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A study of fractional differential equations with proportional boundary conditions

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Abstract

This article deals with a new concept of nonlocal proportional segmental boundary conditions with respect to initial and terminal sections of the given domain. Equipped with these conditions, we investigate the existence and uniqueness of solutions for a Caputo-type fractional differential equation with the nonlinearity depending upon the unknown function together with its lower order fractional derivative. We apply the standard tools of the fixed point theory to accomplish the desired results. Our study is useful in the given configuration as it helps to comprehend the fractional boundary value problems in the sense of proportional boundary conditions.

1 Introduction

We introduce a novel concept of nonlocal proportional boundary conditions with respect to the initial and terminal segments $[0, \nu_1]$ and $[\nu_2, \mathcal{K}]$ respectively of the interval $[0, \mathcal{K}]$ in the sense that

$$x(\nu_1) - x(0) = \lambda_1(x(\mathcal{K}) - x(\nu_2)), \ \int_0^{\nu_1} x(s)ds = \lambda_2 \int_{\nu_2}^{\mathcal{K}} x(s)ds,$$

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where $0 < \nu_1 < \nu_2 < \mathcal{K}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ are constants of proportionality. Here, the first condition defines the proportionality between the difference of values of the unknown function at the end-points of the initial and terminal strips (segments), while the second condition states that the values of the unknown function within the strips $(0, \nu_1)$ and $(\nu_2, \mathcal{K})$ are proportional. Such boundary conditions may find applications in the heat conduction problems when the sources/ sinks are present at the end-points of the initial and terminal segments of a heating rod and the temperature distributions vary differently within the two segments of the heating rod.

Equivalently, the proportional boundary conditions can be expressed as

$$\int_0^{\nu_1} x'(s)ds = \lambda_1 \int_{\nu_2}^{\mathcal{K}} x'(s)ds, \ \int_0^{\nu_1} x(s)ds = \lambda_2 \int_{\nu_2}^{\mathcal{K}} x(s)ds,$$

which can be interpreted as the proportionality of the flux (rate of flow of a quantity $x(t)$) and the quantity itself across the segments $[0, \nu_1]$ and $[\nu_2, \mathcal{K}]$ of the domain $[0, \mathcal{K}]$. We can also interpret these boundary conditions as an integral variant of periodic/anti-periodic boundary conditions with respect to the segments $(0, \nu_1)$ and $(\nu_2, \mathcal{K})$ with the length of these segments small enough to observe periodic/anti-periodic phenomenon. Now we dwell on some examples related to periodicity with respect to the initial and final segments of a specific time interval. A periodic event or pattern within the distinct initial and final sections of a specified range likely helps to understand its characteristics, verify its periodicity, or examine its behavior at critical points. A tiny segment of the sound wave reveals the pattern of changing air pressure. A brief recording of an electrocardiogram (EKG) will display the electrical activity associated with each heartbeat [1]. Both NN and RR intervals represent the time between heartbeats in heart rate variability (HRV) analysis. The RR interval represents the time between each detected heartbeat, measured from peak (R) to peak (R) on the QRS complex while the NN interval represents the RR interval data but with added filtering to remove artifacts and noise present in the data making some RR intervals unreliable (NN intervals are subsets of RR intervals). The daily temperature fluctuations often follow a somewhat periodic pattern, and analyzing temperature data for a brief interval can reveal a part of the daily cycle. Some other examples include temporary oscillations that can occur in response to a sudden change in a system, and occurrence of spectral edges for periodic operators inside the Brillouin zone [2] and $\mathbb{Z}$-periodic media [3].

As an application of the aforementioned segmental boundary conditions, we consider a boundary value problem for Caputo-type fractional differential

equations supplemented with these conditions given by

$$\begin{cases} {}^{c}D^{\sigma}x(t) = \varphi(t, x(t), {}^{c}D^{\varrho}x(t)), \ \ 1 < \sigma \leq 2, \ , \ 0 < \varrho \leq 1, \ t \in [0, \mathcal{K}], \\ x(\nu_1) - x(0) = \lambda_1(x(\mathcal{K}) - x(\nu_2)), \\ \displaystyle\int_0^{\nu_1} x(s)ds = \lambda_2 \int_{\nu_2}^{\mathcal{K}} x(s)ds, \ 0 < \nu_1 < \nu_2 < \mathcal{K}, \end{cases} \tag{1}$$

where ${}^{c}D^{\vartheta}$ denotes the Caputo fractional derivative operator of order $\vartheta \in \{\sigma, \varrho\}$, $\varphi : [0, \mathcal{K}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and both λ_1, λ_2 are real constants different from $\nu_1/(\mathcal{K} - \nu_2)$.

Let us now dwell on some recent work on nonlinear nonlocal fractional boundary value problems. Fractional differential equations arise in a variety of disciplines of applied sciences. Examples include physics and engineering [4], financial economics [5], ecology [6], infection of epithelial cells [7], relaxation filtration processes [8], chaos synchronization [9], etc. For theoretical aspects of fractional calculus, for instance, see [10, 11].

Nonlocal boundary conditions can model the physical phenomena experiencing the changes happening at arbitrary positions (nonlocal points and segments) inside the domain. Nonlocal integral boundary conditions can describe non-uniformities on the curved structures. For application details, see fluid flow problems [12], biomedical sciences [13], etc. One can find the engineering applications of strip type integral boundary conditions in the articles [14, 15]. For a variety of recent results on the topic, we refer the reader to the books [16, 17] and articles [18]-[27].

The rest of the article is structured as follows. Section 2 contains definitions related to our work and an auxiliary lemma dealing with the linear version of the nonlinear problem (1). In Section 3, we explore several existence criteria for solutions of the given problem by means of Schaefer's fixed point theorem, Leray-Schauder's nonlinear alternative and Krasnosel'skiĭ's fixed point theorem, while the uniqueness of its solutions is obtained with the aid of the Banach fixed point theorem. Section 4 demonstrates the application of the main results with the help of examples. Though we employ the standard fixed point theorems to study the given problem, yet the application of these tools of the fixed point theory provide a useful methodology to explore its solution characteristics.

2 Preliminaries

This section contains definitions of fractional calculus related to our work and an auxiliary lemma dealing with the linear version of the problem (1).

Definition 2.1 ([10]) For $\chi \in L_1[a,b]$, the (left) Riemann–Liouville fractional integral of order $\delta \in \mathbb{R}^+$ is given by

$$I^{\delta}_{a^+}\chi(\zeta) = \frac{1}{\Gamma(\delta)}\int_a^{\zeta}(\zeta-\widehat{\zeta})^{\delta-1}\chi(\widehat{\zeta})d\widehat{\zeta},$$

where Γ represents the usual Euler gamma function.

Definition 2.2 ([10]) Let $\chi, \chi^{(n)} \in L_1[a,b], a,b \in \mathbb{R}$, we define the Riemann–Liouville fractional derivative of order $\delta \in (n-1,n]$ for χ as

$$D^{\delta}_{a^+}\chi(\zeta) = \frac{d^n}{d\zeta^n}I^{n-\delta}_{a^+}\chi(\zeta) = \frac{1}{\Gamma(n-\delta)}\frac{d^n}{d\zeta^n}\int_a^{\zeta}(\zeta-\widehat{\zeta})^{n-\delta-1}\chi(\widehat{\zeta})d\widehat{\zeta}.$$

The Caputo fractional derivative ${}^cD^{\delta}_{a^+}\chi$ of order $\delta \in (n-1,n]$ in terms of the Riemann–Liouville fractional derivative is given by

$${}^cD^{\delta}_{a^+}\chi(\zeta) = D^{\delta}_{a^+}\left[\chi(\zeta) - \sum_{p=0}^{n-1}\chi^{(p)}(a)\frac{(\zeta-a)^p}{p!}\right].$$

Remark 2.3 An alternative definition for the (left) Caputo fractional derivative for a function $\chi \in AC^n[a,b]$ of order $\delta \in (n-1,n]$ can also be defined as

$${}^cD^{\delta}_{a^+}\chi(\zeta) = \int_a^{\zeta}\frac{(\zeta-\widehat{\zeta})^{n-\delta-1}}{\Gamma(n-\delta)}\chi^{(n)}(\widehat{\zeta})d\widehat{\zeta}.$$

In the current work, we write $I^{\delta}_{a^+}$ and ${}^cD^{\delta}_{a^+}$ for $a=0$ as I^{δ} and ${}^cD^{\delta}$, respectively. We solve the linear version of the problem (1) in the following lemma, which is used to transform the given problem into a fixed point problem.

