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Quaternion-Based Representation of Rotation Minimizing Motions in Euclidean 3-space

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Abstract

This paper presents a quaternion-based framework for constructing rotation-minimizing motions in Euclidean 3-space, formulated via quaternion operator. By introducing a novel quaternion operator, we derive angular velocity representations directly from the quaternion derivative and its conjugate, enabling smooth and minimal-rotation motion. The proposed approach generates rotation-minimizing motions whose trajectories are aligned with the orbits of a given spatial curve, and it offers a convenient mechanism to compute the corresponding quaternion representation when the orbit and a spatial position are specified. The effectiveness of the method is demonstrated through numerical experiments involving the spherical indicatricestangent, normal, and binormal-of space curves. Additionally, we provide a geometric characterization of quaternionic helical curves with respect to the tangential image T, highlighting the theoretical and practical implications of the proposed model in motion design and spatial kinematics.

1 Introduction

The study of motion in three-dimensional space is fundamental to numerous fields, particularly in robotics, computer-aided geometric design (CAGD), and animation. In these domains, minimizing unnecessary rotational movement is

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essential to ensure smooth, stable, and energy-efficient trajectories. A powerful tool for achieving such motion is the *Rotation Minimizing Frame* (RMF), which offers an alternative to the classical Frenet frame by eliminating excessive twisting, especially near inflection points.

Bishop first introduced the concept of RMFs for spatial curves as a smooth, well-behaved alternative to Frenet frames [3]. Klok later proposed computationally efficient moving frames for trajectory sweeping in 3D applications [8], while Wang et al. developed the double reflection method, enhancing the stability and precision of RMF computation [16]. In a more general setting, Etayo extended the RMF framework to Riemannian manifolds by characterizing normal vector fields that are parallel with respect to the normal connection [6]. Farouki et al. further generalized the concept with rotation-minimizing osculating frames (RMOFs), showing their relevance in rigid-body motions free from yaw [7].

In parallel, quaternion algebra, introduced by Hamilton [10], has gained prominence as an efficient means to represent and compute 3D rotations. Unlike Euler angles, quaternions avoid singularities and gimbal lock, offering numerical stability and simplicity in interpolating rotations [15]. The integration of quaternions into RMF theory has been explored by Jttler [12], who used them to approximate rotation-minimizing motions (RMMs), and by Bayro-Corrochano, who employed Clifford geometric algebra in modeling biological kinematics [4]. Recently, Aslan and Yayl [2] applied quaternion-based geometric operators to describe motions on curves and surfaces.

In this paper, we introduce a quaternion operator-based formulation for constructing rotation-minimizing motions (RMMs) directly from a given space curve and its RMF. This framework offers a novel method for computing the angular velocity of motion as the product of a quaternion derivative and its conjugate, ensuring minimal and geometrically consistent rotation. Inspired by the operator-based approach of Aslan and Yayl, we derive explicit expressions for the quaternion representations of RMMs and investigate their geometric behavior along spherical indicatrices.

Furthermore, we provide numerical examples demonstrating the efficiency and smoothness of the proposed quaternion-based RMM method, highlighting its advantages in spatial kinematics, robotic motion planning, and geometric modeling.

By bridging the gap between quaternion algebra and RMF theory, this work provides a unified, efficient, and geometrically faithful approach to minimal-rotation motion design in Euclidean 3-space.

2 Preliminaries

This section introduces the fundamental concepts and algebraic structures that underpin the quaternion-based rotation-minimizing motions (RMMs) developed in this study. We provide the definitions and properties of rotation-minimizing frames (RMFs), quaternion algebra, and their connections to geometric (Clifford) algebra.

2.1 Rotation-Minimizing Motions and Frames

A spherical motion $U(s)$ is called a *rotation-minimizing motion* (RMM) along a trajectory $z(s)$ if the angular velocity $\omega(s)$ is minimized for all $s \in [0,1]$. More precisely, the total angular displacement is minimized in the following variational sense:

$$\int_0^1 \|\omega(s)\|\, ds.$$

This condition ensures that the motion requires the least possible rotational effort while remaining consistent with the prescribed trajectory [12].

A *rotation-minimizing frame* (RMF) is an orthonormal moving frame $\{T(s), U(s), V(s)\}$ along a spatial curve $\gamma(s)$ parameterized by arc-length (so that $\|\gamma'(s)\| = 1$). In this case, the unit tangent vector is simply

$$T(s) = \gamma'(s),$$

and the vectors $U(s)$ and $V(s)$ span the normal plane $T(s)^{\perp}$. The defining property of an RMF is that the angular velocity vector ω has no component in the tangent direction:

$$\langle \omega, T(s) \rangle \equiv 0.$$

[7].

Equivalently, the derivatives of the normal vectors satisfy

$$U'(s) = \lambda(s)T(s), \quad V'(s) = \mu(s)T(s),$$

for some scalar functions $\lambda(s)$, $\mu(s) \in \mathbb{R}$ [7].

When the RMF is constructed from the Frenet frame $\{T(s), N(s), B(s)\}$, where $N(s)$ and $B(s)$ are the principal normal and binormal vectors respectively, the RMF vectors are obtained via a rotation around the tangent vector:

$$U(s) = \cos\phi(s)\, N(s) + \sin\phi(s)\, B(s),$$
$$V(s) = -\sin\phi(s)\, N(s) + \cos\phi(s)\, B(s),$$

where the rotation angle $\phi(s)$ satisfies

$$\phi(s) = \phi_0 - \int_0^s \tau(\sigma)\, d\sigma,$$

and $\tau(\sigma)$ is the torsion of the curve [7].

The angular velocity of the RMF is then expressed as

$$\omega(s) = \beta(s)U(s) + \gamma(s)V(s),$$

where $\beta(s)$, $\gamma(s)$ are scalar functions determined by the geometry of the curve [6, 7].

The following result, adapted from [1], describes a construction of rotation-minimizing motions associated with spherical curves using RMFs.

