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Energetics in the Charney Problem with a Generic Stratosphere

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ABSTRACT

This paper reports a complete diagnosis of energetics of the three representative structurally distinct unstable modes in the Charney problem with a generic stratosphere presented in Mak et al (2021). The diagnosis of the vertically-local-energetics of these modes (Charney+ mode, Green mode and Tropopause mode) helps delineate the physical nature of the impacts of a tropopause upon their vertical structures. The total-energetics diagnosis reveals that the computational noise is negligible in the significantly unstable modes, but that in a weakly unstable mode is measurable. Treating the top boundary as a sufficiently high free-surface adequately permits the existence of vertically propagating modes with negligibly small energy passing through it. The analysis of this generalized Charney problem could be relevant to the study of troposphere-stratosphere interaction capable of inducing anomalous tropospheric weather regimes.

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1. INTRODUCTION

The study of extratropical cyclogenesis with the most-general-minimal model of baroclinic instability by Charney (1947) is referred to as the “Charney problem”. Mak et al (2021) sought to further elucidate the basic impacts of a stratosphere upon fundamental baroclinic instability. Although the zonal average zonal wind and associated temperature fields in the actual stratosphere have been known for a long time to have quite complex 2D structures maintained by various radiative and mechanical energy sources (Dickinson, 1975), Mak et al (2021, referred to below as * for convenience) adopts a generic stratosphere that contains the two well-established signature features of a typical tropopause. Across such a tropopause there is stepwise change in the basic vertical shear as well as stratification. That model, referred to as the Charney+ model, is arguably the simplest framework for assessing the impacts of an observationally compatible tropopause upon baroclinic instability. Among the key results in * are the existence of three types of structurally distinct modes among the unlimited branches of unstable modes. The objective of this follow-up analysis is to deduce better insight into the physical processes underlying the intriguing vertical structure of those unstable modes.

An extensive discussion of the background literature has been elaborated in * and is therefore not repeated here for brevity. It is however pertinent to note that in the arguably most comprehensive textbook of large scale geophysical dynamics, Vallis (2017, p.351) states: “... the Charney problem is in some respects more complete (for example in allowing a β -effect). The Charney problem in its entirety is also quite mathematically opaque, ...we look at some aspects of the Charney problem approximately.” Vallis’s discussion is based on his analyses of two approximate models. One model incorporates the β parameter alone in an Eady model. The other model incorporates an additional stratospheric basic shear and stratification, but still with $\beta = 0$, in a domain between two rigid surfaces. In contrast, the paper * is a self-contained analysis of a generalized Charney problem that brings to light the possible impacts of a stratosphere.

The nature of baroclinic instability in a model can be delineated in terms of the energetics of the unstable modes (Phillips, 1954). We can also gain some insight into the nature of the atmospheric general circulation as a whole from an energetics perspective (Lorenz, 1967). Diagnoses of energetics are usually performed by evaluating the volume integrals of various components of energetics for the entire system under consideration. In addition, we may diagnose the local-energetics to get some understanding about the local structure of a disturbance (e.g., Mak and Cai, 1989). In this study, we will first perform a diagnosis of the vertically-local energetics of the three interesting unstable modes: Charney+ mode, Green mode and Tropopause mode. It

will be supplemented by a diagnosis of the total-energetics of those modes partly for independently checking the trustworthiness of the energetics results and partly for ascertaining the role of the upper boundary condition used in the instability analysis.

2. VERTICALLY-LOCAL ENERGETICS OF NORMAL MODES

The basic state in the Charney+ model has a particularly simple zonal shear flow $U(z)$ and an observationally compatible stratification profile, $N(z)$. The basic density of the fluid decreases exponentially with increasing height. The fluid is on a beta-plane between two rigid lateral boundaries in a vertical domain between a rigid surface at the ground and a free-surface at the top, $0 \leq z \leq z_{top}$. The model has 10 external dimensional parameters: $-Y, D, H, z_{top}, \rho_o, \lambda, N_1, N_2, f_o, \beta$ (defined in *). Superimposed on such basic state is a quasi-geostrophic (QG) perturbation consisting of a geopotential ϕ and vertical velocity w . It is governed by the linearized QG vorticity equation together with the linearized QG thermodynamic equation. Written in terms of non-dimensional variables ($\tilde{x} = \frac{x f_o}{D N_1}, \tilde{y} = \frac{y f_o}{D N_1}, \tilde{z} = \frac{z}{D}, \tilde{t} = \frac{t f_o}{N_1}, \tilde{\phi} = \frac{\phi f_o}{\lambda N_1 D^2}, \tilde{w} = \frac{w N_1}{D f_o \lambda}, \tilde{U} = \frac{U}{\lambda D}, \tilde{N} = \frac{N}{N_1}, \tilde{\rho}_s = \frac{\rho_s}{\rho_o}$), those two equations have five non-dimensional external parameters: $-\tilde{Y}, \tilde{z}_{top}, A = \frac{N_1^2}{N_2^2}, B = \frac{D}{H}, C = \frac{\beta D N_1^2}{\lambda f_o^2}$. After dropping the tilde from the notations, those equations are

$$\nabla^2 \phi_t + U \nabla^2 \phi_x = \frac{1}{\rho_s} \frac{\partial(\rho_s w)}{\partial z} + C \phi_x \quad (1)$$

$$\phi_{zt} + U \phi_{zx} - \phi_x U_z + w N^2 = 0 \quad (2)$$

with $\rho_s = \exp(-Bz)$ and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

For $0 \leq z \leq 1, U_z = 1, N^2 = 1$

For $1 \leq z \leq z_{top}, U_z = 0, N^2 = A^{-1}$

Domain: $0 \leq x \leq \infty, 0 \leq y \leq Y, 0 \leq z \leq z_{top}$

Boundary conditions: $\phi = 0$ @ $y = 0, Y; w = 0$ @ $z = 0;$

$\phi_z = 0, w_z = 0$ @ $z = z_{top}$

Equivalent to (1) and (2) are a prognostic QG potential vorticity equation governing a single unknown, ϕ , and a diagnostic QG omega-equation relating w to ϕ . A modal instability analysis is performed in * with the former and determined the corresponding w with the latter. However, it is most convenient to directly work with (1) and (2) in a formulation of energetics analysis. The normal mode solution of (1) and (2) has the form

$$\phi = \chi(z) \exp(i(kx - \sigma t)) \sin(l y) \quad (3)$$

$$w = \xi(z) \exp(i(kx - \sigma t)) \sin(l y) \quad (4)$$

with $\sigma = \sigma_r + i\sigma_i; \chi = \chi_r + i\chi_i; \xi = \xi_r + i\xi_i; i = \sqrt{-1}; k$ and l the zonal and meridional wavenumbers; σ eigenvalue; χ eigenfunction.