Lemma 2.4 Let $\lambda_1, \lambda_2 \neq \nu_1/(\mathcal{K}-\nu_2)$. For any $q \in C[0,\mathcal{K}]$, the integral solution of the linear fractional differential equation

$${}^cD^{\sigma}x(t) = q(t),\ 1 < \sigma \leq 2, \tag{2}$$

supplemented with the boundary conditions of the problem (1) is

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} q(s)ds \\ &+ \frac{1}{\nu_1 - \lambda_2(\mathcal{K}-\nu_2)} \left[\lambda_2 \int_{\nu_2}^{\mathcal{K}} \int_0^s \frac{(s-\xi)^{\sigma-1}}{\Gamma(\sigma)} q(\xi)d\xi ds \right. \\ &\left. - \int_0^{\nu_1} \frac{(\nu_1 - s)^{\sigma}}{\Gamma(\sigma+1)} q(s)ds \right] \\ &+ \left(\frac{t + \omega(\nu_1 - \lambda_1(\mathcal{K}-\nu_2))}{\nu_1 - \lambda_1(\mathcal{K}-\nu_2)} \right) \times \\ &\times \left[\lambda_1 \left(\int_0^{\mathcal{K}} \frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)} q(s)ds - \int_0^{\nu_2} \frac{(\nu_2 - s)^{\sigma-1}}{\Gamma(\sigma)} q(s)ds \right) \right. \\ &\left. - \int_0^{\nu_1} \frac{(\nu_1 - s)^{\sigma-1}}{\Gamma(\sigma)} q(s)ds \right], \end{aligned} \tag{3}$$

where

$$\omega = \frac{\lambda_2(\mathcal{K}^2 - \nu_2^2) - \nu_1^2}{2(\nu_1 - \lambda_1(\mathcal{K}-\nu_2))(\nu_1 - \lambda_2(\mathcal{K}-\nu_2))}. \tag{4}$$

Proof. Operating the integral operator I^σ on both sides of (2) and using the formula (3.5.13) in [10], we get

$$x(t) = I^\sigma q(t) + c_0 + c_1 t, \tag{5}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants. Substituting (5) in the boundary conditions of (1) and solving the resulting systems of equations for c_0 and c_1, we find that

$$\begin{aligned} c_0 &= \frac{1}{\nu_1 - \lambda_2(\mathcal{K}-\nu_2)} \left[\lambda_2 \int_{\nu_2}^{\mathcal{K}} \int_0^s \frac{(s-\xi)^{\sigma-1}}{\Gamma(\sigma)} q(\xi)d\xi ds - \int_0^{\nu_1} \frac{(\nu_1 - s)^{\sigma}}{\Gamma(\sigma+1)} q(s)ds \right] \\ &+ \omega \left[\lambda_1 \left(\int_0^{\mathcal{K}} \frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)} q(s)ds - \int_0^{\nu_2} \frac{(\nu_2 - s)^{\sigma-1}}{\Gamma(\sigma)} q(s)ds \right) \right. \\ &\left. - \int_0^{\nu_1} \frac{(\nu_1 - s)^{\sigma-1}}{\Gamma(\sigma)} q(s)ds \right], \\ c_1 &= \frac{1}{\nu_1 - \lambda_1(\mathcal{K}-\nu_2)} \left[\lambda_1 \left(\int_0^{\mathcal{K}} \frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)} q(s)ds - \int_0^{\nu_2} \frac{(\nu_2 - s)^{\sigma-1}}{\Gamma(\sigma)} q(s)ds \right) \right. \\ &- \left. \int_0^{\nu_1} \frac{(\nu_1 - s)^{\sigma-1}}{\Gamma(\sigma)} q(s)ds \right]. \end{aligned}$$

Inserting the values of c_0 and c_1 in (5) leads to the solution (3). The converse of the lemma can be completed by direct computation.

□

3 Main Results

By Lemma 2.4, we can introduce an operator $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ to convert the problem (1) into a fixed problem $x = \mathcal{F}x$, where

$$\begin{aligned}
\mathcal{F}x(t) &= \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)}\varphi(s,x(s),{}^c D^{\varrho}x(s))ds \\
&+\frac{1}{\nu_1-\lambda_2(\mathcal{K}-\nu_2)}\left[\lambda_2\int_{\nu_2}^{\mathcal{K}}\int_0^s \frac{(s-\xi)^{\sigma-1}}{\Gamma(\sigma)}\varphi(\xi,x(\xi),{}^c D^{\varrho}x(\xi))d\xi ds\right. \\
&\left.-\int_0^{\nu_1}\frac{(\nu_1-s)^{\sigma}}{\Gamma(\sigma+1)}\varphi(s,x(s),{}^c D^{\varrho}x(s))ds\right] \\
&+\left(\frac{t+\omega(\nu_1-\lambda_1(\mathcal{K}-\nu_2))}{\nu_1-\lambda_1(\mathcal{K}-\nu_2)}\right)\left[\lambda_1\left(\int_0^{\mathcal{K}}\frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)}\varphi(s,x(s),{}^c D^{\varrho}x(s))ds\right.\right. \\
&- \left.\int_0^{\nu_2}\frac{(\nu_2-s)^{\sigma-1}}{\Gamma(\sigma)}\varphi(s,x(s),{}^c D^{\varrho}x(s))ds\right) \\
&- \left.\int_0^{\nu_1}\frac{(\nu_1-s)^{\sigma-1}}{\Gamma(\sigma)}\varphi(s,x(s),{}^c D^{\varrho}x(s))ds\right], \qquad (6)
\end{aligned}$$

and $\mathcal{A} = \{x : x, {}^c D^{\varrho}x \in C([0,\mathcal{K}],\mathbb{R})\}$ is a Banach space endowed with the norm $\|x\|_{\mathcal{A}} = \sup\{|x(t)| + |{}^c D^{\varrho}x(t)|, t \in [0,\mathcal{K}]\}$.

For computational convenience, we set the notation:

$$\begin{aligned}
\widehat{\rho}_1 &= \frac{1}{\Gamma(\sigma+1)}\max_{t\in[0,\mathcal{K}]}\left\{t^{\sigma}+\left|\frac{t+\omega(\nu_1-\lambda_1(\mathcal{K}-\nu_2))}{\nu_1-\lambda_1(\mathcal{K}-\nu_2)}\right|\left(\nu_1^{\sigma}+|\lambda_1|(\mathcal{K}^{\sigma}+\nu_2^{\sigma})\right)\right\} \\
&+\frac{1}{|\nu_1-\lambda_2(\mathcal{K}-\nu_2)|}\left[\frac{\nu_1^{\sigma+1}+|\lambda_2|(\mathcal{K}^{\sigma+1}-\nu_2^{\sigma+1})}{\Gamma(\sigma+2)}\right], \qquad (7) \\
\widehat{\rho}_2 &= \frac{\mathcal{K}^{\sigma-\varrho}}{\Gamma(\sigma-\varrho+1)}+\frac{\mathcal{K}^{1-\varrho}(\nu_1^{\sigma}+|\lambda_1|(\mathcal{K}^{\sigma}+\nu_2^{\sigma}))}{|\nu_1-\lambda_1(\mathcal{K}-\nu_2)|\Gamma(2-\varrho)\Gamma(\sigma+1)}. \qquad (8)
\end{aligned}$$

3.1 Existence results

This subsection is devoted to three existence results for the problem (1). We make use of Schaefer's fixed point theorem [28] to prove our first existence result.

Theorem 3.1. Let $\varphi : [0,\mathcal{K}]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that there exists a function $\Upsilon \in C([0,\mathcal{K}],\mathbb{R}^+)$ with $\|\Upsilon\| = \max_{t\in[0,\mathcal{K}]}|\Upsilon(t)|$ such that $|\varphi(t,x(t),{}^c D^{\varrho}x(t))| \le \Upsilon(t)$. Then, there exists at least one solution to the problem (1) on $[0,\mathcal{K}]$.

Proof. Let us verify that the operator $\mathcal{F}:\mathcal{A}\to\mathcal{A}$ defined in (6) satisfies the assumptions of Schaefer's fixed point theorem [28]. We first prove that the operator $\mathcal{F}:\mathcal{A}\to\mathcal{A}$ is completely continuous. From (6), we find that

$$\begin{aligned}
{}^{c}D^{\varrho}\mathcal{F}x(t) &= \int_0^t \frac{(t-s)^{\sigma-\varrho-1}}{\Gamma(\sigma-\varrho)}\varphi(s,x(s),{}^{c}D^{\varrho}x(s))ds \\
&\quad + \frac{t^{1-\varrho}}{(\nu_1-\lambda_1(\mathcal{K}-\nu_2))\Gamma(2-\varrho)}\times \\
&\quad \times\Bigg[\lambda_1\Bigg(\int_0^{\mathcal{K}}\frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)}\varphi(s,x(s),{}^{c}D^{\varrho}x(s))ds \\
&\quad - \int_0^{\nu_2}\frac{(\nu_2-s)^{\sigma-1}}{\Gamma(\sigma)}\varphi(s,x(s),{}^{c}D^{\varrho}x(s))ds\Bigg) \\
&\quad - \int_0^{\nu_1}\frac{(\nu_1-s)^{\sigma-1}}{\Gamma(\sigma)}\varphi(s,x(s),{}^{c}D^{\varrho}x(s))ds\Bigg]. \qquad (9)
\end{aligned}$$