Theorem 1. *Let $\beta(u) \subset S^2 \subset \mathbb{R}^3$ be a unit-speed spherical curve and let $\{\beta(u), M_1(u), M_2(u)\}$ be a rotation-minimizing frame (RMF) along the arc-length parameterized trajectory $\int \beta(u)\,du$. Define the matrix*

$$B(u) = \begin{bmatrix} \beta(u) & M_1(u) & M_2(u) \end{bmatrix},$$

and fix the initial point $e_0 = (1, 0, 0)^T$. Then the motion defined by $\beta(u) = B(u)e_0$ is a rotation-minimizing motion (RMM).

2.2 Real Quaternions

Let $\mathbb{H} = \{q = a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ be the algebra of real quaternions, which forms a 4-dimensional real vector space with the basis $\{1, i, j, k\}$ satisfying the multiplication rules:

$$i^2 = j^2 = k^2 = ijk = -1.$$

Quaternion multiplication is associative but not commutative. A quaternion $q \in \mathbb{H}$ is decomposed as:

$$q = \mathrm{S}(q) + \mathrm{V}(q),$$

where $\mathrm{S}(q) = a_0 \in \mathbb{R}$ is the scalar part and $\mathrm{V}(q) = a_1 i + a_2 j + a_3 k \in \mathbb{R}^3$ is the vector part. If $\mathrm{S}(q) = 0$, then q is called a *pure quaternion.*

The product $q * p$ of two quaternions $q = \mathrm{S}(q) + \mathrm{V}(q)$ and $p = \mathrm{S}(p) + \mathrm{V}(p)$ is given by:

$$q * p = \mathrm{S}(q)\mathrm{S}(p) - \langle \mathrm{V}(q), \mathrm{V}(p) \rangle + \mathrm{S}(q)\mathrm{V}(p) + \mathrm{S}(p)\mathrm{V}(q) + \mathrm{V}(q) \times \mathrm{V}(p).$$

The conjugate, norm, modulus, and inverse of q are defined as:

$$\begin{aligned} \bar{q} &= a_0 - a_1 i - a_2 j - a_3 k, \\ N_q &= \bar{q} * q = |q|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2, \\ q^{-1} &= \frac{\bar{q}}{N_q}, \quad N_q \neq 0. \end{aligned}$$

A unit quaternion satisfies $N_q = 1$ and it is represented in trigonometric form as:

$$q = \cos\theta + \sin\theta\, v, \quad \text{where } v \in \mathbb{R}^3, \|v\| = 1.$$

[9, 15, 17, 5]

2.3 Rotations via Unit Quaternions

Let $w \in \mathbb{R}^3$ be a pure quaternion and p a unit quaternion. Then the linear transformation $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\varphi(w) = p * w * p^{-1}$$

represents a rotation in $\mathbb{R}^3$. The matrix representation R of this rotation is orthogonal and explicitly constructed from the components of p [15].

2.4 Geometric Product and Geometric Algebra

In the framework of geometric algebra, the product of two vectors $a, b \in \mathbb{R}^3$ is called the *geometric product* and is defined by:

$$a * b = \langle a, b \rangle + a \times b,$$

where the symmetric part $\langle a, b \rangle$ is the inner (dot) product and the antisymmetric part $a \times b$ is the vector product.

The vector product $a \times b$ is a bivector, representing an oriented area element in the plane spanned by a and b. The inner and vector products are computed via:

$$\langle a, b \rangle = \frac{1}{2}(a * b + b * a), \quad a \times b = \frac{1}{2}(a * b - b * a).$$

This formalism provides a unified and coordinate-free approach to geometric transformations, including rotations and reflections, and is particularly useful in the modeling of kinematic motions.[4]

2.5 Quaternion Operator

Let $a, b \in \mathbb{R}^3$ be non-zero vectors. The *quaternion operator* Q that transforms a into b is defined as:

$$Q = \frac{1}{\|a\|^2} \left(\langle a, b \rangle + a \times b \right), \tag{1}$$

where $\langle a, b \rangle$ is the scalar inner product and $a \times b$ is the vector (cross) product.

Applying this operator to a yields:

$$Q * a = b, \tag{2}$$

where the multiplication is the quaternion (geometric) product. This transformation rotates a into b about the axis defined by $a \times b$ and scales by $\|b\|/\|a\|$ [2].

2.6 Quaternionic Frenet Frame and Helices

In $\mathbb{R}^4$, quaternionic curves are described using a generalized Frenet frame. Let $\alpha(s)$ be a unit-speed quaternion-valued curve parameterized by arc-length, so that $\|\alpha'(s)\| = 1$. The Frenet frame $\{T, N_1, N_2, N_3\}$ is then completely determined by the curve $\alpha(s)$, and the generalized Frenet equations are given by [19]:

$$\begin{aligned} T' &= KN_1, \\ N_1' &= -KT + kN_2, \\ N_2' &= -kN_1 + (r-K)N_3, \\ N_3' &= -(r-K)N_2, \end{aligned} \tag{3}$$

where T, N_1, N_2, N_3 form an orthonormal frame and K, k, r are curvature and torsion functions.

A generalized quaternionic helix satisfies the condition

$$\left(\frac{K}{k}\right)^2 + \frac{1}{(r-K)^2}\left(\left(\frac{K}{k}\right)'\right)^2 = \text{constant}, \tag{4}$$

which characterizes constant-angle properties between the tangent vector $T(s)$ and a fixed direction in four-dimensional space.

Quaternionic helices are important in the study of the rotational behavior of curves and rigid body motions in both Euclidean and Minkowski geometries.

3 Rotation Minimizing Motions with Quaternion Operator

In many applications of kinematics and geometric modeling, describing motions with minimal angular velocity is of both theoretical and practical interest, especially in reducing unnecessary twisting along a given trajectory. One such motion is the *Rotation Minimizing Motion* (RMM), which is particularly useful when the orbit lies on a sphere.

Let $\alpha(t) \subset S^2$ be a spherical curve. A spherical motion $U(t)$ is called a rotation minimizing motion (RMM) if its angular velocity vector is given by

$$w(t) = \alpha(t) \times \alpha'(t),$$

ensuring that the angular speed is minimized, as originally characterized by Jttler [12].