A sufficiently high z_{top} is set at 100 km. We depict the domain by 800 uniform levels in the numerical computation of eigenmodes. Our focus is first on getting more insight of the vertical structure of the major unstable modes in the context of their energetics. Let us consider the horizontally integrated energy of a normal mode within one-unit vertical distance as a function of height. We denote such integration over one wavelength in x-direction of a normal mode, Θ , by an angular bracket

$$\langle \Theta \rangle = \int_0^{2\pi} \int_0^\pi \Theta d(kx) d(ly) \quad (5)$$

where Θ stands for K , P or E respectively for the kinetic energy, available potential energy and total energy per unit volume of air. They are defined by $K = \frac{1}{2} \rho_s ((\phi_x)^2 + (\phi_y)^2)$, $P = \frac{\rho_s}{2N^2} (\phi_z)^2$ and $E = K + P$. The vertically-local energy tendency equations can be readily obtained from (1) and (2) as

$$\langle K \rangle_t = - \left\langle \frac{\partial(\rho_s w \phi)}{\partial z} \right\rangle + \rho_s \langle w \phi_z \rangle \quad (6)$$

$$\langle P \rangle_t = \frac{\rho_s U_z}{N^2} \langle v \phi_z \rangle - \rho_s \langle w \phi_z \rangle \quad (7)$$

$$\langle E \rangle_t = - \left\langle \frac{\partial(\rho_s w \phi)}{\partial z} \right\rangle + \frac{\rho_s U_z}{N^2} \langle v \phi_z \rangle \quad (8)$$

Eq. (6) says that the rate of change of the vertically-local K is the sum of local conversion from P to K , $\rho_s \langle w \phi_z \rangle$ and convergence of vertical wave flux, $-\langle \frac{\partial(\rho_s w \phi)}{\partial z} \rangle$. The latter is a measure of vertical redistribution of kinetic energy by virtue of the wave motion in each unstable mode. Eq. (7) says that the rate of change of the vertically-local P is equal to the sum of a generation of P from the basic state, $\frac{\rho_s U_z}{N^2} \langle v \phi_z \rangle$, and a conversion from K to P , $(-\rho_s \langle w \phi_z \rangle)$. The explicit expressions to be used for computation are

$$\langle K \rangle = \left[\frac{\pi^2}{4} (k^2 + l^2) \rho_s (\chi_r^2 + \chi_i^2) \right] \exp(2\sigma_i t) \equiv K \exp(2\sigma_i t) \quad (9a)$$

$$\langle P \rangle = \frac{\pi^2 \rho_s}{4 N^2} \left[\left(\frac{d\chi_r}{dz} \right)^2 + \left(\frac{d\chi_i}{dz} \right)^2 \right] \exp(2\sigma_i t) \equiv P \exp(2\sigma_i t) \quad (9b)$$

$$\rho_s \langle w \phi \rangle = \frac{\pi^2 \rho_s}{2} (\xi_r \chi_r + \xi_i \chi_i) \exp(2\sigma_i t) \equiv F \exp(2\sigma_i t) \quad (9c)$$

$$\left\langle - \frac{\partial(\rho_s w \phi)}{\partial z} \right\rangle = - \frac{\pi^2}{2} \frac{d}{dz} (\rho_s (\xi_r \chi_r + \xi_i \chi_i)) \exp(2\sigma_i t) \equiv R \exp(2\sigma_i t) \quad (9d)$$

$$\rho_s \langle w \frac{\partial \phi}{\partial z} \rangle = \frac{\pi^2 \rho_s}{2} \left(\xi_r \frac{d\chi_r}{dz} + \xi_i \frac{d\chi_i}{dz} \right) \exp(2\sigma_i t) \equiv S \exp(2\sigma_i t) \quad (9e)$$

$$\left\langle \frac{\rho_s U_z}{N^2} v \phi_z \right\rangle = \frac{\pi^2 k \rho_s U_z}{2} \left(\chi_r \frac{d\chi_i}{dz} - \chi_i \frac{d\chi_r}{dz} \right) \exp(2\sigma_i t) \equiv G \exp(2\sigma_i t) \quad (9f)$$

We refer to (6) and (7) in symbolic form as

$$\langle K \rangle_t = (R + S) \exp(2\sigma_i t) \quad (10)$$

$$\langle P \rangle_t = (G - S) \exp(2\sigma_i t) \quad (11)$$

The physical nature of the vertical structure of a normal mode may be interpreted on the basis of the vertical distributions in R , S and G , from the perspective of vertically-local energetic processes.

3. RESULTS OF THE VERTICALLY-LOCAL ENERGETICS

We now examine the vertically-local energetics of three specific unstable modes: Charney+ mode, Green mode and Tropopause mode under the following parametric conditions:

Charney+ mode: $A = 0.25, B = 1, 1/C = 0.6$ for $l = 1, k = 2$

Green mode: $A = 0.25, B = 1, 1/C = 0.4$ for $l = 1, k = 1$

Tropopause mode: $A = 0.25, B = 1, 1/C = 0.08$ for $l = 1, k = 4$

We begin by taking a look at the signature characteristics in the vertical profile of the eigenfunctions of these modes, $|\chi|$ (**Figure 1**). The Charney+ mode and the Tropopause mode are essentially trapped in the troposphere. It suffices to show their vertical structure and related properties of energetics from $z = 0$ to $z = 2.5$. Since the Green mode spans the entire vertical domain, we need to present its structure from $z = 0$ to $z = 10$. The values of $|\chi|$ of the three modes are all of order unity intrinsically set by MATLAB. We see that the maximum magnitude of Charney+ mode (top) is located at $z = 0$, that of Green mode (mid) at z_{top} and that of the Tropopause mode (bottom) near $z = 1$. The increase of $|\chi|$ of the Green mode with height stems from the density decreasing exponentially with increasing height. For the purpose of making a comparison, we could renormalize the three eigenfunctions such that the three modes would have the same total energy. However, the shape of the vertical profile of any energetics property is independent of renormalization. We will henceforth present the energetics results of the three modes without applying renormalization.