Let $\Lambda\subset\mathcal{A}$ be bounded, then for any $x\in\Lambda$, we have

$$\begin{aligned}
\|\mathcal{F}x\| &= \max_{t\in[0,\mathcal{K}]}|\mathcal{F}x(t)| \\
&\le \max_{t\in[0,\mathcal{K}]}\Bigg\{\int_0^t\frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)}|\varphi(s,x(s),{}^{c}D^{\varrho}x(s))|ds \\
&+ \frac{1}{|\nu_1-\lambda_2(\mathcal{K}-\nu_2)|}\Bigg[|\lambda_2|\int_{\nu_2}^{\mathcal{K}}\int_0^s\frac{(s-\xi)^{\sigma-1}}{\Gamma(\sigma)}|\varphi(\xi,x(\xi),{}^{c}D^{\varrho}x(\xi))|d\xi ds \\
&+ \int_0^{\nu_1}\frac{(\nu_1-s)^{\sigma}}{\Gamma(\sigma+1)}|\varphi(s,x(s),{}^{c}D^{\varrho}x(s))|ds\Bigg] \\
&+ \left|\frac{t+\omega(\nu_1-\lambda_1(\mathcal{K}-\nu_2))}{\nu_1-\lambda_1(\mathcal{K}-\nu_2)}\right|\Bigg[|\lambda_1|\Bigg(\int_0^{\mathcal{K}}\frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)}|\varphi(s,x(s),{}^{c}D^{\varrho}x(s))|ds \\
&+ \int_0^{\nu_2}\frac{(\nu_2-s)^{\sigma-1}}{\Gamma(\sigma)}|\varphi(s,x(s),{}^{c}D^{\varrho}x(s))|ds\Bigg) \\
&+ \int_0^{\nu_1}\frac{(\nu_1-s)^{\sigma-1}}{\Gamma(\sigma)}|\varphi(s,x(s),{}^{c}D^{\varrho}x(s))|ds\Bigg]\Bigg\} \\
&\le \|\Upsilon\|\max_{t\in[0,\mathcal{K}]}\Bigg\{\int_0^t\frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)}ds \\
&\quad + \frac{1}{|\nu_1-\lambda_2(\mathcal{K}-\nu_2)|}\Bigg[|\lambda_2|\int_{\nu_2}^{\mathcal{K}}\int_0^s\frac{(s-\xi)^{\sigma-1}}{\Gamma(\sigma)}d\xi ds+\int_0^{\nu_1}\frac{(\nu_1-s)^{\sigma}}{\Gamma(\sigma+1)}ds\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{t+\omega(\nu_1-\lambda_1(\mathcal{K}-\nu_2))}{\nu_1-\lambda_1(\mathcal{K}-\nu_2)}\right|\left[|\lambda_1|\left(\int_0^{\mathcal{K}}\frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)}ds+\int_0^{\nu_2}\frac{(\nu_2-s)^{\sigma-1}}{\Gamma(\sigma)}ds\right)\right.\\
&\left.+\int_0^{\nu_1}\frac{(\nu_1-s)^{\sigma-1}}{\Gamma(\sigma)}ds\right]\Bigg\}\\
=\;&\|\Upsilon\|\left(\max_{t\in[0,\mathcal{K}]}\left\{\frac{t^\sigma}{\Gamma(\sigma+1)}+\left|\frac{t+\omega(\nu_1-\lambda_1(\mathcal{K}-\nu_2))}{\nu_1-\lambda_1(\mathcal{K}-\nu_2)}\right|\left[\frac{\nu_1^\sigma+|\lambda_1|(\mathcal{K}^\sigma+\nu_2^\sigma)}{\Gamma(\sigma+1)}\right]\right\}\right.\\
&\left.+\frac{1}{|\nu_1-\lambda_2(\mathcal{K}-\nu_2)|}\left[\frac{\nu_1^{\sigma+1}+|\lambda_2|(\mathcal{K}^{\sigma+1}-\nu_2^{\sigma+1})}{\Gamma(\sigma+2)}\right]\right)=\|\Upsilon\|\widehat{\rho}_1,
\end{aligned}
$$

where $\widehat{\rho}_1$ is given by (7). Analogously, we can find that

$$
\begin{aligned}
\|{}^cD^\varrho\mathcal{F}x\| &\leq \|\Upsilon\|\left[\frac{\mathcal{K}^{\sigma-\varrho}}{\Gamma(\sigma-\varrho+1)}+\frac{\mathcal{K}^{1-\varrho}(\nu_1^\sigma+|\lambda_1|(\mathcal{K}^\sigma+\nu_2^\sigma))}{|\nu_1-\lambda_1(\mathcal{K}-\nu_2)|\Gamma(2-\varrho)\Gamma(\sigma+1)}\right]\\
&=\|\Upsilon\|\widehat{\rho}_2,
\end{aligned}
$$

where $\widehat{\rho}_2$ is given by (8). It follows from the foregoing inequalities that the operator $\mathcal{F}$ is uniformly bounded. Next, for $0<\eta_1<\eta_2<\mathcal{K}$, we have

$$
\begin{aligned}
&|\mathcal{F}x(\eta_2)-\mathcal{F}x(\eta_1)|\\
\leq\;&\|\Upsilon\|\left\{\int_0^{\eta_1}\frac{|(\eta_2-s)^{\sigma-1}-(\eta_1-s)^{\sigma-1}|}{\Gamma(\sigma)}ds+\int_{\eta_1}^{\eta_2}\frac{|(\eta_2-s)^{\sigma-1}|}{\Gamma(\sigma)}ds\right.\\
&+\frac{|\eta_2-\eta_1|}{|\nu_1-\lambda_1(\mathcal{K}-\nu_2)|}\left[|\lambda_1|\left(\int_0^{\mathcal{K}}\frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)}ds+\int_0^{\nu_2}\frac{(\nu_2-s)^{\sigma-1}}{\Gamma(\sigma)}ds\right)\right.\\
&\left.\left.+\int_0^{\nu_1}\frac{(\nu_1-s)^{\sigma-1}}{\Gamma(\sigma)}ds\right]\right\}\\
\leq\;&\|\Upsilon\|\frac{1}{\Gamma(\sigma+1)}\left\{2|(\eta_2-\eta_1)^\sigma|+|\eta_2^\sigma-\eta_1^\sigma|+\frac{|\eta_2-\eta_1|(\nu_1^\sigma+|\lambda_1|(\nu_2^\sigma+\mathcal{K}^\sigma)}{|\nu_1-\lambda_1(\mathcal{K}-\nu_2)|}\right\}.
\end{aligned}
$$

Likewise, we find that

$$
\begin{aligned}
|{}^cD^\varrho\mathcal{F}x(\eta_2)-{}^cD^\varrho\mathcal{F}x(\eta_1)| &\leq \|\Upsilon\|\left\{\frac{2|(\eta_2-\eta_1)^{\sigma-\varrho}|+|\eta_2^{\sigma-\varrho}-\eta_1^{\sigma-\varrho}|}{\Gamma(\sigma-\varrho+1)}\right.\\
&\left.+\frac{|\eta_2^{1-\varrho}-\eta_1^{1-\varrho}|(\nu_1^\sigma+|\lambda_1|(\mathcal{K}^\sigma+\nu_2^\sigma))}{|\nu_1-\lambda_1(\mathcal{K}-\nu_2)|\Gamma(\sigma+1)\Gamma(2-\varrho)}\right\}.
\end{aligned}
$$

Hence, $\{\mathcal{F}x:x\in\Lambda\}$ and $\{{}^cD^\varrho\mathcal{F}x:x\in\Lambda\}$ are relatively compact in $\mathcal{A}$ by the application of Arzelá-Ascoli theorem. Therefore, $\mathcal{F}(\Lambda)$ is relatively compact subset of $\mathcal{A}$.

In the final step, we consider a set $K = \{x \in \mathcal{A} : x = \widehat{\omega}\mathcal{F}x, 0 < \widehat{\omega} < 1\}$ and establish that it is bounded. Letting $x \in K, t \in [0, \mathcal{K}]$, we have $x(t) = \widehat{\omega}\mathcal{F}x(t)$. Employing the arguments used in the previous step, we find that $\|x\| \leq \|\mathcal{F}x\| \leq \|\Upsilon\|\widehat{\rho}_1$ and $\|{}^cD^{\varrho}x\| \leq \|{}^cD^{\varrho}\mathcal{F}x\| \leq \|\Upsilon\|\widehat{\rho}_2$. Thus, the set K is bounded. From the foregoing steps, we deduce that the hypothesis of Schaefer's fixed point theorem [28] is satisfied and hence its conclusion implies that the operator $\mathcal{F}$ has a fixed point. Thus, at least one solution to the problem (1) exists on $[0, \mathcal{K}]$. □

In our next result, we apply Leray-Schauder's nonlinear alternative [28] to prove the existence of solutions to the problem (1).