Assuming that $\alpha(s)$ is parameterized by arc-length, we define the Sabban frame $\{\alpha(s), T(s), S(s)\}$, where $T(s) = \alpha'(s)$ is the unit tangent vector and $S(s) = \alpha(s) \times T(s)$ is the unit normal vector on the sphere. This orthonormal frame satisfies the following differential system:

$$\begin{bmatrix} \alpha'(s) \\ T'(s) \\ S'(s) \end{bmatrix} = \begin{bmatrix} 0 & m & m \\ -m & 0 & n \\ 0 & -n & 0 \end{bmatrix} \begin{bmatrix} \alpha(s) \\ T(s) \\ S(s) \end{bmatrix} \tag{5}$$

Here, the geodesic curvature is given by $k_g = \frac{n}{m}$. For more details, see [14].

It is rotated about the position vector $\alpha(s)$ by an angle $\theta(s) = \int n(s)\, ds$, where $n(s)$ is the geodesic curvature, to obtain a Rotation Minimizing Frame (RMF) $\{\alpha(s), M_1(s), M_2(s)\}$.

According to Theorem 1, the corresponding spherical motion

$$U(s) = \begin{bmatrix} \alpha & M_1 & M_2 \end{bmatrix}$$

defines a rotation minimizing motion whose angular velocity norm $\|w(s)\|$ is minimal. For further mathematical details and derivations, we refer the reader to [1].

Remark 1. *In the context of the rotation minimizing motion described by the frame $U(s) = \{\alpha(s), M_1(s), M_2(s)\}$, where*

$$M_1(s) = \cos\theta(s)\, T(s) + \sin\theta(s)\, S(t), \quad M_2(s) = -\sin\theta(s)\, T(s) + \cos\theta(s)\, S(s),$$

it follows from the angular velocity formula that

$$w(s) = mS(s) \tag{6}$$

with $\|w(s)\| = m$. Since the norm of the angular velocity vector $w(s)$ of this motion satisfies

$$\|w(s)\| \leq \|W(s)\|$$

for any other differentiable orthonormal frame along $\alpha(s)$, the motion defined by $U(s)$ minimizes the angular speed.

Therefore, the scalar m coincides with the minimal angular speed of the motion represented by the rotation minimizing frame $U(s)$.

Now, in the following theorem, we give the quaternion corresponding to the RMM using the quaternion operator.

Theorem 2. *Assume that $\alpha(t)$ is a spherical curve in $\mathbb{R}^3$ and*

$$U(t) = \begin{bmatrix}\alpha(t) & M_1(t) & M_2(t)\end{bmatrix}$$

defines a rotation minimizing motion (RMM) along $\alpha(t)$. Suppose further that the motion of a fixed point $P \in \mathbb{R}^3$ under $U(t)$ satisfies

$$\alpha(t) = U(t)P.$$

Then the corresponding quaternion operator representing this motion is given by

$$Q(t) = \langle P, \alpha(t)\rangle + P \times \alpha(t), \tag{7}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product and $\times$ denotes the cross product.

Moreover, this quaternion $Q(t)$ characterizes the Rotation Minimizing Motion (RMM) defined by $U(t)$.

Proof. Let $Q(t) = \langle P, \alpha(t)\rangle + P \times \alpha(t)$ be the quaternion operator associated with the point P and the orbit curve $\alpha(t)$. We aim to show that

$$Q(t) * P = \alpha(t),$$

where $*$ denotes the quaternion product, and P is considered as a pure quaternion.

We begin by substituting the expression of $Q(t)$:

$$Q(t) * P = (\langle P, \alpha(t)\rangle + P \times \alpha(t)) * P.$$

Using quaternion multiplication rules for pure quaternions, we recall the identity:

$$(a + A) * P = aP + A \times P,$$

where a is a scalar, and A, P are pure quaternions (vectors). Applying this to our case, we get:

$$Q(t) * P = \langle P, \alpha(t)\rangle P + (P \times \alpha(t)) \times P.$$

Now we use the vector triple product identity:

$$(P \times \alpha(t)) \times P = \alpha(t) - \langle P, \alpha(t)\rangle P.$$

Substituting this back, we obtain:

$$\begin{aligned} Q(t) * P &= \langle P, \alpha(t)\rangle P + [\alpha(t) - \langle P, \alpha(t)\rangle P] \\ &= \langle P, \alpha(t)\rangle P + \alpha(t) - \langle P, \alpha(t)\rangle P \\ &= \alpha(t). \end{aligned}$$

Hence, the orbit curve $\alpha(t)$ is obtained by the action of the quaternion operator $Q(t)$ on the point P via quaternion multiplication.

□

Theorem 3. *Assume that $\alpha(t)$ is a spherical curve in $\mathbb{R}^3$ and*

$$U(t) = \begin{bmatrix} \alpha(t) & M_1(t) & M_2(t) \end{bmatrix}$$

defines a rotation minimizing motion (RMM) along $\alpha(t)$ with orbit $\alpha(t)$ of a fixed point $P \in \mathbb{R}^3$. Suppose that the quaternion operator $Q(t)$ is defined as in (7).

Then the angular velocity vector $w(t)$ associated with the RMF $\{\alpha(t), M_1(t), M_2(t)\}$ is given by

$$\dot{Q}(t) * \bar{Q}(t) = w(t), \tag{8}$$

where $\dot{Q}(t)$ denotes the time derivative of $Q(t)$, $\bar{Q}(t)$ is its quaternionic conjugate, and $*$ denotes quaternion multiplication.

Proof. The quaternion operator $Q(t)$ is defined by:

$$Q(t) = \langle P, \alpha(t)\rangle + P \times \alpha(t),$$

where $P \in \mathbb{R}^3$ is a fixed unit pure quaternion.

