,\varphi}(X,\bar{x})$, it holds that

$$\left(d^T \nabla^*_\varepsilon f(\bar{x})\right)_{h,\varphi} \geq 0_\varphi.$$

Consequently,

$$Z^1_{h,\varphi}(X,\bar{x}) \cap Z^2_{h,\varphi}(X,\bar{x}) = \varnothing.$$

Thus, the proof from Lemma 4.1 is concluded.

5. NECESSARY CONDITIONS FOR $(h,\varphi)_\varepsilon$-MULTI-OBJECTIVE PROGRAMMING

Consider that vectorial optimization problem is represented as follows:
(VHFP) $\min f(x) = (f_1(x), f_2(x), \ldots, f_p(x))^T$

$$\text{s.t. } g_i(x) \leq 0_\varphi \text{ for } i = 1, 2, \ldots, m \tag{5.1}$$

$$h_j(x) = 0_\varphi \text{ for } j = 1, 2, \ldots, l$$

where $0_\varphi = \varphi^{-1}(0)$.
Obviously, the feasible region for (VHFP) is still X.
Define the $(h,\varphi)_\varepsilon$-Lagrangian function for (VHPF) as
$VL_{h,\varphi}(x,\lambda,u,v) = \left[\sum_{i=1}^{p}\right] \lambda_i[\cdot]f_i(x)[+]\left[\sum_{j=1}^{m}\right] u_j[\cdot]g_j(x)[+]\left[\sum_{k=1}^{l}\right] v_k[\cdot]h_k(x)$
where $\lambda \in R_p, u \in R_m, v \in R_l$.
Let $\bar{x} \in X$, denote the single-objective program associated with (VHFP) at $\bar{x}$ by (SVHFP) ($\bar{x}$), which is represented as follows:

$$\begin{aligned} (\text{SVHFP})(\bar{x}) \min q(x) = \left[\sum_{i=1}^{p}\right] & f_i(x) \\ \text{s.t. } & g_j(x) \leq 0_\varphi \text{ for } j = 1, 2, \ldots, m \\ & h_k(x) = 0_\varphi \text{ for } k = 1, 2, \ldots, l, \\ & f_i(x)[-1]f_i(\bar{x}) \leq 0_\varphi \text{ for } i = 1, 2, \ldots, p \end{aligned} \tag{5.2}$$

Where $0_\varphi = \varphi^{-1}(0)$.
Let SXV denote the feasible region for $(SVHFP)(\bar{x})$. Evidently, $SXV \subseteq X$
First recall the concept below, which is needed later.
A solution $\bar{x}$ is efficient for (VHFP) if and only if there is no other feasible x for (VHFP) such that, for some $i_0 \in \{i = 1, 2, \ldots, p\}, f_{i0}(x) < f_{i0}(\bar{x}), \forall i \neq i_0, f_i(x) < f_i(\bar{x})$.

At this point, it is presented the following lemma, which will be invoked to prove Theorem 5.1.

Lemma 5.1[28] Let $\bar{x} \in X$. If $\bar{x}$ is an efficient solution for (VHFP), then $\bar{x}$ is optimal for (SVHFP)($\bar{x}$).

Theorem 5.1 Let f_i for $i = 1, 2, \ldots, p, g_j$ for $j = 1, 2, \ldots, m, h_k$ for $k = 1, 2, \ldots, l$ be $(h,\varphi)_\varepsilon-$ differentiable on $R_n, \bar{x} \in X, \bar{x}$, be sufficient for (VHFP) and $\left[Z^1_{h,\varphi}(SVX,\bar{x})\right]^+_{h,\varphi} = [T_{h,\varphi}(SVX,\bar{x})]^+_{h,\varphi}$. Then there exist vectors $\bar{\lambda} \in R_p^{++}, \bar{u} \in R_m^+$and $\bar{v} \in R_l$ such that