Theorem 3.2. Let $\varphi : [0, \mathcal{K}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. In addition, we assume that

(Q_1) There exist functions $\psi : [0, \infty) \to (0, \infty)$ and $P \in C([0, \mathcal{K}], \mathbb{R}^+)$ such that ψ is continuous nondecreasing and $|\varphi(t, x, {}^c D^{\varrho}x)| \leq P(t)\psi(\|x\|_{\mathcal{A}})$ for each $(t, x, {}^c D^{\varrho}x) \in [0, \mathcal{K}] \times \mathbb{R} \times \mathbb{R}$;

(Q_2) We can find a positive constant $\mathcal{M}$ satisfying

$$\frac{\mathcal{M}}{\psi(\mathcal{M})\|P\|(\widehat{\rho}_1 + \widehat{\rho}_2)} > 1.$$

Then, at least one solution to the problem (1) exists on $[0, \mathcal{K}]$.

Proof. We complete the proof in several steps.

(*i*) *The operator $\mathcal{F}$ maps bounded sets (balls) into bounded sets in $C([0, \mathcal{K}], \mathbb{R})$.*

Define a bounded ball $B_{\upsilon} = \{x \in C([0, \mathcal{K}], \mathbb{R}) : \|x\|_{\mathcal{A}} \leq \upsilon\}$ in $C([0, \mathcal{K}], \mathbb{R})$ and take $x \in B_{\upsilon}$ such that

$$\begin{aligned}
\|\mathcal{F}x\| &\leq \|P\|\psi(\|x\|_{\mathcal{A}}) \max_{t\in[0,\mathcal{K}]} \left\{ \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} ds \right. \\
&+ \frac{1}{|\nu_1 - \lambda_2(\mathcal{K} - \nu_2)|} \left[|\lambda_2| \int_{\nu_2}^{\mathcal{K}} \int_0^s \frac{(s-\xi)^{\sigma-1}}{\Gamma(\sigma)} d\xi ds + \int_0^{\nu_1} \frac{(\nu_1 - s)^{\sigma}}{\Gamma(\sigma+1)} ds \right] \\
&+ \left| \frac{t + \omega(\nu_1 - \lambda_1(\mathcal{K} - \nu_2))}{\nu_1 - \lambda_1(\mathcal{K} - \nu_2)} \right| \left[|\lambda_1| \left(\int_0^{\mathcal{K}} \frac{(\mathcal{K} - s)^{\sigma-1}}{\Gamma(\sigma)} ds \right. \right. \\
&+ \left. \left. \int_0^{\nu_2} \frac{(\nu_2 - s)^{\sigma-1}}{\Gamma(\sigma)} ds \right) + \int_0^{\nu_1} \frac{(\nu_1 - s)^{\sigma-1}}{\Gamma(\sigma)} ds \right] \Bigg\} \\
&\leq \|P\|\psi(\upsilon) \left(\max_{t\in[0,\mathcal{K}]} \left\{ \frac{t^{\sigma}}{\Gamma(\sigma+1)} + \left| \frac{t + \omega(\nu_1 - \lambda_1(\mathcal{K} - \nu_2))}{\nu_1 - \lambda_1(\mathcal{K} - \nu_2)} \right| \times \right. \right. \\
&\times \left. \left. \left[\frac{\nu_1^{\sigma} + |\lambda_1|(\mathcal{K}^{\sigma} + \nu_2^{\sigma})}{\Gamma(\sigma+1)} \right] \right\} \right.
\end{aligned}$$

$$+\frac{1}{|\nu_1-\lambda_2(\mathcal{K}-\nu_2)|}\left[\frac{\nu_1^{\sigma+1}+|\lambda_2|(\mathcal{K}^{\sigma+1}-\nu_2^{\sigma+1})}{\Gamma(\sigma+2)}\right]\Bigg) = \|P\|\psi(\upsilon)\widehat{\rho}_1,$$

where $\widehat{\rho}_1$ is given by (7). Analogously, we can find that

$$\begin{aligned}\|{}^cD^{\varrho}\mathcal{F}x\| &\leq \|P\|\psi(\upsilon)\left[\frac{\mathcal{K}^{\sigma-\varrho}}{\Gamma(\sigma-\varrho+1)}+\frac{\mathcal{K}^{1-\varrho}(\nu_1^{\sigma}+|\lambda_1|(\mathcal{K}^{\sigma}+\nu_2^{\sigma}))}{|\nu_1-\lambda_1(\mathcal{K}-\nu_2)|\Gamma(2-\varrho)\Gamma(\sigma+1)}\right]\\ &= \|P\|\psi(\upsilon)\widehat{\rho}_2,\end{aligned}$$

where $\widehat{\rho}_2$ is given by (8). From the last two inequalities, we get

$$\|\mathcal{F}x\|_{\mathcal{A}} = \|\mathcal{F}x\| + \|{}^cD^{\varrho}\mathcal{F}x\| \leq \|P\|\psi(\upsilon)(\widehat{\rho}_1+\widehat{\rho}_2).$$

Therefore, $\mathcal{F}$ maps bounded sets (balls) into bounded sets in $C([0,\mathcal{K}],\mathbb{R})$.

(*ii*) *The operator* $\mathcal{F}$ *maps bounded sets into equicontinuous sets of* $C([0,\mathcal{K}],\mathbb{R})$.

Taking $\gamma_1,\gamma_2\in[0,\mathcal{K}]$ with $\gamma_1<\gamma_2$ and $x\in B_{\upsilon}$, we obtain

$$\begin{aligned}|\mathcal{F}x(\gamma_2)-\mathcal{F}x(\gamma_1)| &\leq \frac{\|P\|\psi(\upsilon)}{\Gamma(\sigma+1)}\Big\{2|(\gamma_2-\gamma_1)^{\sigma}|+|\gamma_2^{\sigma}-\gamma_1^{\sigma}|\\ &+ \frac{|\gamma_2-\gamma_1|(\nu_1^{\sigma}+|\lambda_1|(\nu_2^{\sigma}+\mathcal{K}^{\sigma})}{|\nu_1-\lambda_1(\mathcal{K}-\nu_2)|}\Big\},\end{aligned}$$

$$\begin{aligned}|{}^cD^{\varrho}\mathcal{F}x(\gamma_2)-{}^c\,D^{\varrho}\mathcal{F}x(\gamma_1)| &\leq \|P\|\psi(\upsilon)\Big\{\frac{2|(\gamma_2-\gamma_1)^{\sigma-\varrho}|+|\gamma_2^{\sigma-\varrho}-\gamma_1^{\sigma-\varrho}|}{\Gamma(\sigma-\varrho+1)}\\ &+\frac{|\gamma_2^{1-\varrho}-\gamma_1^{1-\varrho}|(\nu_1^{\sigma}+|\lambda_1|(\mathcal{K}^{\sigma}+\nu_2^{\sigma}))}{|\nu_1-\lambda_1(\mathcal{K}-\nu_2)|\Gamma(\sigma+1)\Gamma(2-\varrho)}\Big\}.\end{aligned}$$

Notice the right-hand sides of the foregoing inequalities tend to zero as $\gamma_2\to\gamma_1$ independently of $x\in B_{\upsilon}$. So, an application of the Arzelá-Ascoli theorem [28] implies that $\mathcal{F}:C([0,\mathcal{K}],\mathbb{R})\to C([0,\mathcal{K}],\mathbb{R})$ is completely continuous.

(*iii*) In the final step, it will be shown that there exists an open set $\mathcal{W}$ such that $x\in\partial\mathcal{W}$ and $x\neq\lambda\mathcal{F}x,\lambda\in(0,1)$. Suppose that there exists $x\in C([0,\mathcal{K}],\mathbb{R})$ satisfying $x=\lambda\mathcal{F}x,\lambda\in(0,1)$. As in the first part of the proof, we can get

$$\frac{\|x\|_{\mathcal{A}}}{\|P\|\psi(\|x\|_{\mathcal{A}})(\widehat{\rho}_1+\widehat{\rho}_2)}\leq 1.$$

By the assumption (Q_2), we can find $\mathcal{M}>0$ such that $\|x\|_{\mathcal{A}}\neq\mathcal{M}$. Consider $\mathcal{W}=\{x\in\mathcal{A}:\|x\|_{\mathcal{A}}<\mathcal{M}\}$ and note that the operator $\mathcal{F}:\overline{\mathcal{W}}\to C([0,\mathcal{K}],\mathbb{R})$ is continuous and completely continuous, where $\overline{\mathcal{W}}$ denotes the closure of $\mathcal{W}$.

According to the definition of the set $\mathcal{W}$, there does not exist any $x \in \partial\mathcal{W}$ (boundary of $\mathcal{W}$) such that $x = \lambda\mathcal{F}x$ for some $\lambda \in (0,1)$. So, by the conclusion of Leray-Schauder's nonlinear alternative, there exists a fixed point $x \in \overline{\mathcal{W}}$. In consequence, we obtain at least one solution to the problem (1) on $[0,\mathcal{K}]$. □

Our last existence result for the problem (1) relies on Krasnosel'skiĭ's fixed point theorem [28].