Taking the derivative with respect to t:

$$\dot{Q}(t) = \langle P, \alpha'(t)\rangle + P \times \alpha'(t).$$

Using $\alpha'(t) = mT(t)$ in (5), we get:

$$\dot{Q}(t) = m\left(\langle P, T(t)\rangle + P \times T(t)\right).$$

The conjugate of $Q(t)$ is:

$$\bar{Q}(t) = \langle P, \alpha(t)\rangle - P \times \alpha(t).$$

So the quaternion product becomes:

$$\dot{Q}(t) * \bar{Q}(t) = m\left(\langle P, T\rangle + P \times T\right) * \left(\langle P, \alpha\rangle - P \times \alpha\right).$$

Letting $P = (1, 0, 0)$, and writing $\alpha(t) = (\alpha_1, \alpha_2, \alpha_3)$, $T(t) = (t_1, t_2, t_3)$, then:

$$Q(t) = \alpha_1 + \alpha_3 j - \alpha_2 k, \quad \dot{Q}(t) = m(t_1 - t_3 j + t_2 k).$$

Compute the quaternion product:

$$\begin{aligned}\dot{Q}(t) * \bar{Q}(t) &= m(t_1 - t_3 j + t_2 k)(\alpha_1 + \alpha_3 j - \alpha_2 k) \\ &= m\left[(\text{scalar part}) + (\text{vector part})\right].\end{aligned}$$

The scalar part is:

$$t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3 = \langle T, \alpha \rangle = 0$$

The vector part simplifies to:

$$\alpha(t) \times T(t) = S(t).$$

Thus:

$$\dot{Q}(t) * \bar{Q}(t) = mS(t) = w(t).$$

Hence, the angular velocity vector of the RMM is given by the quaternionic product:

$$\dot{Q}(t) * \bar{Q}(t) = w(t). \qquad \square$$

□

Remark 2. *Jttler [12] calculated the angular velocity of the RMM as follows:*

$$w(t) = 2\dot{Q}(t) * \bar{Q}(t) \tag{9}$$

where $Q(t)$ is a unit quaternion corresponding to the spherical motion $U = U(t)$. He also used the equality as follows:

$$Up_0 = \dot{Q}(t) * p_0 * \bar{Q}(t) \tag{10}$$

with this method, it is quite difficult to find the unit quaternion corresponding to the spherical motion.

In summary, the proposed operator-based method is computationally more direct and easier to apply, since obtaining $Q(t)$ requires only the knowledge of the orbit curve $\alpha(t)$, without any additional normalization or auxiliary equations.

Corollary 1. *Rotation minimizing motion (RMM) in Euclidean 3-space is represented with $U(t) = \begin{bmatrix} \alpha & M_1 & M_2 \end{bmatrix}$ and $Q(t)$ is unit quaternion corresponding to this motion. Thus, it is written as*

$$\begin{bmatrix} \alpha & M_1 & M_2 \end{bmatrix} P = Q(t) * P = \alpha(t) \tag{11}$$

Example 1. *For the circular curve on S^2 defined by*

$$\alpha(t) = (\sin(\psi)\cos(t\phi), \sin(\psi)\sin(t\phi), \cos(\psi)) \tag{12}$$

with the suitable angles ψ and ϕ. The quaternion corresponding to the RMM of the given curve $\alpha(t)$ is defined as $Q(t) = Q_1(t) * Q_2(t)$. Its quaternion representation is computed as

$$Q(t) = \Bigg(\cos\left(\frac{\psi}{2}\right)\cos\left(\frac{\phi t(\cos(\psi)-1)}{2}\right), -\sin\left(\frac{\psi}{2}\right)\sin\left(\frac{\phi t(\cos(\psi)+1)}{2}\right), \sin\left(\frac{\psi}{2}\right)\cos\left(\frac{\phi t(\cos(\psi)+1)}{2}\right), -\sin\left(\frac{\phi t(\cos(\psi)-1)}{2}\right)\cos\left(\frac{\psi}{2}\right)\Bigg) \tag{13}$$

From (9) and (13), angular velocity of the spherical motion $U(t)$ is computed as follows:

$$w(t) = \big(\cos(t\phi)\phi\sin(\psi)\cos(\psi), -\sin(t\phi)\phi\sin(\psi)\cos(\psi), \phi\sin^2(\psi)\big) \tag{14}$$

For more details, see in [12].

Now, in the present paper, we can achieve the same result using the quaternion operator that is the subject of this study. Using the quaternion operator defined as in (7), we obtain the representation quaternion of the RMM as follows:

$$\mathfrak{Q}(t) = (\sin(\psi)\cos(t\phi), 0, -\cos(\psi), \sin(\psi)\sin(t\phi)) \tag{15}$$

where $P = (1, 0, 0)$. Now we can find the angular velocity of the RMF defined as $\{\alpha, M_1, M_2\}$ with the help of Theorem 3. By differentiating (15) with respect to t, we get

$$\dot{\mathfrak{Q}}(t) = (-\phi\sin(\psi)\sin(t\phi), 0, 0, \phi\sin(\psi)\cos(t\phi)) \tag{16}$$

If we take the conjugate of (15), we obtain

$$\bar{\mathfrak{Q}}(t) = (\sin(\psi)\cos(t\phi), 0, \cos(\psi), -\sin(\psi)\sin(t\phi)) \tag{17}$$

Then by using (16) and (17), we get the angular velocity of the RMF as

$$\begin{aligned} \dot{\mathfrak{Q}}(t) * \bar{\mathfrak{Q}}(t) &= \phi\sin(\psi)\, S \\ &= mS \\ &= w(t) \end{aligned} \tag{18}$$

where $S = \alpha \times T$ is binormal which is defined as in (5) and $\times$ denotes cross product in $\mathbb{R}^3$. The result here is identical to (14).

Remark 3. *For the frames $\{\alpha(t), \alpha'(t), S(t)\}$ and $\{\alpha, M_1, M_2\}$, we have*

$$A(t) = \begin{bmatrix} \alpha(t) & \alpha'(t) & S(t) \end{bmatrix} \tag{19}$$

and

$$B(t) = \begin{bmatrix} \alpha(t) & M_1 & M_2 \end{bmatrix} \tag{20}$$

where $A(t)$ is spherical motion matrix with the orbit $\alpha(t) = A(t)e_1$ and $e_1 = (1,0,0)$ is a basis for $\mathbb{R}^3$. $B(t)$ is rotation minimizing motion (RMM) with the orbit $\alpha(t) = B(t)e_1$. Evidently, it proves the equation $\bar{w} = \alpha \times \alpha'$ given in [12] with Proposition 1. We can easily see that

$$\begin{aligned} \alpha \times \alpha' &= \alpha \times (m\cos\theta M_1 - m\sin\theta M_2) \\ &= m\cos\theta M_2 + m\sin\theta M_1 \\ &= \bar{w} \end{aligned} \tag{21}$$

4 On the Applications of Spherical Indicatrices

In this section, we derive the RMM motion for spherical indicatrices of a given curve with orbits T, N, B respectively, using the quaternion operator.