$$\begin{aligned}&\frac{1}{\varepsilon}\otimes\left(\varepsilon\bar{\lambda}_1\otimes\nabla_\varepsilon^* f_1(\bar{x})\oplus_\varepsilon\bar{\lambda}_2\otimes\nabla_\varepsilon^* f_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon\bar{\lambda}_p\otimes\nabla_\varepsilon^* f_p(\bar{x})\right)\oplus_\varepsilon\\&\oplus_\varepsilon\frac{1}{\varepsilon}\otimes(\varepsilon\bar{u}_1\otimes\nabla_\varepsilon^* g_1(\bar{x})\oplus_\varepsilon\bar{u}_2\otimes\nabla_\varepsilon^* g_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon\bar{u}_m\otimes\nabla_\varepsilon^* g_m(\bar{x}))\oplus_\varepsilon\\&\oplus_\varepsilon\frac{1}{\varepsilon}\otimes(\varepsilon\bar{v}_1\otimes\nabla_\varepsilon^* h_1(\bar{x})\oplus_\varepsilon\bar{v}_2\otimes\nabla_\varepsilon^* h_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon\bar{v}_l\otimes\nabla_\varepsilon^* h_l(\bar{x}))=0_h\end{aligned}\tag{5.3}$$

$\bar{u}_j[\cdot]g_i(\bar{x})=0_\varphi$ for $j=1,2,\ldots,m.$

Proof: Since $\bar{x}$ is efficient for (VHFP), from Lemma 5.1 it is deduced that it is an optimal solution for (SVHFP) $(\bar{x})$. Hence by hypothesis $\left[Z^1_{h,\varphi}(SVX,\bar{x})\right]^+_{h,\varphi}=$ $[T_{h,\varphi}(SVX,\bar{x})]^+_{h,\varphi}$and Theorem 3.2, one concludes that there exist vectors $\lambda\in R_p^{++}, u\in R_m^+$and $v\in R_l$

$$\begin{aligned}&\nabla_\varepsilon^* q(\bar{x})\oplus_\varepsilon\frac{1}{\varepsilon^2}\otimes(\varepsilon\lambda_1\otimes\nabla_\varepsilon^* c_1(\bar{x})\oplus_\varepsilon\lambda_2\otimes{\nabla_\varepsilon}^* c_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon\lambda_p\otimes\nabla_\varepsilon^* c_p(\bar{x}))\oplus_\varepsilon\\&\oplus_\varepsilon\frac{1}{\varepsilon^2}\otimes(\varepsilon u_1\otimes\nabla_\varepsilon^* g_1(\bar{x})\oplus_\varepsilon u_2\otimes\nabla_\varepsilon^* g_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon u_m\otimes\nabla_\varepsilon^* g_m(\bar{x}))\oplus_\varepsilon\\&\oplus_\varepsilon\frac{1}{\varepsilon^2}\otimes(\varepsilon v_1\otimes\nabla_\varepsilon^* h_1(\bar{x})\oplus_\varepsilon v_2\otimes\nabla_\varepsilon^* h_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon v_l\otimes\nabla_\varepsilon^* h_l(\bar{x}))=0_h\end{aligned}\tag{5.4}$$

$$\begin{aligned}\lambda_i[\cdot]c_i(\bar{x})=0_\varphi \text{ for } i=1,2,\ldots,p\\ u_j[\cdot]g_i(\bar{x})=0_\varphi \text{ for } j=1,2,\ldots,m\end{aligned}\tag{5.5}$$

where $q(x)=\left[\sum_{i=1}^p\right]f_i(x), c_i(x)=f_i(x)[-1]f_i(\bar{x}), i=1,2,\ldots,p.$
It follows from Lemma 2.2 (ii) that

$$\nabla_\varepsilon^* q(x)=\frac{1}{\varepsilon}\otimes(\varepsilon\otimes\nabla_\varepsilon^* f_1(\bar{x})\oplus_\varepsilon\nabla_\varepsilon^* f_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon{\nabla_\varepsilon}^* f_p(\bar{x}))\tag{5.6}$$