Theorem 3.3. Let $\varphi : [0,\mathcal{K}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

(Q_3) For all $x_i, y_i \in \mathbb{R}$, there exists $L > 0$ satisfying

$$|\varphi(t,x_1,x_2) - \varphi(t,y_1,y_2)| \leq L(|x_1 - y_1| + |x_2 - y_2|);$$

(Q_4) There exists a function $\varpi \in C([0,\mathcal{K}])$ such that

$$|\varphi(t,x(t),\, {}^cD^{\varrho}x(t))| \leq \varpi(t),\ \forall (t,x,\, {}^cD^{\varrho}x) \in [0,\mathcal{K}] \times \mathbb{R}^2.$$

If $\delta L < 1$, then at least one solution to the problem (1) exists on $[0,\mathcal{K}]$, where

$$\delta = (\widehat{\rho}_1 + \widehat{\rho}_2) - \left(\frac{\mathcal{K}^{\sigma}}{\Gamma(\sigma+1)} + \frac{\mathcal{K}^{\sigma-\varrho}}{\Gamma(\sigma-\varrho+1)}\right), \tag{10}$$

$\widehat{\rho}_1$ and $\widehat{\rho}_2$ are given in (7) and (8), respectively.

Proof. We define a closed ball $B_\theta = \{x \in \mathcal{A} : \|x\|_{\mathcal{A}} \leq \theta\}$ with $\theta \geq \|\varpi\|(\widehat{\rho}_1 + \widehat{\rho}_2)$ and $\max\limits_{t\in[0,\mathcal{K}]} |\varpi(t)| = \|\varpi\| < \infty$. Next, we introduce the operators $\mathcal{F}_1, \mathcal{F}_2 : B_\theta \to \mathcal{A}$ as

$$\begin{aligned}
(\mathcal{F}_1 x)(t) &= \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)}\varphi(s,x(s),{}^c D^{\varrho}x(s))ds,\\
(\mathcal{F}_2 x)(t) &= \frac{1}{\nu_1 - \lambda_2(\mathcal{K}-\nu_2)}\left[\lambda_2 \int_{\nu_2}^{\mathcal{K}}\int_0^s \frac{(s-\xi)^{\sigma-1}}{\Gamma(\sigma)}\varphi(\xi,x(\xi),{}^c D^{\varrho}x(\xi))d\xi ds\right.\\
&\quad \left. - \int_0^{\nu_1} \frac{(\nu_1-s)^{\sigma}}{\Gamma(\sigma+1)}\varphi(s,x(s),{}^c D^{\varrho}x(s))ds\right]\\
&\quad + \left(\frac{t + \omega(\nu_1 - \lambda_1(\mathcal{K}-\nu_2))}{\nu_1 - \lambda_1(\mathcal{K}-\nu_2)}\right) \times\\
&\quad \times \left[\lambda_1\left(\int_0^{\mathcal{K}} \frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)}\varphi(s,x(s),{}^c D^{\varrho}x(s))ds\right.\right.\\
&\quad \left.\left. - \int_0^{\nu_2} \frac{(\nu_2-s)^{\sigma-1}}{\Gamma(\sigma)}\varphi(s,x(s),{}^c D^{\varrho}x(s))ds\right)\right.\\
&\quad \left. - \int_0^{\nu_1} \frac{(\nu_1-s)^{\sigma-1}}{\Gamma(\sigma)}\varphi(s,x(s),{}^c D^{\varrho}x(s))ds\right].
\end{aligned}$$

Observe that $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$. Now, we verify that $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy the assumptions of Krasnosel'skiĭ's fixed point theorem. Taking $x, y \in B_\theta$, we obtain

$$\begin{aligned}
&\|\mathcal{F}_1 x + \mathcal{F}_2 y\| \\
= &\max_{t\in[0,\mathcal{K}]} |\mathcal{F}_1 x(t) + \mathcal{F}_2 y(t)| \\
\leq &\max_{t\in[0,\mathcal{K}]} \Bigg\{ \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(s, x(s), {}^c D^\varrho x(s))| ds \\
&+ \frac{1}{|\nu_1 - \lambda_2(\mathcal{K}-\nu_2)|} \Bigg[|\lambda_2| \int_{\nu_2}^{\mathcal{K}} \int_0^s \frac{(s-\xi)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(\xi, y(\xi), {}^c D^\varrho y(\xi))| d\xi ds \\
&+ \int_0^{\nu_1} \frac{(\nu_1 - s)^\sigma}{\Gamma(\sigma+1)} |\varphi(s, y(s), {}^c D^\varrho y(s))| ds \Bigg] \\
&+ \left| \frac{t + \omega(\nu_1 - \lambda_1(\mathcal{K}-\nu_2))}{\nu_1 - \lambda_1(\mathcal{K}-\nu_2)} \right| \Bigg[|\lambda_1| \Bigg(\int_0^{\mathcal{K}} \frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(s, y(s), {}^c D^\varrho y(s))| ds \\
&+ \int_0^{\nu_2} \frac{(\nu_2 - s)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(s, y(s), {}^c D^\varrho y(s))| ds \Bigg) \\
&+ \int_0^{\nu_1} \frac{(\nu_1 - s)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(s, y(s), {}^c D^\varrho y(s))| ds \Bigg] \Bigg\}
\end{aligned}$$

$$\begin{aligned}
\leq &\|\varpi\| \Bigg(\max_{t\in[0,\mathcal{K}]} \left\{ \frac{t^\sigma}{\Gamma(\sigma+1)} + \left| \frac{t + \omega(\nu_1 - \lambda_1(\mathcal{K}-\nu_2))}{\nu_1 - \lambda_1(\mathcal{K}-\nu_2)} \right| \left[\frac{\nu_1^\sigma + |\lambda_1|(\mathcal{K}^\sigma + \nu_2^\sigma)}{\Gamma(\sigma+1)} \right] \right\} \\
&+ \frac{1}{|\nu_1 - \lambda_2(\mathcal{K}-\nu_2)|} \left[\frac{\nu_1^{\sigma+1} + |\lambda_2|(\mathcal{K}^{\sigma+1} - \nu_2^{\sigma+1})}{\Gamma(\sigma+2)} \right] \Bigg) \\
= &\|\varpi\| \widehat{\rho}_1,
\end{aligned}$$

where $\widehat{\rho}_1$ is given in (7). In the same way, we find that

$$\begin{aligned}
&\|({}^c D^\varrho \mathcal{F}_1 x) + ({}^c D^\varrho \mathcal{F}_2 y)\| \\
= &\max_{t\in[0,\mathcal{K}]} |({}^c D^\varrho \mathcal{F}_1 x)(t) + ({}^c D^\varrho \mathcal{F}_2 y)(t)| \\
\leq &\max_{t\in[0,\mathcal{K}]} \Bigg\{ \int_0^t \frac{(t-s)^{\sigma-\varrho-1}}{\Gamma(\sigma-\varrho)} |\varphi(s, x(s), {}^c D^\varrho x(s))| ds \\
&+ \frac{t^{1-\varrho}}{|\nu_1 - \lambda_1(\mathcal{K}-\nu_2)|\Gamma(2-\varrho)} \Bigg[|\lambda_1| \Bigg(\int_0^{\mathcal{K}} \frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(s, y(s), {}^c D^\varrho y(s))| ds \\
&+ \int_0^{\nu_2} \frac{(\nu_2 - s)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(s, y(s), {}^c D^\varrho y(s))| ds \Bigg) \\
&+ \int_0^{\nu_1} \frac{(\nu_1 - s)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(s, y(s), {}^c D^\varrho y(s))| ds \Bigg] \Bigg\} = \|\varpi\| \widehat{\rho}_2,
\end{aligned}$$

where $\widehat{\rho}_2$ is represented by (8). Thus, $\|\mathcal{F}_1x + \mathcal{F}_2y\|_{\mathcal{A}} \leq \|\varpi\|(\widehat{\rho}_1 + \widehat{\rho}_2) \leq \theta$. Therefore, $\mathcal{F}_1x + \mathcal{F}_2y \in B_\theta$. Hence, the first assumption of Krasnosel'skiĭ's fixed point theorem holds true.