Figure 2 shows a comparison of the distributions of the vertically-local P , and K . The maximum value of P , of each mode is found in a thin layer next to the ground surface indicating that the surface baroclinicity is indeed the source of the instability in each case (left panels). The Green mode (mid-left) has appreciable values of P in the lower half of the stratospheric layer diminishing towards the top boundary even though $|\chi|$ itself is larger at greater heights. On the other hand, these three modes have quite different distributions of K , in z . The Charney+

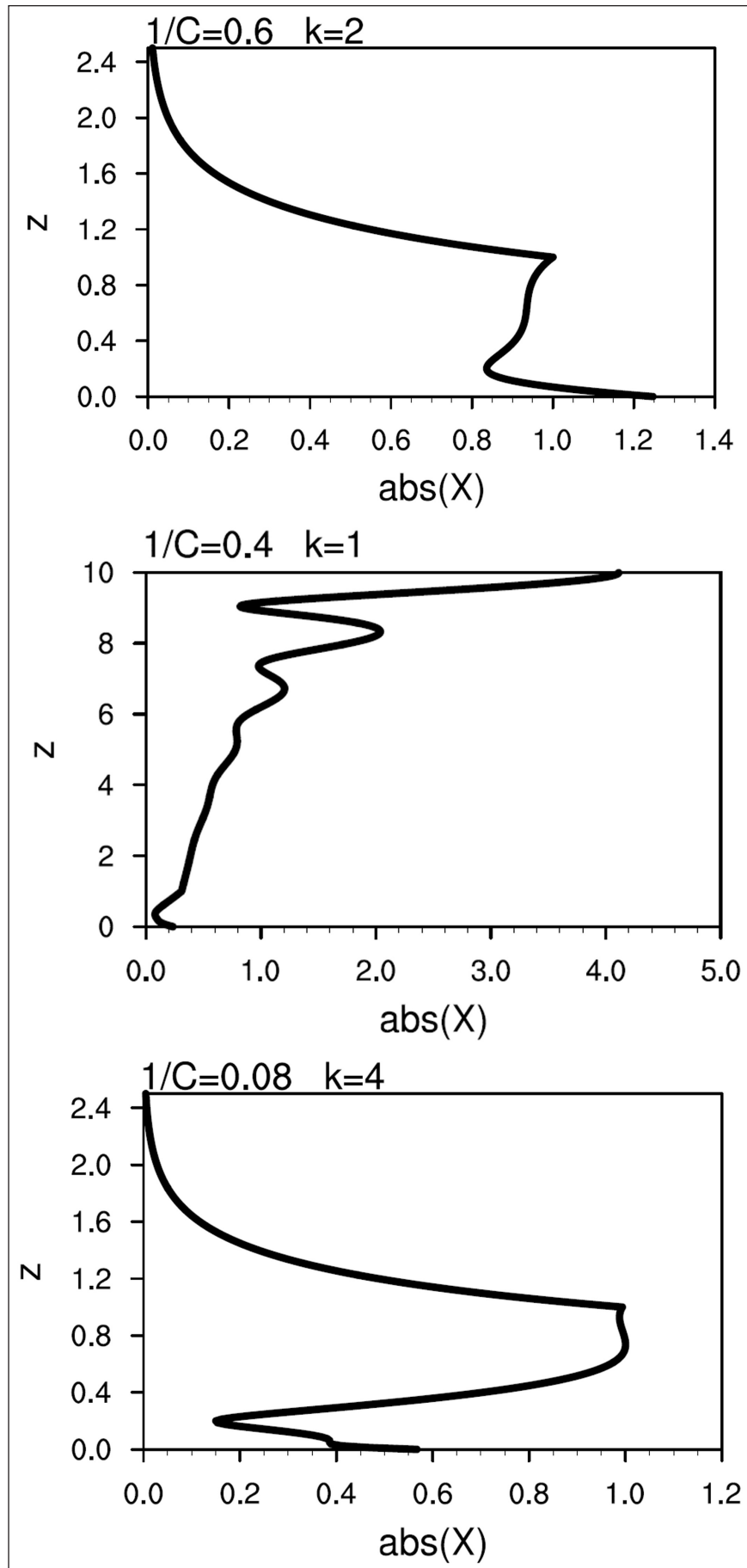


Figure 1 Vertical variation of the magnitude of the eigenfunction χ of (top) Charney+ mode, (middle) Green mode and (bottom) Tropopause mode. Charney+ and Tropopause modes are shown from $z = 0$ to 2.5; their values are virtually zero higher up. Green mode is shown from $z = 0$ to 10.