By Lemma 2.2 (iii), it is obtained

$$\nabla_\varepsilon^* c_i(\bar{x})=\nabla_\varepsilon^* f_i(\bar{x})\tag{5.7}$$

Substituting (5.6) and (5.7) into (5.4), it is obtained

$$\begin{aligned}
&\frac{1}{\varepsilon}\otimes(\varepsilon\otimes\nabla_\varepsilon^* f_1(\bar{x})\oplus_\varepsilon\nabla_\varepsilon^* f_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon\nabla_\varepsilon^* f_p(\bar{x}))\oplus_\varepsilon\\
&\oplus_\varepsilon\frac{1}{\varepsilon^2}\otimes(\varepsilon\lambda_1\otimes\nabla_\varepsilon^* f_1(\bar{x})\oplus_\varepsilon\lambda_2\otimes\nabla_\varepsilon^* f_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon\lambda_p\otimes\nabla_\varepsilon^* f_p(\bar{x}))\oplus_\varepsilon\\
&\oplus_\varepsilon\frac{1}{\varepsilon^2}\otimes(\varepsilon u_1\otimes\nabla_\varepsilon^* g_1(\bar{x})\oplus_\varepsilon u_2\otimes\nabla_\varepsilon^* g_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon u_p\otimes\nabla_\varepsilon^* g_p(\bar{x}))\oplus_\varepsilon\\
&\oplus_\varepsilon\frac{1}{\varepsilon^2}\otimes(\varepsilon v_1\otimes\nabla_\varepsilon^* h_1(\bar{x})\oplus_\varepsilon v_2\otimes\nabla_\varepsilon^* h_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon v_l\otimes\nabla_\varepsilon^* h_l(\bar{x}))=0_h
\end{aligned}\tag{5.8}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{\varepsilon}\otimes\Bigg[\Big(\varepsilon\otimes\nabla_\varepsilon^* f_1(\bar{x})\oplus_\varepsilon\nabla_\varepsilon^* f_2(\bar{x})\oplus_\varepsilon\ldots\\
&\qquad\oplus_\varepsilon\nabla_\varepsilon^* f_p(\bar{x})\Big)\oplus_\varepsilon\frac{1}{\varepsilon^2}\otimes\Big(\varepsilon\lambda_1\otimes\nabla_\varepsilon^* f_1(\bar{x})\oplus_\varepsilon\lambda_2\otimes\nabla_\varepsilon^* f_2(\bar{x})\\
&\qquad\oplus_\varepsilon\ldots\oplus_\varepsilon\lambda_p\otimes\nabla_\varepsilon^* f_p(\bar{x})\Big)\Bigg]\\
&=\frac{1}{\varepsilon}\otimes\Bigg[\Big(\varepsilon\otimes\nabla_\varepsilon^* f_1(\bar{x})\oplus_\varepsilon\nabla_\varepsilon^* f_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon\nabla_\varepsilon^* f_p(\bar{x})\Big)\oplus_\varepsilon\\
&\Big(\lambda_1\otimes\nabla_\varepsilon^* f_1(\bar{x})\oplus_\varepsilon\frac{\lambda_2}{\varepsilon}\otimes\nabla_\varepsilon^* f_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon\frac{\lambda_p}{\varepsilon}\otimes\nabla_\varepsilon^* f_p(\bar{x})\Big)\Bigg]\\
&=\frac{1}{\varepsilon}\otimes\Big(\varepsilon(1+\lambda_1)\otimes\nabla_\varepsilon^* f_1(\bar{x})\oplus_\varepsilon(1+\lambda_2)\otimes\nabla_\varepsilon^* f_2(\bar{x})\\
&\qquad\oplus_\varepsilon\ldots\oplus_\varepsilon(1+\lambda_p)\otimes\nabla_\varepsilon^* f_p(\bar{x})\Big)
\end{aligned}$$

A substitution of (5.9) into (5.8) yields:

$$\begin{aligned}
&\frac{1}{\varepsilon}\otimes\Big(\varepsilon\bar{\lambda}_1\otimes\nabla_\varepsilon^* f_1(\bar{x})\oplus_\varepsilon\bar{\lambda}_2\otimes\nabla_\varepsilon^* f_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon\bar{\lambda}_p\otimes\nabla_\varepsilon^* f_p(\bar{x})\Big)\\
&\oplus_\varepsilon\frac{1}{\varepsilon}\otimes\Big(\varepsilon u_1\otimes\nabla_\varepsilon^* g_1(\bar{x})\oplus_\varepsilon u_2\otimes\nabla_\varepsilon^* g_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon u_m\otimes\nabla_\varepsilon^* g_p(\bar{x})\Big)\\
&\oplus_\varepsilon\frac{1}{\varepsilon}\otimes\Big(\varepsilon v_1\otimes\nabla_\varepsilon^* h_1(\bar{x})\oplus_\varepsilon v_2\otimes\nabla_\varepsilon^* h_2(\bar{x})\oplus_\varepsilon\ldots\oplus_\varepsilon v_l\otimes\nabla_\varepsilon^* h_l(\bar{x})\Big)=0_h
\end{aligned}$$