4.1 Applications of Tangent Indicatrix (T)

Let $\alpha(s)$ be a unit-speed regular space curve in the Euclidean 3-space $\mathbb{E}^3$, and let $\{T(s), N(s), B(s)\}$ be its classical Frenet frame. The differential equations of the Frenet frame is written in matrix form as follows:

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{22}$$

Here, $\kappa(s)$ and $\tau(s)$ denote the curvature and torsion of the curve $\alpha(s)$, respectively. The corresponding *Darboux vector* is given by

$$\omega(s) = \tau(s)T(s) + \kappa(s)B(s). \tag{23}$$

Now, let $P = (1,0,0)$ be a fixed unit vector. We define a unit quaternion $Q(s)$ associated with the tangent vector $T(s)$ as follows:

$$Q(s) = \langle P, T(s)\rangle + P \times T(s). \tag{24}$$

Then, the tangent indicatrix $T(s)$ is described by a *Rotation minimizing motion (RMM)* on the unit sphere, where $T(s)$ traces a trajectory induced by the quaternion $Q(s)$. This motion has angular velocity:

$$\dot{Q}(s) * \overline{Q}(s) = \kappa(s)B(s) = \bar{\omega}(s), \tag{25}$$

where $\bar{\omega}(s)$ corresponds to the angular velocity vector of the RMM in the *Bishop frame* $\{T(s), N_1(s), N_2(s)\}$.

The Bishop frame is an alternative orthonormal frame that remains well-defined even at points where the curvature vanishes. It is given by the orthonormal set $\{T(s), N_1(s), N_2(s)\}$, where N_1 and N_2 are relatively parallel vector fields perpendicular to T. The evolution equations are:

$$\begin{aligned} T'(s) &= k_1(s)N_1(s) + k_2(s)N_2(s), \\ N_1'(s) &= -k_1(s)T(s), \\ N_2'(s) &= -k_2(s)T(s), \end{aligned} \tag{26}$$

where $k_1(s)$ and $k_2(s)$ are the natural curvatures of the frame.

Additionally, if $A(s) = [T(s)\ N_1(s)\ N_2(s)]$ is the matrix representing the Bishop frame, then the RMM satisfies:

$$A(s)P = T(s),$$

indicating that $T(s)$ is the image of the fixed vector P under a rotation determined by the Bishop frame.

Since:

$$Q(s) * P = T(s),$$

the quaternion $Q(s)$ also determines the same RMM as that generated by the Bishop frame.

4.2 Applications of Normal Indicatrix (N)

Let $\alpha : I \to \mathbb{E}^3$ be a unit-speed regular curve with Frenet frame

$$\{T(s), N(s), B(s)\}$$

, and let $N(s)$ denote the principal normal vector. The *normal indicatrix* of α is defined as the curve traced by $N(s)$ on the unit sphere S^2.

To study the geometry of this normal indicatrix, consider the integral curve:

$$\gamma(s) = \int N(s)\, ds.$$

Along $\gamma(s)$, we define the *N-Bishop frame*, which is obtained by rotating an alternative orthonormal frame $\{N(s), C(s), W(s)\}$ about the normal vector $N(s)$ by an angle $\theta(s)$. This rotation yields the orthonormal frame

$$\{N(s), \tilde{N}_1(s), \tilde{N}_2(s)\}$$

whose evolution is governed by the system of equations [13]:

$$\begin{bmatrix} N'(s) \\ \tilde{N}_1'(s) \\ \tilde{N}_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ \tilde{N}_1(s) \\ \tilde{N}_2(s) \end{bmatrix}.$$

The functions k_1 and k_2 are called the *N-Bishop curvatures*, and $\theta(s) = \arctan(k_2/k_1)$ denotes the rotation angle. This frame minimizes rotation around the normal direction, and is particularly useful in characterizing slant helices and spherical curves.

Now, let $\{N, M_1, M_2\}$ denote a Bishop frame along the normal direction curve $\gamma(s)$, such that M_1 and M_2 are orthonormal vectors orthogonal to N. Then, we define the motion matrix:

$$A(s) = \begin{bmatrix} N(s) & M_1(s) & M_2(s) \end{bmatrix},$$

and consider the trajectory of a point $P = (1, 0, 0)^T$. The curve $A(s)P = N(s)$ shows that the point P traces the normal indicatrix under a *rotation minimizing motion (RMM)*. The Darboux vector of this motion is:

$$D = fW,$$

where $f = \sqrt{\kappa^2 + \tau^2}$ and $W = \frac{\tau T + \kappa B}{f}$ is a unit vector aligned with the rotation axis.

This RMM can also be described using unit quaternions. Let $Q(s)$ be the unit quaternion operator defined by:

$$Q(s) = \langle P, N(s) \rangle + P \times N(s),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product and $\times$ is the cross product in $\mathbb{R}^3$. Then the angular velocity of this motion satisfies:

$$\dot{Q}(s) * \overline{Q}(s) = fW = D.$$

Thus, the quaternion $Q(s)$ realizes the RMM such that the orbit of point P coincides with the normal indicatrix $N(s)$. This elegant representation connects the geometry of the normal indicatrix to quaternionic kinematics and enhances our understanding of N-Bishop frame applications.

4.3 Applications of Binormal Indicatrix (B)

Let $\gamma : I \to \mathbb{E}^3$ be a unit-speed regular space curve, and let $B(s)$ denote its binormal vector field. The trajectory traced by $B(s)$ on the unit sphere is known as the *binormal indicatrix*, which provides valuable geometric information about the torsional behavior of the curve.

Consider a Bishop frame $\{B(s), L_1(s), L_2(s)\}$ constructed along the binormal indicatrix, where $L_1(s)$ and $L_2(s)$ are orthonormal vector fields perpendicular to $B(s)$. This structure forms a *Type-2 Bishop frame*, as introduced by Ylmaz and Turgut [18]. In this frame, $B(s)$ is treated as a fixed direction, and the remaining frame vectors rotate within the normal plane.