Next, we use the assumption (Q_3) and (10) to establish that $\mathcal{F}_2$ is a contraction. Let $x, y \in B_\theta$, then

$$
\begin{aligned}
& \|\mathcal{F}_2x - \mathcal{F}_2y\| \\
= \; & \max_{t\in[0,\mathcal{K}]} |\mathcal{F}_2x(t) - \mathcal{F}_2y(t)| \\
\leq \; & \max_{t\in[0,\mathcal{K}]} \Bigg\{ \frac{1}{|\nu_1 - \lambda_2(\mathcal{K}-\nu_2)|} \Bigg[|\lambda_2| \int_{\nu_2}^{\mathcal{K}} \int_0^s \frac{(s-\xi)^{\sigma-1}}{\Gamma(\sigma)} \\
& \times |\varphi(\xi, x(\xi), {}^c D^{\varrho}x(\xi)) - \varphi(\xi, y(\xi), {}^c D^{\varrho}y(\xi))| d\xi ds \\
& + \int_0^{\nu_1} \frac{(\nu_1 - s)^{\sigma}}{\Gamma(\sigma+1)} |\varphi(s, x(s), {}^c D^{\varrho}x(s)) - \varphi(s, y(s), {}^c D^{\varrho}y(s))| ds \Bigg] \\
& + \left| \frac{t + \omega(\nu_1 - \lambda_1(\mathcal{K}-\nu_2))}{\nu_1 - \lambda_1(\mathcal{K}-\nu_2)} \right| \Bigg[|\lambda_1| \Bigg(\int_0^{\mathcal{K}} \frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)} \\
& \times |\varphi(s, x(s), {}^c D^{\varrho}x(s)) - \varphi(s, y(s), {}^c D^{\varrho}y(s))| ds \\
& + \int_0^{\nu_2} \frac{(\nu_2 - s)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(s, x(s), {}^c D^{\varrho}x(s)) - \varphi(s, y(s), {}^c D^{\varrho}y(s))| ds \Bigg) \\
& + \int_0^{\nu_1} \frac{(\nu_1 - s)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(s, x(s), {}^c D^{\varrho}x(s)) - \varphi(s, y(s), {}^c D^{\varrho}y(s))| ds \Bigg] \Bigg\} \\
\leq \; & L\|x - y\| \max_{t\in[0,\mathcal{K}]} \Bigg\{ \frac{1}{|\nu_1 - \lambda_2(\mathcal{K}-\nu_2)|} \Bigg[|\lambda_2| \int_{\nu_2}^{\mathcal{K}} \int_0^s \frac{(s-\xi)^{\sigma-1}}{\Gamma(\sigma)} d\xi ds \\
& + \int_0^{\nu_1} \frac{(\nu_1 - s)^{\sigma}}{\Gamma(\sigma+1)} ds \Bigg] \\
& + \left| \frac{t + \omega(\nu_1 - \lambda_1(\mathcal{K}-\nu_2))}{\nu_1 - \lambda_1(\mathcal{K}-\nu_2)} \right| \Bigg[|\lambda_1| \Bigg(\int_0^{\mathcal{K}} \frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)} ds + \int_0^{\nu_2} \frac{(\nu_2 - s)^{\sigma-1}}{\Gamma(\sigma)} ds \Bigg) \\
& + \int_0^{\nu_1} \frac{(\nu_1 - s)^{\sigma-1}}{\Gamma(\sigma)} ds \Bigg] \Bigg\} \\
= \; & L\|x - y\| \left(\widehat{\rho}_1 - \frac{\mathcal{K}^{\sigma}}{\Gamma(\sigma+1)} \right).
\end{aligned}
$$

Similarly, one can obtain that

$$
\|{}^c D^{\varrho}\mathcal{F}_2x - {}^c D^{\varrho}\mathcal{F}_2y\| = \max_{t\in[0,\mathcal{K}]} |{}^c D^{\varrho}\mathcal{F}_2x(t) - {}^c D^{\varrho}\mathcal{F}_2y(t)|
$$

$$
\begin{aligned}
\leq & \max_{t\in[0,\mathcal{K}]}\left\{\frac{t^{1-\varrho}}{|\nu_1-\lambda_1(\mathcal{K}-\nu_2)|\Gamma(2-\varrho)}\times\right.\\
& \left[|\lambda_1|\Big(\int_0^{\mathcal{K}}\frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)}|\varphi(s,x(s),{}^c D^{\varrho}x(s))-\varphi(s,y(s),{}^c D^{\varrho}y(s))|ds\right.\\
& +\int_0^{\nu_2}\frac{(\nu_2-s)^{\sigma-1}}{\Gamma(\sigma)}|\varphi(s,x(s),{}^c D^{\varrho}x(s))-\varphi(s,y(s),{}^c D^{\varrho}y(s))|ds\Big)\\
& \left.\left.+\int_0^{\nu_1}\frac{(\nu_1-s)^{\sigma-1}}{\Gamma(\sigma)}|\varphi(s,x(s),{}^c D^{\varrho}x(s))-\varphi(s,y(s),{}^c D^{\varrho}y(s))|ds\right]\right\}\\
\leq & L\|x-y\|\max_{t\in[0,\mathcal{K}]}\left\{\frac{t^{1-\varrho}}{|\nu_1-\lambda_1(\mathcal{K}-\nu_2)|\Gamma(2-\varrho)}\left[|\lambda_1|\Big(\int_0^{\mathcal{K}}\frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)}ds\right.\right.\\
& \left.\left.+\int_0^{\nu_2}\frac{(\nu_2-s)^{\sigma-1}}{\Gamma(\sigma)}ds\Big)+\int_0^{\nu_1}\frac{(\nu_1-s)^{\sigma-1}}{\Gamma(\sigma)}ds\right]\right\}\\
= & L\|x-y\|\left(\widehat{\rho}_2-\frac{\mathcal{K}^{\sigma-\varrho}}{\Gamma(\sigma-\varrho+1)}\right).
\end{aligned}
$$

In consequence, we get

$$\|\mathcal{F}_2x-\mathcal{F}_2y\|_{\mathcal{A}}\leq L\delta\|x-y\|,$$

where δ is given in (10). By the given condition $\delta L<1$, it follows that $\mathcal{F}_2$ is a contraction. This verifies the second hypothesis of Krasnosel'skiĭ's fixed point theorem.

Finally, we establish that $\mathcal{F}_1$ is compact and continuous. Notice that continuity of $\varphi(t,x(t),{}^c D^{\varrho}x(t))$ yields that of the operator $\mathcal{F}_1$. For any $x\in B_\theta, t\in[0,\mathcal{K}]$, we have

$$
\begin{aligned}
\|\mathcal{F}_1x\| &= \max_{t\in[0,\mathcal{K}]}|\mathcal{F}_1x(t)|\\
&\leq \max_{t\in[0,\mathcal{K}]}\int_0^t\frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)}|\varphi(s,x(s),{}^c D^{\varrho}x(s))|ds\leq\|\varpi\|\frac{\mathcal{K}^{\sigma}}{\Gamma(\sigma+1)},\\
\|{}^c D^{\varrho}\mathcal{F}_1x\| &= \max_{t\in[0,\mathcal{K}]}|{}^c D^{\varrho}\mathcal{F}_1x(t)|\\
&\leq \max_{t\in[0,\mathcal{K}]}\int_0^t\frac{(t-s)^{\sigma-\varrho-1}}{\Gamma(\sigma-\varrho)}|\varphi(s,x(s),{}^c D^{\varrho}x(s))|ds\\
&\leq \|\varpi\|\frac{\mathcal{K}^{\sigma-\varrho}}{\Gamma(\sigma-\varrho+1)}.
\end{aligned}
$$

Therefore, $\mathcal{F}_1$ is uniformly bounded.

Next, we take $0 < \eta_1 < \eta_2 < \mathcal{K}$. Then, we have

$$\begin{aligned} & |\mathcal{F}_1 x(\eta_2) - \mathcal{F}_1 x(\eta_1)| \\ = & \Big| \int_0^{\eta_1} \frac{[(\eta_2 - s)^{\sigma-1} - (\eta_1 - s)^{\sigma-1}]}{\Gamma(\sigma)} \varphi(s, x(s), {}^c D^{\varrho} x(s)) ds \\ & + \int_{\eta_1}^{\eta_2} \frac{(\eta_2 - s)^{\sigma-1}}{\Gamma(\sigma)} \varphi(s, x(s), {}^c D^{\varrho} x(s)) ds \Big| \\ \leq & \frac{\|\varpi\|}{\Gamma(\sigma+1)} \Big\{ 2|(\eta_2 - \eta_1)^{\sigma}| + |\eta_2^{\sigma} - \eta_1^{\sigma}| \Big\} \to 0, \end{aligned}$$

when $(\eta_2 - \eta_1) \to 0$, independently of $x \in B_\theta$. Likewise, we find that

$$|{}^c D^{\varrho} \mathcal{F} x(\eta_2) - {}^c D^{\varrho} \mathcal{F} x(\eta_1)| \leq \|\varpi\| \Big\{ \frac{2|(\eta_2 - \eta_1)^{\sigma-\varrho}| + |\eta_2^{\sigma-\varrho} - \eta_1^{\sigma-\varrho}|}{\Gamma(\sigma - \varrho + 1)} \Big\} \to 0,$$

as $(\eta_2 - \eta_1) \to 0$ independently of $x \in B_\theta$. Thus, $\mathcal{F}_1$ is relatively compact on B_θ. Therefore, it follows by the Arzelá-Ascoli theorem that $\mathcal{F}_1$ is compact on B_θ. As all the assumptions of Krasnosel'skiĭ's fixed point theorem are verified, therefore, its conclusion implies that at least one solution to the problem (1) exists on $[0, \mathcal{K}]$.