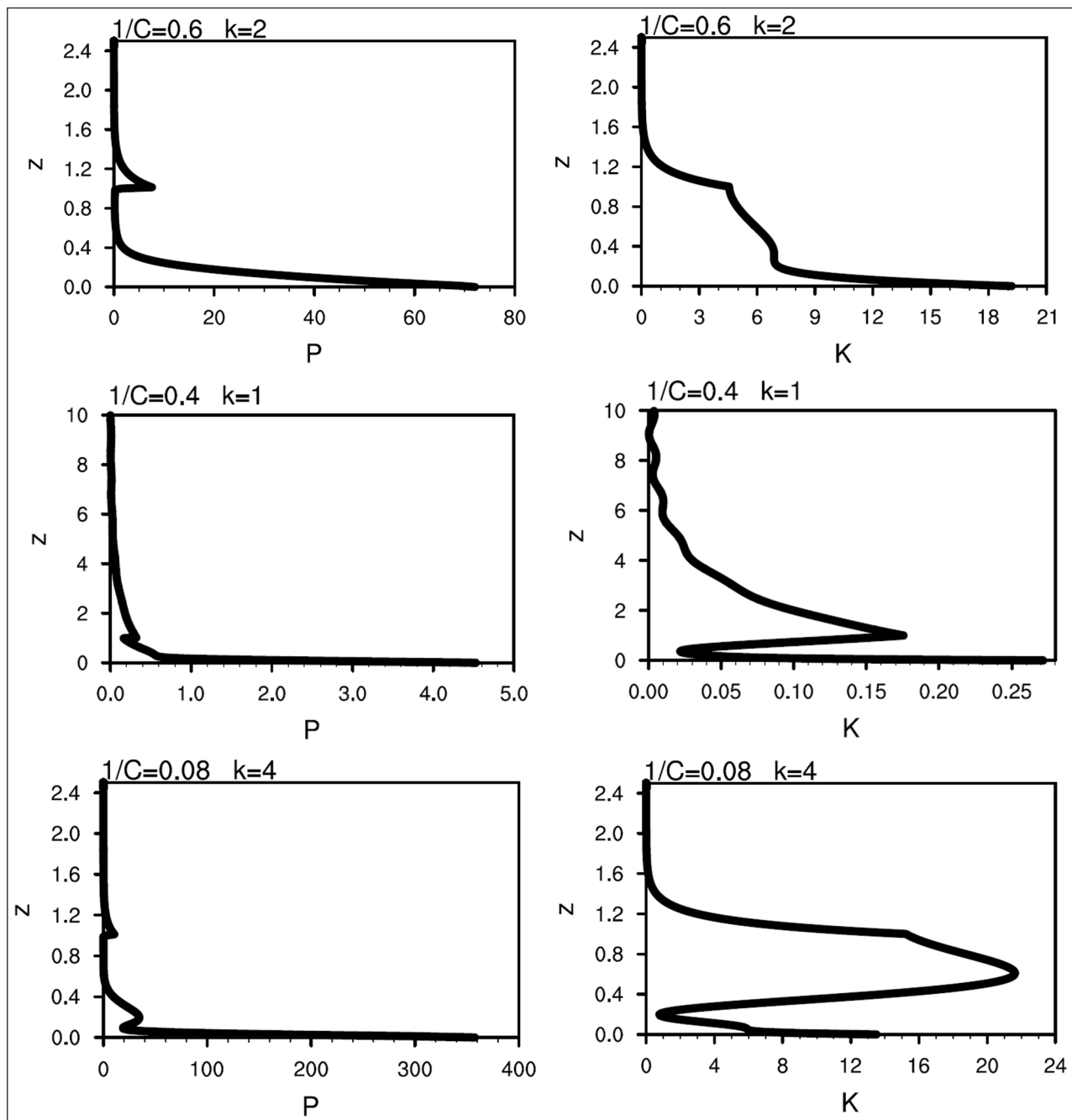


Figure 2 Vertical distribution of the non-dimensional horizontal integrated eddy available potential energy EAPE, P , for (top-left) Charney+ mode, (mid-left) Green mode and (bot-left) Tropopause mode ; That of the eddy kinetic energy EKE, K , for (top-right) Charney+ mode, (mid-right) Green mode and (bot-right) Tropopause mode. The non-dimensional parameters, $1/C$ and k , associated with these modes are indicated at the top of each panel with the remaining parameters having common values: $l = 1$, $A = 0.25$, $B = 1$. Charney+ and Tropopause modes are shown from $z = 0$ to 2.5 ; their values are virtually zero higher up. Green mode is shown from $z = 0$ to 10 .

mode and Tropopause mode are essentially bounded in the troposphere. The former (top-right) has greatest amount of K , at the surface and moderate values in the rest of the troposphere. The Tropopause mode (bottom-right) has distinctly large values of K , in an upper tropospheric layer and considerably smaller values in a very thin surface layer. In contrast, the Green mode (mid-right) has substantial values of K , in the lower half of the stratosphere.

We next show in [Figure 3](#) the vertically-local conversion rate from the basic P to the wave P , $\frac{\rho_s U_z}{N^2} \langle v \phi_z \rangle$, labeled

as ZAPE to EAPE. Such conversion rate in the Charney+ mode and the Tropopause mode is large only in a shallow layer next to the surface. The Green mode also has modest values in the rest of the troposphere. These results confirm that most of the potential energy of the normal modes is released from the basic state near the surface. This conversion rate is zero for all modes above the tropopause level because of $U_z = 0$ in our generic stratosphere.

[Figure 4](#) shows the vertically-local conversion rate from P to K , $\rho_s \langle w \phi_z \rangle$, labeled as EAPE to EKE. [Figure 5](#)