The differential equations for the Type-2 Bishop frame

$$\{N_1(s), N_2(s), B(s)\}$$

are given by:

$$\begin{aligned} N_1'(s) &= -k_1(s)B(s), \\ N_2'(s) &= -k_2(s)B(s), \\ B'(s) &= k_1(s)N_1(s) + k_2(s)N_2(s), \end{aligned} \tag{27}$$

where $k_1(s)$ and $k_2(s)$ are the Bishop curvatures of the second kind.

These curvatures are related to the torsion $\tau(s)$ and a smooth angle function $\theta(s)$ as follows:

$$\begin{aligned} k_1(s) &= -\tau(s)\cos\theta(s), \\ k_2(s) &= -\tau(s)\sin\theta(s). \end{aligned} \tag{28}$$

The transformation between the classical Frenet frame $\{T, N, B\}$ and the Type-2 Bishop frame is given by:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sin\theta(s) & -\cos\theta(s) & 0 \\ \cos\theta(s) & \sin\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_1(s) \\ N_2(s) \\ B(s) \end{bmatrix}. \tag{29}$$

To describe the motion of $B(s)$ using quaternion representation, let $P = (1, 0, 0)$ be a fixed vector, and define the quaternion:

$$Q(s) = \langle P, B(s) \rangle + P \times B(s). \tag{30}$$

This unit quaternion defines a rotation minimizing motion (RMM) on the unit sphere. The derivative of $Q(s)$ satisfies:

$$\dot{Q}(s) * \bar{Q}(s) = \tau(s)T(s), \tag{31}$$

and it rotates the vector P as:

$$Q(s) * P = B(s) \tag{32}$$

Hence, the matrix motion defined by:

$$A(s) = \begin{bmatrix} B(s) & L_1(s) & L_2(s) \end{bmatrix} \tag{33}$$

corresponds to a rotational motion whose orbit is the binormal field of the curve. The angular velocity vector of this motion is the Darboux vector $F(s) = \tau(s)T(s)$.

Therefore, the motion $A(t)$ generates an RMM (Rotation minimizing motion) whose trajectory is B.

Example 2. *We consider the space curve*

$$\rho(s) = \left(-\frac{3}{4}\left(\frac{\cos(3s)}{9} + \cos(s)\right), -\frac{3}{4}\left(\frac{\sin(3s)}{9} + \sin(s)\right), -\frac{\sqrt{3}}{2}\cos(s) \right), \tag{34}$$

which is parameterized by arc-length s.
This curve is a unit-speed slant helix, as its principal normal vector maintains a constant angle with a fixed direction in space [11]. The graphic of curve $\rho(s)$ is illustrated in Figure 1.

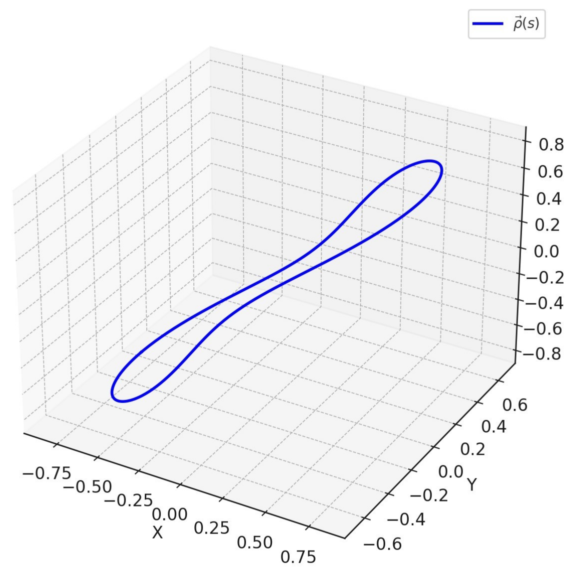

Figure 1: Slant helix $\rho = \rho(s)$

The Frenet vectors and the curvatures of the curve $\rho(s)$ are as:

$$\begin{aligned} T(s) &= \left(\frac{3}{4}\left(\frac{\sin(3s)}{3}+\sin(s)\right),\ -\frac{3}{4}\left(\frac{\cos(3s)}{3}+\cos(s)\right),\ \frac{\sqrt{3}}{2}\sin(s)\right), \\ N(s) &= \left(\frac{\sqrt{3}}{2}\cos(2s), \frac{\sqrt{3}}{2}\sin(2s), \frac{1}{2}\right), \\ B(s) &= \Bigg(-\frac{3}{8}\left(\frac{\cos(3s)}{3}+\cos(s)\right)-\frac{3}{4}\sin(2s)\sin(s), -\frac{3}{8}\left(\frac{\sin(3s)}{3}+\sin(s)\right) \\ &\quad +\frac{3}{4}\sin(s)\cos(2s), \frac{\sqrt{3}}{2}\cos(s)\Bigg), \end{aligned} \tag{35}$$

and

$$\begin{aligned} \kappa(s) &= \sqrt{3}\cos(s), \\ \tau(s) &= \sqrt{3}\sin(s) \end{aligned} \tag{36}$$

Based on the definition of the Bishop frame given in (26), the Bishop motion $A(s) = [T(s)\ N_1(s)\ N_2(s)]$ represents a rotation minimizing motion whose orbit corresponds to the tangential indicatrix T, satisfying the condition $AP = T$ for the fixed point $P = (1, 0, 0)$. From (25), we can determine $\bar{\omega}$ the angular velocity of the RMM as

$$\begin{aligned} \bar{\omega} = \sqrt{3}\cos(s)\Bigg(&-\frac{3}{8}\left(\frac{\cos(3s)}{3}+\cos(s)\right)-\frac{3}{4}\sin(2s)\sin(s), \\ &-\frac{3}{8}\left(\frac{\sin(3s)}{3}+\sin(s)\right)+\frac{3}{4}\sin(s)\cos(2s), \frac{\sqrt{3}}{2}\cos(s)\Bigg), \end{aligned} \tag{37}$$

By appliying the unit quaternion operator in (24), we get

$$Q(s) = \left(\frac{3}{4}\left(\frac{\sin(3s)}{3}+\sin(s)\right), 0, -\frac{\sqrt{3}}{2}\sin(s), -\frac{3}{4}\left(\frac{\cos(3s)}{3}+\cos(s)\right)\right), \tag{38}$$