□

3.2 Uniqueness result

In this subsection, we present a uniqueness result for the problem (1), which is based on Banach's fixed point theorem.

Theorem 3.4. Let $\varphi : [0, \mathcal{K}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. If the condition (Q_3) holds and that $L(\widehat{\rho}_1 + \widehat{\rho}_2) < 1$, where $\widehat{\rho}_1$ and $\widehat{\rho}_2$ are respectively given by (7) and (8), then a unique solution to the problem (1) exists on $[0, \mathcal{K}]$.

Proof. Let us first consider a ball $B_\gamma = \{x \in \mathcal{A} : \|x\|_{\mathcal{A}} \leq \gamma\}$ with

$$\gamma \geq \frac{N(\widehat{\rho}_1 + \widehat{\rho}_2)}{1 - L(\widehat{\rho}_1 + \widehat{\rho}_2)}, \ \max_{t \in [0, \mathcal{K}]} |\varphi(t, 0, 0)| = N < \infty$$

and show that $\mathcal{F} B_\gamma \subset B_\gamma$, where the operator $\mathcal{F}$ is defined in (6). For any $x \in B_\gamma$, it follows by using the condition (Q_3) that

$$\begin{aligned} |\varphi(t, x(t), {}^c D^{\varrho} x(t))| &\leq |\varphi(t, x(t), {}^c D^{\varrho} x(t)) - \varphi(t, 0, 0)| + |\varphi(t, 0, 0)| \\ &\leq L(|x(t)| + |{}^c D^{\varrho} x(t)|) + |\varphi(t, 0, 0)| \\ &\leq L(\|x\| + \|{}^c D^{\varrho} x\|) + N = L\|x\|_{\mathcal{A}} + N \leq L\gamma + N. \end{aligned}$$

Using the above estimate, we obtain

$$\begin{aligned}
\|\mathcal{F}x\| &\leq \max_{t\in[0,\mathcal{K}]}\Bigg\{\int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)}|\varphi(s,x(s),{}^c D^{\varrho}x(s))|ds \\
&\quad+\frac{1}{|\nu_1-\lambda_2(\mathcal{K}-\nu_2)|}\Bigg[|\lambda_2|\int_{\nu_2}^{\mathcal{K}}\int_0^s \frac{(s-\xi)^{\sigma-1}}{\Gamma(\sigma)}|\varphi(\xi,x(\xi),{}^c D^{\varrho}x(\xi))|d\xi ds \\
&\quad+\int_0^{\nu_1}\frac{(\nu_1-s)^{\sigma}}{\Gamma(\sigma+1)}|\varphi(s,x(s),{}^c D^{\varrho}x(s))|ds\Bigg] \\
&\quad+\left|\frac{t+\omega(\nu_1-\lambda_1(\mathcal{K}-\nu_2))}{\nu_1-\lambda_1(\mathcal{K}-\nu_2)}\right|\Bigg[|\lambda_1|\Bigg(\int_0^{\mathcal{K}}\frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)}|\varphi(s,x(s),{}^c D^{\varrho}x(s))|ds \\
&\quad+\int_0^{\nu_2}\frac{(\nu_2-s)^{\sigma-1}}{\Gamma(\sigma)}|\varphi(s,x(s),{}^c D^{\varrho}x(s))|ds\Bigg) \\
&\quad+\int_0^{\nu_1}\frac{(\nu_1-s)^{\sigma-1}}{\Gamma(\sigma)}|\varphi(s,x(s),{}^c D^{\varrho}x(s))|ds\Bigg]\Bigg\} \\
&\leq (L\gamma+N)\max_{t\in[0,\mathcal{K}]}\Bigg\{\int_0^t\frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)}ds \\
&\quad+\frac{1}{|\nu_1-\lambda_2(\mathcal{K}-\nu_2)|}\Bigg[|\lambda_2|\int_{\nu_2}^{\mathcal{K}}\int_0^s\frac{(s-\xi)^{\sigma-1}}{\Gamma(\sigma)}d\xi ds+\int_0^{\nu_1}\frac{(\nu_1-s)^{\sigma}}{\Gamma(\sigma+1)}ds\Bigg] \\
&\quad+\left|\frac{t+\omega(\nu_1-\lambda_1(\mathcal{K}-\nu_2))}{\nu_1-\lambda_1(\mathcal{K}-\nu_2)}\right|\Bigg[|\lambda_1|\Bigg(\int_0^{\mathcal{K}}\frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)}ds+\int_0^{\nu_2}\frac{(\nu_2-s)^{\sigma-1}}{\Gamma(\sigma)}ds\Bigg) \\
&\quad+\int_0^{\nu_1}\frac{(\nu_1-s)^{\sigma-1}}{\Gamma(\sigma)}ds\Bigg]\Bigg\} \\
&= (L\gamma+N)\Bigg(\max_{t\in[0,\mathcal{K}]}\Bigg\{\frac{t^{\sigma}}{\Gamma(\sigma+1)} \\
&\quad+\left|\frac{t+\omega(\nu_1-\lambda_1(\mathcal{K}-\nu_2))}{\nu_1-\lambda_1(\mathcal{K}-\nu_2)}\right|\Bigg[\frac{\nu_1^{\sigma}+|\lambda_1|(\mathcal{K}^{\sigma}+\nu_2^{\sigma})}{\Gamma(\sigma+1)}\Bigg]\Bigg\} \\
&\quad+\frac{1}{|\nu_1-\lambda_2(\mathcal{K}-\nu_2)|}\Bigg[\frac{\nu_1^{\sigma+1}+|\lambda_2|(\mathcal{K}^{\sigma+1}-\nu_2^{\sigma+1})}{\Gamma(\sigma+2)}\Bigg]\Bigg) \\
&= (L\gamma+N)\widehat{\rho}_1.
\end{aligned}$$

Analogously, we find that

$$\begin{aligned}
\|{}^c D^{\varrho}\mathcal{F}x\| &\leq (L\gamma+N)\Bigg[\frac{\mathcal{K}^{\sigma-\varrho}}{\Gamma(\sigma-\varrho+1)}+\frac{\mathcal{K}^{1-\varrho}(\nu_1^{\sigma}+|\lambda_1|(\mathcal{K}^{\sigma}+\nu_2^{\sigma}))}{|\nu_1-\lambda_1(\mathcal{K}-\nu_2)|\Gamma(2-\varrho)\Gamma(\sigma+1)}\Bigg] \\
&= (L\gamma+N)\widehat{\rho}_2.
\end{aligned}$$

In consequence, we have

$$\|\mathcal{F}x\|_{\mathcal{A}}=\|\mathcal{F}x\|+\|{}^c D^{\varrho}\mathcal{F}x\|\leq(L\gamma+N)(\widehat{\rho}_1+\widehat{\rho}_2)\leq\gamma,$$

which implies that $\mathcal{F}x \in B_\gamma$. Since $x \in B_\gamma$ is an arbitrary element, therefore $\mathcal{F}B_\gamma \subset B_\gamma$.

In order to establish that $\mathcal{F}$ is a contraction, we take $x, y \in \mathbb{R}$ and obtain

$$\begin{aligned}
&\|\mathcal{F}x - \mathcal{F}y\| \\
= &\sup_{t\in[0,\mathcal{K}]} |\mathcal{F}x(t) - \mathcal{F}y(t)| \\
\leq &\sup_{t\in[0,\mathcal{K}]} \Bigg\{ \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(s,x(s),{}^c D^\varrho x(s)) - \varphi(s,y(s),{}^c D^\varrho y(s))| ds \\
&+ \frac{1}{|\nu_1 - \lambda_2(\mathcal{K}-\nu_2)|} \Bigg[|\lambda_2| \int_{\nu_2}^{\mathcal{K}} \int_0^s \frac{(s-\xi)^{\sigma-1}}{\Gamma(\sigma)} \\
&\times |\varphi(\xi,x(\xi),{}^c D^\varrho x(\xi)) - \varphi(\xi,y(\xi),{}^c D^\varrho y(\xi))| d\xi ds \\
&+ \int_0^{\nu_1} \frac{(\nu_1-s)^{\sigma}}{\Gamma(\sigma+1)} |\varphi(s,x(s),{}^c D^\varrho x(s)) - \varphi(s,y(s),{}^c D^\varrho y(s))| ds \Bigg] \\
&+ \left| \frac{t + \omega(\nu_1 - \lambda_1(\mathcal{K}-\nu_2))}{\nu_1 - \lambda_1(\mathcal{K}-\nu_2)} \right| \Bigg[|\lambda_1| \Bigg(\int_0^{\mathcal{K}} \frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)} \\
&\times |\varphi(s,x(s),{}^c D^\varrho x(s)) - \varphi(s,y(s),{}^c D^\varrho y(s))| ds \\
&+ \int_0^{\nu_2} \frac{(\nu_2-s)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(s,x(s),{}^c D^\varrho x(s)) - \varphi(s,y(s),{}^c D^\varrho y(s))| ds \Bigg) \\
&+ \int_0^{\nu_1} \frac{(\nu_1-s)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(s,x(s),{}^c D^\varrho x(s)) - \varphi(s,y(s),{}^c D^\varrho y(s))| ds \Bigg] \Bigg\} \\
\leq &\ L\widehat{\rho}_1 \|x - y\|.
\end{aligned}$$