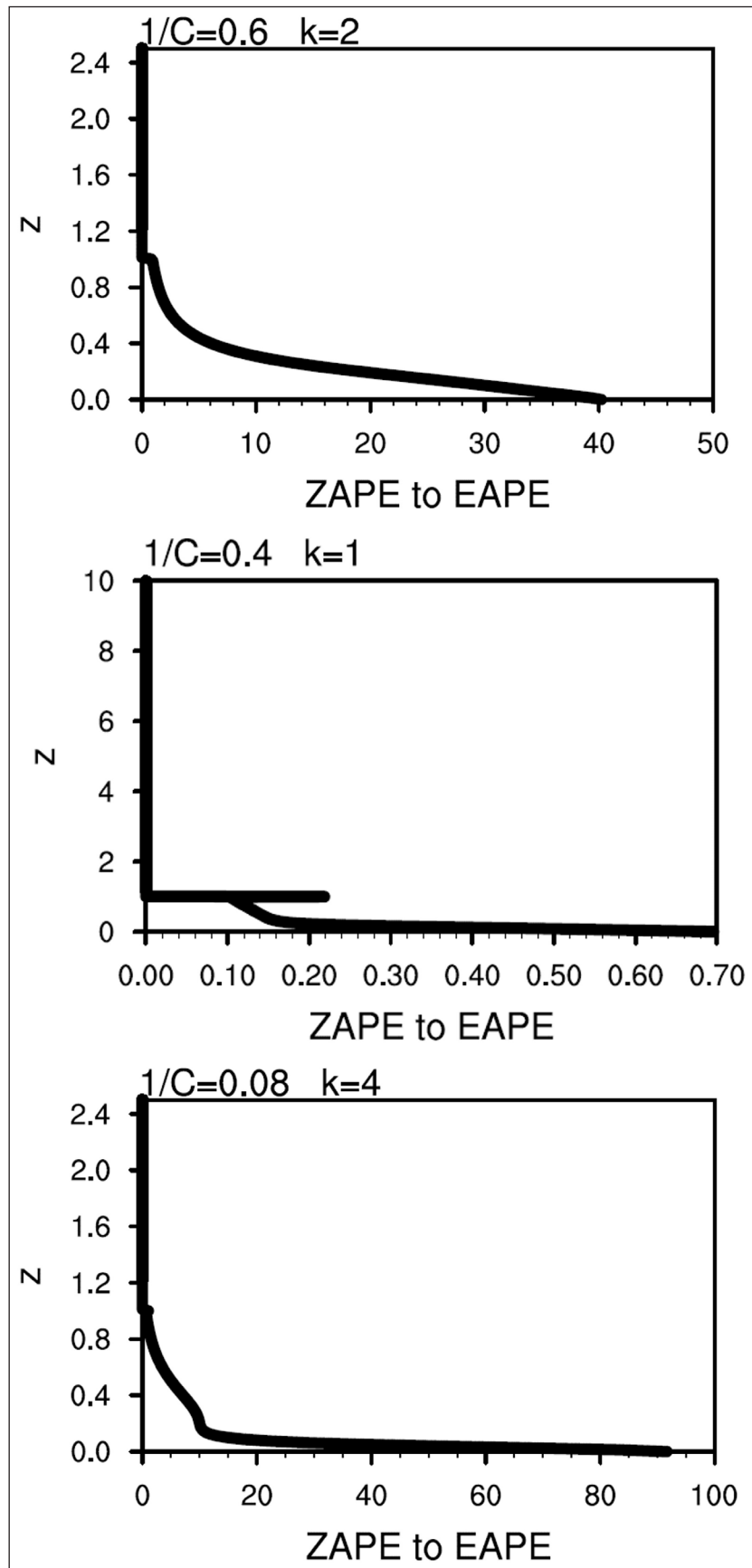


Figure 3 Vertical distribution of the non-dimensional horizontal integrated conversion rate from the basic available potential energy ZAPE to eddy available potential energy EAPE, $\frac{\rho_s U_z}{N^2} \langle v \phi_z \rangle$ for Charney+ mode (top), Green mode (middle) and Tropopause mode (bottom). The non-dimensional parameters, $1/C$ and k , associated with these modes are indicated at the top of each panel with $l = 1$, $A = 0.25$, $B = 1$, as the common values of the remaining parameters. Charney+ and Tropopause modes are shown from $z = 0$ to 2.5 ; their values are virtually zero higher up. Green mode is shown from $z = 0$ to 10 .

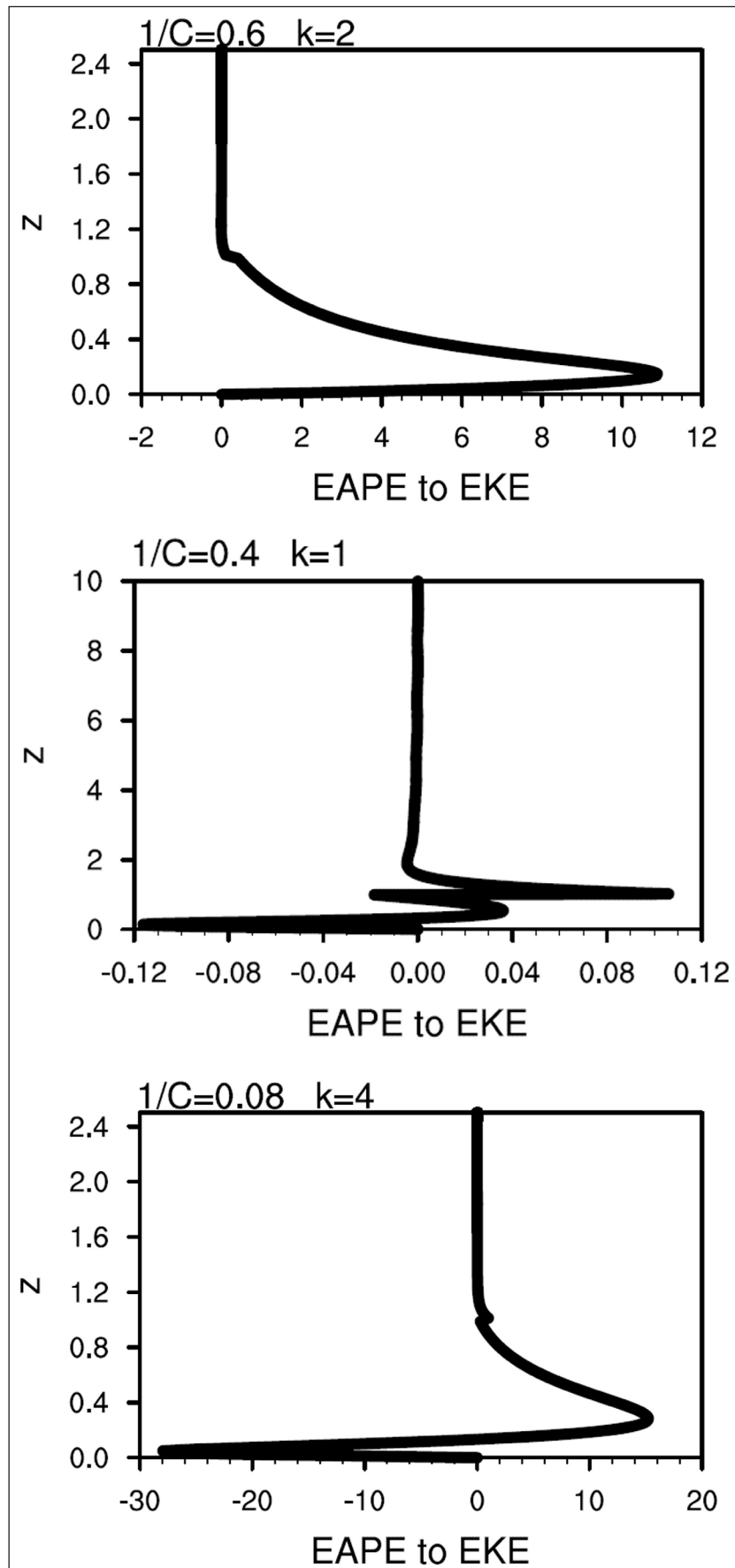


Figure 4 Vertical distribution of the non-dimensional horizontal integrated conversion rate from the eddy available potential energy to eddy kinetic energy, P to K , for Charney+ mode (top), Green mode (middle) and Tropopause mode (bottom). The non-dimensional parameters, $1/C$ and k , associated with these modes are indicated at the top of each panel with $l = 1$, $A = 0.25$, $B = 1$, as the common values of the remaining parameters. Charney+ and Tropopause modes are shown from $z = 0$ to 2.5 ; their values are virtually zero higher up. Green mode is shown from $z = 0$ to 10 .