If we take the derivative of (38), we obtain

$$\dot{Q}(s) = \left(\frac{3}{4}\Big(\cos(3s)+\cos(s)\Big), 0, -\frac{\sqrt{3}}{2}\cos(s), -\frac{3}{4}\Big(\sin(3s)+\sin(s)\Big)\right), \tag{39}$$

We can write the conjugate of (38) as

$$\bar{Q}(s) = \left(\frac{3}{4}\left(\frac{\sin(3s)}{3} + \sin(s)\right), 0, \frac{\sqrt{3}}{2}\sin(s), \frac{3}{4}\left(\frac{\cos(3s)}{3} + \cos(s)\right)\right), \quad (40)$$

By using (25), (39) and (40), we obtain the equality in (25) as follows,

$$\begin{aligned} \dot{Q}(s) * \overline{Q}(s) &= \kappa(s)B(s) = \bar{\omega}(s), \\ &= \sqrt{3}\cos(s)\Big(-\frac{3}{8}\left(\frac{\cos(3s)}{3} + \cos(s)\right) - \frac{3}{4}\sin(2s)\sin(s), \\ &-\frac{3}{8}\left(\frac{\sin(3s)}{3} + \sin(s)\right) + \frac{3}{4}\sin(s)\cos(2s), \frac{\sqrt{3}}{2}\cos(s)\Big), \end{aligned} \quad (41)$$

Thus, since the unit quaternion operator $Q(s)$ satisfies the relation $Q * P = T$, the point $P = (1, 0, 0)$ undergoes a rotation minimizing motion (RMM) whose orbit is the curve T. In the following graphics, we give the comparison of Frenet frame vectors (N, B) and the Bishop frame vectors (N_1, N_2) on the unit slant helix $\rho(s)$ in Figure 2 and Figure 3. Finally, in the following figure, we give the graphic of tangent indicatrix (T) of unit slant helix $\rho(s)$ on the unit sphere in Figure 4.

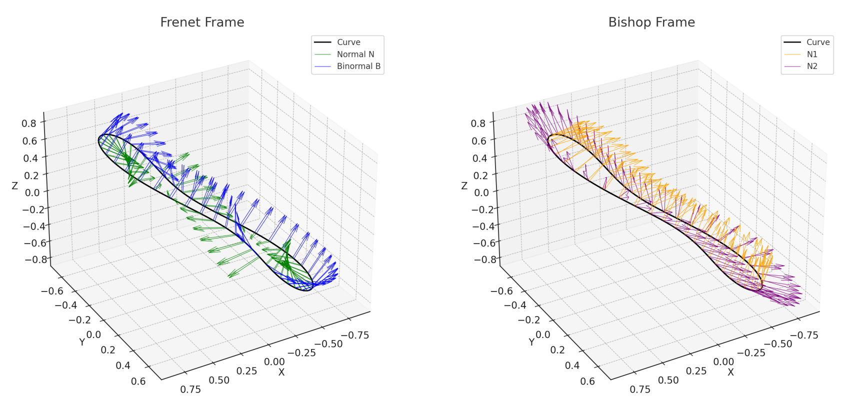

Figure 2: Comparison of the Frenet frame and the Bishop frame on the unit slant helix $\rho(s)$

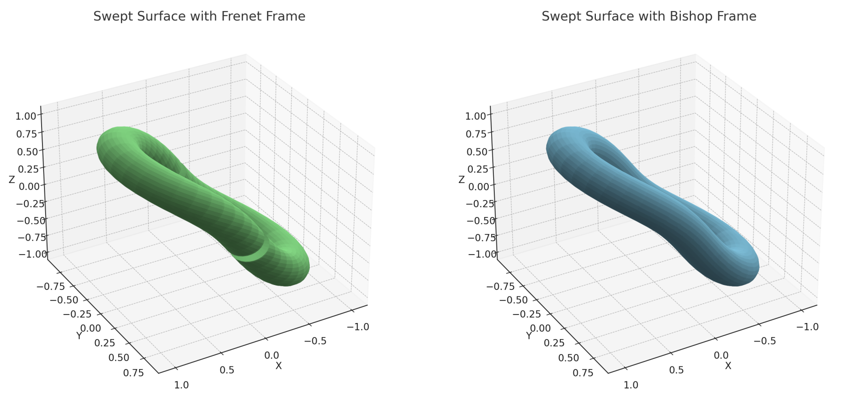

Figure 3: Comparison of swept surfaces generated using the Frenet frame (left) and the Bishop frame (right) on the unit slant helix $\rho(s)$

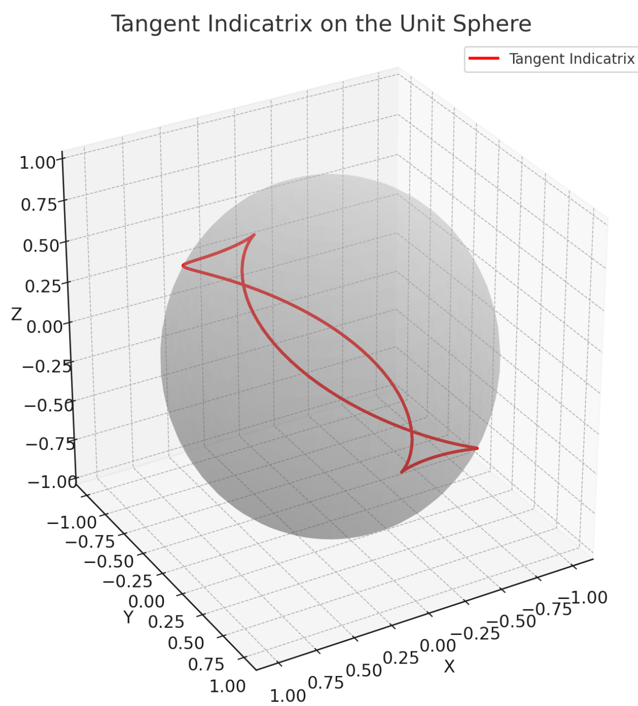

Figure 4: The tangent indicatrix T of the slant helix $\rho(s)$ on the unit sphere

5 Quaternionic Characterization of Helices via Rotation Minimizing Motions

In this section, we define the Rotation Minimizing Motion (RMM) along a given space curve $\gamma(t) \subset \mathbb{R}^3$ using its tangential trajectory $T(t)$. Moreover, we provide a characterization of the helical structure of the motion within the quaternionic framework.