But

$$\begin{aligned}
\bar{\lambda}_i&=1+\lambda_i, &&\text{for } i=1,2,\ldots,p,\\
\bar{u}_j&=u_j, &&\text{for } j=1,2,\ldots,m,\\
\bar{v}_k&=v_k, &&\text{for } k=1,2,\ldots,l.
\end{aligned}$$

Consequently, $\lambda \in R_p^{++}, \bar{u} \in R_m^{+}$and $\bar{v}_k \in R_l$ and

$$\frac{1}{\varepsilon} \otimes \left(\varepsilon\bar{\lambda}_1 \otimes \nabla_\varepsilon^* f_1(\bar{x}) \oplus_\varepsilon \bar{\lambda}_2 \otimes \nabla_\varepsilon^* f_2(\bar{x}) \oplus_\varepsilon \ldots \oplus_\varepsilon \bar{\lambda}_p \otimes \nabla_\varepsilon^* f_p(\bar{x})\right) \oplus_\varepsilon$$
$$\oplus_\varepsilon \frac{1}{\varepsilon} \otimes (\varepsilon\bar{u}_1 \otimes \nabla_\varepsilon^* g_1(\bar{x}) \oplus_\varepsilon \bar{u}_2 \otimes \nabla_\varepsilon^* g_2(\bar{x}) \oplus_\varepsilon \ldots \oplus_\varepsilon \bar{u}_m \otimes \nabla_\varepsilon^* g_m(\bar{x})) \oplus_\varepsilon, \tag{5.10}$$
$$\oplus_\varepsilon \frac{1}{\varepsilon} \otimes (\varepsilon\bar{v}_1 \otimes \nabla_\varepsilon^* h_1(\bar{x}) \oplus_\varepsilon \bar{v}_2 \otimes \nabla_\varepsilon^* h_2(\bar{x}) \oplus_\varepsilon \ldots \oplus_\varepsilon \bar{v}_l \otimes \nabla_\varepsilon^* h_l(\bar{x})) = 0_h$$

Combining (5.5) and (5.6), the proof is completed.

6. Cryptographic application with Ben-Tal type operators

In 1997, Danish researcher Agner Fog proposed a new class of pseudorandom number generators called RANROT, also based on the Fibonacci recurrence [8]. The innovation introduced by Agner Fog consisted in perturbing the LFG generator by employing circular bitwise rotation operations. Within this new class, four types of generators are defined, as presented below:

$$\begin{aligned}
\text{Type A:}\quad & x_n = \left((x_{n-j} + x_{n-k}) \bmod 2^b\right) \operatorname{rotr} r \\
\text{Type B:}\quad & x_n = ((x_{n-j} \operatorname{rotr} r_1) + (x_{n-k} \operatorname{rotr} r_2)) \bmod 2^b \\
\text{Type B}_3:\quad & x_n = ((x_{n-i} \operatorname{rotr} r_1) + (x_{n-j} \operatorname{rotr} r_2) + (x_{n-k} \operatorname{rotr} r_3)) \bmod 2^b \\
\text{Type W:}\quad & z_n = ((y_{n-j} \operatorname{rotr} r_3) + (y_{n-k} \operatorname{rotr} r_1)) \bmod 2^{b/2} \\
& y_n = ((z_{n-j} \operatorname{rotr} r_4) + (z_{n-k} \operatorname{rotr} r_2)) \bmod 2^{b/2} \\
& x_n = y_n + z_n \cdot 2^{b/2}
\end{aligned} \tag{6.1}$$

where x_n is an unsigned integer represented on b bits (usually the length of the memory word), y_n and z_n are unsigned integers represented on $\frac{b}{2}$ bits , i, j and k are also unsigned integers with the property $0 < i < j < k$ and $r_1,\ r_2,\ r_3,\ r_4$ are circular rotation parameters.