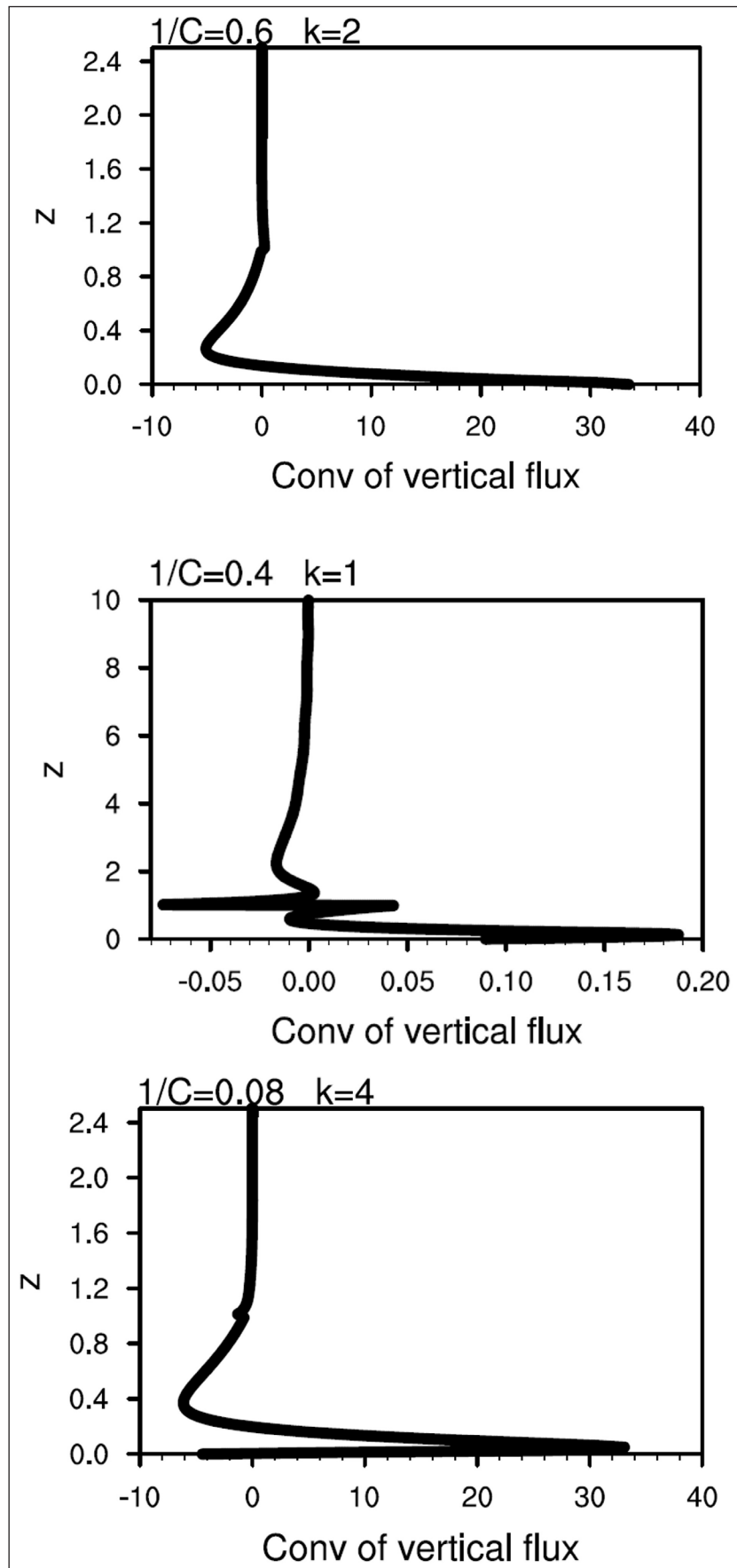


Figure 5 Vertical distribution of the convergence of vertical wave flux, $-\langle \frac{\partial(\rho_s w \phi)}{\partial z} \rangle$, for the Charney+ mode (top), Green mode (middle) and Tropopause mode (bottom). The non-dimensional parameters, $1/C$ and k , associated with these modes are indicated at the top of each panel with $l = 1$, $A = 0.25$, $B = 1$, as the common values of the remaining parameters. Charney+ and Tropopause modes are shown from $z = 0$ to 2.5; their values are virtually zero higher up. Green mode is shown from $z = 0$ to 10.

shows the distribution of convergence of vertical wave flux, $-\langle \frac{\partial(\rho_s w \phi)}{\partial z} \rangle$. In the case of the Charney+ mode, the maximum conversion from P to K takes place at a rather low steering level, $z \approx 0.2$. This conversion necessarily diminishes towards the surface because $w = 0$ is imposed at $z = 0$ as a boundary condition. Nevertheless, the amplitude of these modes is found locally maximum at $z = 0$. It is so only possible because there is a compensating vertical energy flux convergence in that layer (**Figure 5** top panel). The divergence of wave energy flux at higher levels serves to transfer some kinetic energy to the upper troposphere. The downward energy flux towards the surface is even more essential in a Tropopause mode because the conversion process $\rho_s \langle w \phi_z \rangle$, is found actually negative there (**Figure 4**, bottom panel). But the surface maximum of the Tropopause mode is not as strong as the Charney+ mode because the energy flux in the former decreases towards the surface (**Figure 5**, bottom panel). Hence the surface maximum amplitude of the Tropopause mode is not so strong. The positive values of $\rho_s \langle w \phi_z \rangle$ in the upper part of the troposphere in both the Charney+ and Tropopause modes serve to support the large amplitude there.