In a similar manner, we get

$$\begin{aligned}
&\|{}^c D^\varrho \mathcal{F}x - {}^c D^\varrho \mathcal{F}y\| \\
\leq &\sup_{t\in[0,\mathcal{K}]} \Bigg\{ \int_0^t \frac{(t-s)^{\sigma-\varrho-1}}{\Gamma(\sigma-\varrho)} |\varphi(s,x(s),{}^c D^\varrho x(s)) - \varphi(s,y(s),{}^c D^\varrho y(s))| ds \\
&+ \frac{t^{1-\varrho}}{|\nu_1 - \lambda_1(\mathcal{K}-\nu_2)|\Gamma(2-\varrho)} \Bigg[|\lambda_1| \Bigg(\int_0^{\mathcal{K}} \frac{(\mathcal{K}-s)^{\sigma-1}}{\Gamma(\sigma)} \\
&\times |\varphi(s,x(s),{}^c D^\varrho x(s)) - \varphi(s,y(s),{}^c D^\varrho y(s))| ds \\
&+ \int_0^{\nu_2} \frac{(\nu_2-s)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(s,x(s),{}^c D^\varrho x(s)) - \varphi(s,y(s),{}^c D^\varrho y(s))| ds \Bigg) \\
&+ \int_0^{\nu_1} \frac{(\nu_1-s)^{\sigma-1}}{\Gamma(\sigma)} |\varphi(s,x(s),{}^c D^\varrho x(s)) - \varphi(s,y(s),{}^c D^\varrho y(s))| ds \Bigg] \Bigg\} \\
\leq &\ L\widehat{\rho}_2 \|x - y\|.
\end{aligned}$$

Thus, we obtain $\|\mathcal{F}x - \mathcal{F}y\|_{\mathcal{A}} \leq L(\widehat{\rho}_1 + \widehat{\rho}_2)\|x - y\|_{\mathcal{A}}$, which shows that $\mathcal{F}$ is

a contraction since $L(\widehat{\rho}_1 + \widehat{\rho}_2) < 1$. Therefore, $\mathcal{F}$ has a unique fixed point by Banach's fixed point theorem. Consequently, a unique solution to the problem (1) exists on $[0, \mathcal{K}]$. □

Remark 3.5. From (10), one can notice that $\delta < \widehat{\rho}_1 + \widehat{\rho}_2$. So, when the value of δ becomes that of $\delta < \widehat{\rho}_1 + \widehat{\rho}_2$, then, the existence of at least one solution changes to that of a unique solution.

4 Examples

Here, we present the examples illustrating the results obtained in the last section.

Example 4.1. Consider a nonlinear fractional differential equation with proportional boundary conditions given by

$$\begin{cases} {}^cD^{\frac{7}{4}}x(t) = \dfrac{8}{t^3+2}e^{-x^2(t)}\ln\Big(7+4\sin^2 x(t)\Big) + \dfrac{|{}^cD^{\frac{1}{4}}x(t)|}{(t^2+6)(1+|{}^cD^{\frac{1}{4}}x(t)|)}, \\ x\Big(\dfrac{1}{4}\Big) - x(0) = \dfrac{1}{3}\Big(x(1) - x\Big(\dfrac{3}{4}\Big)\Big), \ \displaystyle\int_0^{\frac{1}{4}} x(s)ds = \frac{1}{7}\int_{\frac{3}{4}}^{1} x(s)ds. \end{cases} \tag{11}$$

Here $t \in [0,1]$, that is, $\mathcal{K} = 1$. It is easy to find that

$$|\varphi(t, x(t), {}^c D^{\frac{1}{4}}x(t))| \leq \frac{8}{t^3+2}\ln(11) + \frac{1}{t^2+6} = \Upsilon(t)$$

and $\|\Upsilon\| = \max_{t\in[0,1]}|\Upsilon(t)| \simeq 9.75824776$. Thus, we deduce by the conclusion of Theorem 3.1 that the problem (11) has at least one solution on $[0,1]$.

Example 4.2. Consider the following fractional differential boundary value problem

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) = \varphi(t, x(t), {}^cD^{\frac{1}{2}}x(t)), \ t \in [0,1], \\ x\Big(\dfrac{1}{4}\Big) - x(0) = \dfrac{1}{4}\Big(x(1) - x\Big(\dfrac{1}{2}\Big)\Big), \ \displaystyle\int_0^{\frac{1}{4}} x(s)ds = \frac{1}{3}\int_{\frac{1}{2}}^{1} x(s)ds, \end{cases} \tag{12}$$

where $\lambda_1 = \frac{1}{4}$, $\lambda_2 = \frac{1}{3}$, $\mathcal{K} = 1$, $\sigma = \frac{3}{2}$, $\varrho = \frac{1}{2}$, $\nu_1 = \frac{1}{4}$, $\nu_2 = \frac{1}{2}$ and

$$\varphi(t, x(t), {}^cD^{\frac{1}{2}}x(t)) = \frac{1}{9+t^4}\Big[\frac{1}{\sqrt{t^2+4}}\Big(\frac{|x|^2}{1+|x|} + \sin({}^cD^{\frac{1}{2}}x(t))\Big) + \frac{1}{2}\Big]. \tag{13}$$

Using the given data, we find that $\omega = 9$, $\widehat{\rho}_1 \simeq 7.78187384$, $\widehat{\rho}_2 \simeq 4.14668997$, $\widehat{\rho}_1 + \widehat{\rho}_2 = 11.92856381$ and $\delta \simeq 10.17631103$ (δ is given by (10)).

Observe that

$$|\varphi(t, x(t), {}^cD^{\frac{1}{2}}x(t))| \leq \frac{1}{9+t^4}\Big(\frac{1+\|x\|_{\mathcal{A}}}{2}\Big).$$

Clearly

$$P(t) = \frac{1}{9+t^4}, \quad \psi(\|x\|_{\mathcal{A}}) = \frac{1+\|x\|_{\mathcal{A}}}{2}.$$

By the condition (Q_2), we find that $\mathcal{M} > \mathcal{M}_1$, where $\mathcal{M}_1 \simeq 1.96470216$. Thus, the hypothesis of Theorem 3.2 holds true and hence its conclusion implies that the problem (12) with $\varphi(t, x(t), {}^c D^{\frac{1}{2}}x(t))$ given by (13) has at least one solution on $[0, 1]$.

Example 4.3. Consider the problem (12) with

$$\varphi(t, x(t), {}^cD^{\frac{1}{2}}x(t)) = \frac{e^{-t}}{\sqrt{t^2+400}}\left(\frac{|{}^cD^{\frac{1}{2}}x(t)|}{1+|{}^cD^{\frac{1}{2}}x(t)|} + \tan^{-1}x(t)\right) + \frac{1}{1+t^4}. \quad (14)$$

Clearly, $\varphi(t, x(t), {}^cD^{\frac{1}{2}}x(t))$ satisfies the condition (Q_3) with $L = 1/20$ and $L(\widehat{\rho}_1 + \widehat{\rho}_2) \simeq 0.59642819$. Therefore, by Theorem 3.4, the problem (12) with $\varphi(t, x(t), {}^cD^{\frac{1}{2}}x(t))$ given by (14) has a unique solution on $[0, 1]$.

On the other hand, $|\varphi(t, x(t), {}^cD^{\frac{1}{2}}x(t))| \leq \varpi(t)$, where

$$\varpi(t) = \frac{e^{-t}(2+\pi)}{2\sqrt{t^2+400}} + \frac{1}{1+t^4}, \ \|\varpi\| = (42+\pi)/40.$$

Thus, the assumption (Q_4) is satisfied. Moreover, $L\delta \simeq 0.50881555 < 1$. Thus, all the assumptions of Theorem 3.3 are satisfied and hence its conclusion applies to the problem (12) with $\varphi(t, x(t), {}^cD^{\frac{1}{2}}x(t))$ given by (14).

5 Conclusions

In this article, we introduced a new concept of nonlocal boundary conditions with respect to the initial and final segments of the given domain and derived existence and uniqueness results for a nonlinear Caputo fractional differential equation equipped with these boundary conditions. Though we use the standard fixed point theorems in our study, yet their imposition to the given problem yields new results, which extend the literature on nonlocal boundary value problems in a new direction.

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Conflict of Interest

All authors declare no conflicts of interest in this paper.

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