Let $\gamma(t)$ be a curve in $\mathbb{R}^3$ with its Frenet frame $\{T, N, B\}$. We also consider a Rotation Minimizing Frame (RMF) $\{T, M_1, M_2\}$ along $\gamma(t)$, such that it is defined on the trajectory $\int T\,dt$. The rotation matrix $U = \begin{bmatrix} T & M_1 & M_2 \end{bmatrix}$ satisfies the relation $UP = T$, where $P = (1, 0, 0)$. Thus, U represents the RMM along the trajectory T, with reference direction P. In this case, the Darboux vector of the RMM is given by $w(t) = \kappa B$.

To describe the motion quaternionically, we define the quaternion $Q(t)$ such that

$$Q(t) * P = T(t),$$

where $P = (0, 1, 0, 0)$ denotes the corresponding quaternionic form of the vector $(1, 0, 0)$. By Theorem 2, the unit quaternion $Q(t)$ is written as

$$Q(t) = \langle P, T \rangle + P \times T, \tag{42}$$

and its conjugate is

$$\bar{Q}(t) = \langle P, T \rangle - P \times T. \tag{43}$$

Differentiating $Q(t)$ with respect to t, we obtain:

$$\dot{Q}(t) = \kappa[\langle P, N \rangle + P \times N]. \tag{44}$$

Hence, we have

$$\dot{Q}(t) * \bar{Q}(t) = w(t) = \kappa B. \tag{45}$$

The Frenet frame evolution is written as:

$$\begin{bmatrix} B \\ -N \\ T \end{bmatrix}' = \begin{bmatrix} 0 & \tau & 0 \\ -\tau & 0 & \kappa \\ 0 & -\kappa & 0 \end{bmatrix} \begin{bmatrix} B \\ -N \\ T \end{bmatrix}. \tag{46}$$

By quaternionic multiplication, we find:

$$\begin{aligned} B * Q &= N_1, \\ -N * Q &= N_2, \\ T * Q &= N_3 = P. \end{aligned} \tag{47}$$

This yields:

$$\begin{aligned} Q &= \langle P, T\rangle + P \times T, \\ N_1 &= \langle P, N\rangle + P \times N, \\ N_2 &= \langle P, B\rangle + P \times B, \\ N_3 &= P. \end{aligned} \tag{48}$$

Using this quaternionic frame and letting $K = \kappa$, $k = \tau$, and $r = \kappa$, we obtain the following frame evolution:

$$\begin{bmatrix} Q \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}, \tag{49}$$

where κ and τ are the curvature and torsion of the curve $\gamma(t)$, respectively.

Theorem 4. *Let $\zeta(s) = \int Q(s)\,ds$ be a quaternionic curve in $\mathbb{R}^4$. Then $\zeta(s)$ is a helix with axis*

$$\ell = \cos\theta\, Q + \sin\theta\, N_2 \tag{50}$$

if and only if $\frac{\tau}{\kappa}$ is constant.

Proof. If $\zeta(s)$ is a helix with axis ℓ, then

$$\langle Q, \ell\rangle = \cos\theta \quad \text{and} \quad \ell' = 0. \tag{51}$$

Differentiating ℓ using the frame equations gives:

$$\ell' = \cos\theta\kappa N_1 - \sin\theta\tau N_1 = (\cos\theta\kappa - \sin\theta\tau)N_1. \tag{52}$$

Thus, $\ell' = 0$ if and only if

$$\cos\theta\kappa - \sin\theta\tau = 0, \tag{53}$$

which implies

$$\frac{\tau}{\kappa} = \cot\theta = \text{constant}. \tag{54}$$

Conversely, if $\frac{\tau}{\kappa}$ is constant, then $\ell' = 0$, implying ℓ is a constant vector. Hence, $\zeta(s)$ is a helix with axis ℓ, completing the proof. □

Example 3. *Let us consider circular helix $\gamma(t) = (\cos t, \sin t, at)$ in $\mathbb{R}^3$ for a non-zero $a \in \mathbb{R}$. Its Frenet frame is well-defined with constant curvature and torsion given by*

$$\kappa = \frac{1}{1+a^2}, \quad \tau = \frac{a}{1+a^2}.$$

Therefore, the ratio $\tau/\kappa = a$ is constant, and the quaternionic curve $\zeta(s) = \int Q(s)\, ds$ generated by this curve satisfies the classical helix condition in Theorem 4.

Furthermore, since both κ and τ are constant, it follows that the generalized quaternionic helix condition

$$\left(\frac{K}{k}\right)^2 + \frac{1}{(r-K)^2}\left(\left(\frac{K}{k}\right)'\right)^2 = constant$$

is trivially satisfied. Hence, the quaternionic curve associated with $\gamma(t)$ is both a classical and a generalized quaternionic helix.

6 Conclusions

In this study, we introduced a quaternion operator-based framework for modeling rotation-minimizing motions (RMMs) along space curves using the Rotation Minimizing Frame (RMF) in Euclidean 3-space. By computing the angular velocity via the product of a quaternion's derivative and its conjugate, we achieved a compact and geometrically faithful representation that ensures smooth motion with minimal rotational effort.

Our approach, grounded in geometric algebra and compatible with the structure of Clifford algebras (specifically Cl(0, 2)), enables efficient modeling of RMMs associated with the spherical indicatricesnamely, the tangent, normal, and binormal imagesof a given spatial trajectory. Inspired by the operator-based methods of Aslan and Yayl, the proposed formulation not only provides analytical clarity but also enhances numerical stability and computational efficiency.

The presented numerical experiments validate the theoretical model and demonstrate its effectiveness in generating stable and smooth motion patterns, particularly in applications involving spatial motion planning and differential geometry. Furthermore, the characterization of quaternionic helical curves with respect to the RMF offers a new perspective on the geometry of constrained rotational motion.

Overall, this work contributes to the growing body of research on quaternion-based motion modeling and its interplay with Clifford geometric frameworks, providing a robust foundation for future developments in robotics, spatial kinematics, and geometric computing.

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