In the case of the Green mode, the conversion from P to K and the convergence of vertical energy flux are out of phase in the troposphere (middle panel of **Figures 4 & 5**). That results in having small magnitude in the troposphere. The divergence of vertical energy flux in the lower stratosphere supports the upward propagation of the wave in the stratosphere. The amplitude is larger at greater height due to the exponential decrease of density with increasing height.

Finally, we present in **Figure 6** the vertical distribution of the vertical wave flux itself, $\rho_s \langle w \phi \rangle$, further documenting the subtle role of this process in all three modes in the troposphere noted earlier.

4. TOTAL-ENERGETICS OF NORMAL MODES

Let us denote the total volume integral of a type of energy, Θ , of a normal mode by a curly bracket as

$$\{\Theta\} = \int_0^{2\pi} \int_0^\pi \int_0^{z_{top}} \Theta d(kx) d(ly) dz \quad (12)$$

where Θ could stand for K, P or E. We perform a budget diagnosis separately for $\{K\}$, $\{P\}$, and $\{E\}$ which enables us to make three separate estimates of the growth rate of a normal mode on the basis of such diagnoses. The results are denoted by γ_1 , γ_2 , and γ_3 respectively. It is pertinent to note that the computation of eigenvalue-eigenfunction with MATLAB involving matrices as large as 800×800 in Mak et al (2021) involves taking many arithmetical operations. There is inevitably small random computational noise in the eigenfunction. It

follows that the three energetically based separate estimates of the growth rate of a normal mode would not be identical to the imaginary part of its eigenvalue itself. But if the energetics results presented in the last section are self-consistent, the four measures of the growth rate should be fairly similar. The actual degree of discrepancy would shed light to the nature of differences among the unstable modes under consideration.

The tendency equations for the three forms of total energy can be written as

$$\{K\}_t = (F_o + S_o) \exp(2\gamma_1 t) \quad (13a)$$

$$\{P\}_t = (G_o - S_o) \exp(2\gamma_2 t) \quad (13b)$$

$$\{E\}_t = (G_o + F_o) \exp(2\gamma_3 t) \quad (13c)$$

where

$$F_o = -\frac{\pi^2}{2} (\rho_s (\xi_r \chi_r + \xi_i \chi_i))_{z_{top}} \quad (14a)$$

$$G_o = \left(\frac{\pi^2 k}{2} \int_0^{z_{top}} \frac{\rho_s U_z}{N^2} \left(\chi_r \frac{d\chi_i}{dz} - \chi_i \frac{d\chi_r}{dz} \right) dz \right) \quad (14b)$$

$$S_o = \frac{\pi^2}{2} \int_0^{z_{top}} \rho_s \left(\xi_r \frac{d\chi_r}{dz} + \xi_i \frac{d\chi_i}{dz} \right) dz \quad (14c)$$

Eq. (13a) says that the total kinetic energy of a normal mode would change in time due to two processes: a vertical wave flux through the top boundary, F_o , and generation of K from P, S_o . Eq. (13b) says that the total available potential energy would change in time resulting from two processes: generation of P from the basic state G_o and a conversion from P to K. Eq. (13c) says that the total energy would change in time due to two processes: generation of P from the basic state and a vertical wave flux through the top boundary. The free-surface boundary condition $\phi_z = 0$ at z_{top} in our instability analysis permits a disturbance to propagate through the top boundary although the related energy is expected very small since $\lim_{z \rightarrow \infty} \rho_s = 0$.

By (13a,b,c), the three energetic-based estimates of the growth rate of a normal mode can be determined as

$$\gamma_1 = \frac{F_o + S_o}{2K_o}; \quad \gamma_2 = \frac{G_o - S_o}{2P_o}; \quad \gamma_3 = \frac{G_o + F_o}{2E_o} \quad (15)$$

where

$$K_o = \left(\frac{\pi^2}{4} (k^2 + l^2) \int_0^{z_{top}} \rho_s (\chi_r^2 + \chi_i^2) dz \right)$$

$$P_o = \left(\frac{\pi^2}{4} \int_0^{z_{top}} \frac{\rho_s}{N^2} \left(\left(\frac{d\chi_r}{dz} \right)^2 + \left(\frac{d\chi_i}{dz} \right)^2 \right) dz \right)$$