6.1 Construction of the RANROT BEN-TAL A generator

Consider the Ranrot A type generator.Within this generator, instead of the classical addition operation, it is used the Ben-Tal addition operation.The basic formula of RANROT BEN-TAL A uses the following rule for updating the internal state:

$$x_n = \left((x_{n-j} \oplus x_{n-k}) \bmod 2^b\right) \text{rotr } r \tag{6.2}$$

where:

- x_n is the current state and is an unsigned integer represented on b bits.
- k and j are indices in the internal state.
- r is the number of bits for the circular rotation.
- *rotr* is the operation of rotation to the right.
- $\oplus$ is the Ben-Tal addition operator.

Since the chosen operator uses an invertible function, it is imperative that the selected function satisfies this condition.The choice of the invertible function is arbitrary. Initially, the functions are considered as simple as possible, such as the linear function. The complexity level of these functions can be increased subsequently.It is considered the addition operator of the following form:

$$x \oplus y = h^{-1}(h(x) + h(y)) \tag{6.3}$$

where $h:\mathbb{Z} \to \mathbb{Z}$, $h(x) = mx + n$, $h^{-1}(x)=\left[\frac{n-x}{m}\right]$, $[x]$ represents the floor of x.

6.2 Testing the RANROT BEN-TAL generator with linear function

The generator was subjected to a battery of NIST tests [22] under two conditions: testing the classical RANROT A generator and testing the RANROT BEN-TAL A generator with a linear function.

The classical RANROT A generator failed several tests, indicating possible non-randomness. The poorest results occurred in the Frequency, BlockFrequency, Cumulative Sums, Runs, FFT, NonOverlappingTemplate, ApproximateEntropy, and Serial tests, with very low p-values (< 0.01), indicating significant deviations from randomness, possible repetitive structures in sequences, and the presence of specific patterns.

Although the generator passed some tests such as LongestRun, Rank, Random Excursions, and Linear Complexity, the extremely low values obtained in the basic tests of uniformity, dependence, correlation, and specific pattern tests render this generator extremely weak.

Following the testing of the RANROT BEN-TAL A generator with a linear function, the generator passed the Frequency test (p=0.333754, 1979 out of 2000 sequences passed). However, the failure of the BlockFrequency test indicates an unbalanced distribution of ones within blocks.

Tests that passed include Longest Run of Ones (p=0.414525), Rank Test (p=0.939005), Random Excursions and Random Excursions Variant (most p-values acceptable), and Linear Complexity (p=0.389855).

Failures in Block Frequency, Runs, FFT, and Approximate Entropy suggest detectable structures and the lack of a completely random distribution.

The NonOverlapping Template test shows some critical values, although fewer failures than Ranrot A.

Conclusion:

In the present paper, it is introduced a new generalized pseudo-operation and with the generalized operators introduced by Ben-Tal it is obtained the relation between $(h, \varphi)_\varepsilon$ - generalized directional derivative and Clarke directional derivative and the relation between the generalized gradients. Five kinds of cones are introduced, which are used to establish the constraints qualifications and the generalized Karush-Kuhn-Tucker optimality necessary conditions are developed for a class of generalized $(h, \varphi)_\varepsilon$-differentiable single-objective programming problems and then for multiobjective programming problems,by using this generalized pseudo-operations, an extension of Avriel-Ben-Tal algebraic operations.

By Theorem 3.1 and (6.10), there can be obtained $\nabla_{\varepsilon,x}^{*} VL_{h,\varphi}(\bar{x}, \bar{\lambda}, \bar{u}, \bar{v}) = 0_h$.
If $i = 1, 2, \ldots, m$ are replaced by $i = 1, 2, \ldots$, in (HFP) and (VHFP), we obtain $(h, \varphi)_\varepsilon$-semi-infinite programs which can also be dealt with by using our methods in this paper.

In the final chapter of this article, it is also presented an application of Ben-Tal's operators in cryptography. By using Ben-Tal's operators along with techniques employed in RANROT algorithms, it is developed a pseudo-random number generator that significantly improved the classic RANROT generator. Ben-Tal's operators can be integrated into cryptography either to create mixing functions that are resistant to attacks or to generate non-linear outputs by combining these operators, thus ensuring data integrity. They can assist in key generation through transformations that ensure a uniform and random distribution of values. By using these operators in a specific manner, encryption algorithms can be developed with advanced security properties, including resistance to linear analysis and increased non-linearity. Moreover, these mathematical techniques can contribute to optimizing the efficiency of cryptographic algorithms by reducing the computation time required for encryption and decryption.

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