$$E_o = K_o + P_o$$

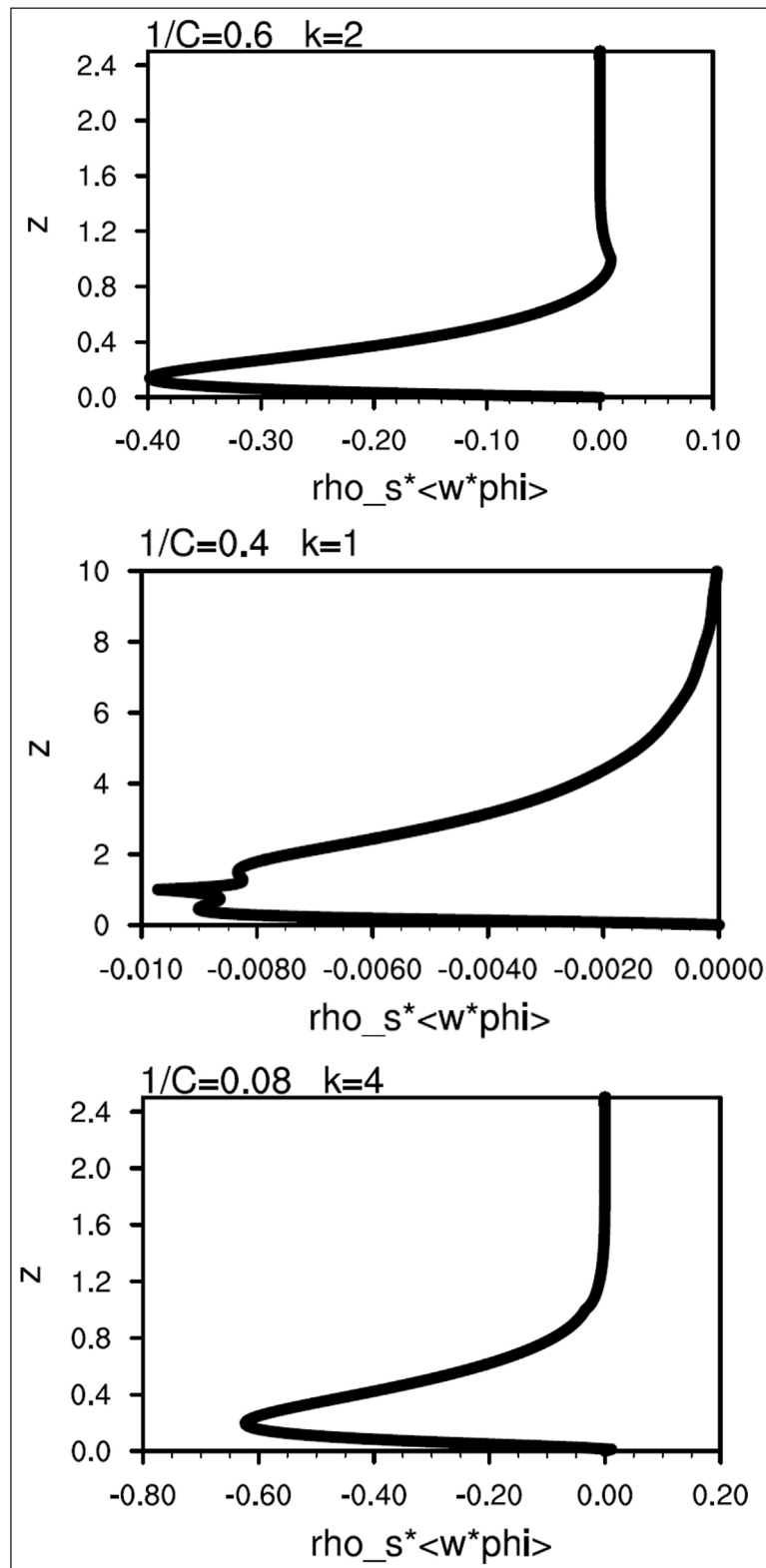


Figure 6 Vertical distribution of vertical wave flux, $\rho_s \langle w \phi \rangle$, for the (top) Charney+ mode, (middle) Green mode and (bot) Tropopause mode. Charney+ and Tropopause modes are shown from $z = 0$ to 2.5; their values are virtually zero higher up, Green mode is shown from $z = 0$ to 10.

The results for the three normal modes under consideration are presented in [Table 1](#) which suggests the following deductions:

1. The Charney+ mode is most unstable of the three modes; the three separately deduced growth rates do agree well with the eigenvalue growth rate.
2. The Green mode is weakly unstable. Only the deduced growth rate on the basis of the APE energetics formulation agrees well with the eigenvalue growth rate. The deduced growth rate is found particularly sensitive to the computational noise in the eigenfunction when the kinetic energy tendency equation is applied. There is greater

GROWTH RATE	$k = 2, 1/C = 0.6$ (CHARNEY+)	$k = 1, 1/C = 0.4$ (GREEN)	$k = 4, 1/C = 0.08$ (TROPopause)
(KE) γ_1	0.275	-0.009	0.095
(APE) γ_2	0.236	0.078	0.150
(E) γ_3	0.265	0.058	0.124
(Eigenvalue) σ_i	0.279	0.076	0.128

Table 1 A comparison of the growth rates for each of the three normal modes as measured by three separate corresponding energetic-based estimates γ_1 , γ_2 , and γ_3 as well as the imaginary part of the eigenvalue, σ_i .

uncertainty in this aspect of the energetics of this mode.

3. The Tropopause mode is intermediately unstable among the three modes. The energetically deduced growth rates using the tendency equations for APE and total energy agree acceptably well with the eigenvalue growth rate.

We are led to the conclusion that the more unstable a normal mode is the more negligible the impact of computational noise would be. The finding for the Green mode is therefore not problematic in principle. In light of the findings, we may confidently state that the detailed energetics results presented in Section 3 are self-consistent and trustworthy.

5. CONCLUDING REMARKS

In this study we further examine the three structurally distinct types of unstable normal modes reported in Mak et al (2021) from the perspective of energetics. The findings of the various results are not recapped here. However, it warrants to reiterate one point. While the Tropopause mode has maximum K at the tropopause due to local conversion from wave P to wave K , the secondary maximum K at the surface actually stems from a significant transfer of energy downward via the process of vertical wave flux. It is sufficiently large to overcompensate a local conversion from wave K to wave P (Figure 4 bottom panel). The process of convergence of wave flux is also responsible for the significant kinetic energy of the Green mode in the stratosphere. The significance of wave flux diminishes towards the top boundary where ρ_s is increasingly small. Treating the upper boundary as a free-surface is necessary and sufficient for the existence of propagating disturbances. The findings of this follow-up analysis help appreciate better why they have such intriguing vertical structures.

The relevance of the unstable modes in the Charney problem with a generic stratosphere to the atmosphere and dynamic meteorology has been discussed in *. The relevance of the generalized Charney problem can be further appreciated in light of the more recent work of Baldwin and Dunkerton (2001). They report that large

variations in the strength of the stratospheric circulation descend to the lowermost stratosphere and are followed by anomalous tropospheric weather regimes. These stratospheric events also precede shifts in a number of tropospheric phenomena including the location of storm tracks and the location of likelihood of mid-latitude storms. As far as the dynamical nature of such events and relationship is concerned, those authors suggest that forced planetary waves emanating from the troposphere could carry zonal momentum upward and thereby could alter the stratospheric circulation when they deposit zonal momentum to the background flow at a critical level at 50 km or higher. The Green mode generated in the troposphere may play a similar role as well, albeit necessarily in the context of quasi-linear or nonlinear dynamics. That is of course beyond the scope of a linear instability analysis under consideration. Therefore, a self-contained instability analysis of the Charney problem under the influence of an observationally compatible stratosphere is a meaningful contribution to dynamic meteorology. It helps lay the groundwork for in-depth nonlinear analysis in future.

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COMPETING INTERESTS

The authors have no competing interests to declare.

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