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Generalized Jacobsthal numbers and restricted k-ary words

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Abstract. We consider a generalization of the problem of counting ternary words of a given length which was recently investigated by Koshy and Grimaldi [10]. In particular, we use finite automata and ordinary generating functions in deriving a k-ary generalization. This approach allows us to obtain a general setting in which to study this problem over a k-ary language. The corresponding class of n-letter k-ary words is seen to be equinumerous with the closed walks of length $n-1$ on the complete graph for k vertices as well as a restricted subset of colored square-and-domino tilings of the same length. A further polynomial extension of the k-ary case is introduced and its basic properties deduced. As a consequence, one obtains some apparently new binomial-type identities via a combinatorial argument.

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1 Introduction

For a positive integer n, let b_n be the number of words $w = w_1w_2\cdots w_n$ over the alphabet $\Sigma_3 = \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$ such that $w_1 = \mathtt{0} = w_n$ and $w_i \neq w_{i+1}$ for $i = 1, 2, \ldots, n-1$. The digits 0, 1 and 2 are ternary digits and the words enumerated by b_n are called *ternary words*. A variant of this interesting counting problem was proposed in the final round of the 1987 Austrian Olympiad [2, 9]. More recently, Koshy and Grimaldi [10] have found a close connection between b_n and the Jacobsthal sequence J_n. In particular, they prove that $b_n = 2J_{n-2}$ for $n \geq 1$. Recall that the Jacobsthal numbers are defined recursively by

$$J_n = J_{n-1} + 2J_{n-2}, \qquad n \geq 1,$$

where $J_{-1} = \frac{1}{2}$ and $J_0 = 0$. The first few terms are given by

$$\{J_n\}_{n\geq 0} = \{0,\ 1,\ 1,\ 3,\ 5,\ 11,\ 21,\ 43,\ 85,\ 171,\ 341,\ 683,\ 1365,\ \ldots\}.$$

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The ordinary generating function for the Jacobsthal numbers is

$$J(z) := \sum_{n=0}^{\infty} J_n z^n = \frac{z}{(1+z)(1-2z)}. \tag{1}$$

Therefore, the corresponding *Binet* formula is given by $J_n = \frac{2^n-(-1)^n}{3}$ for $n \geq 0$. From (1) and the equality $b_n = 2J_{n-2}$, we have

$$\sum_{n=1}^{\infty} b_n z^n = \frac{(1-z)z}{(1+z)(1-2z)}. \tag{2}$$

In [10], some relationships were found between ternary words and the Jacobsthal numbers using a bijective and recurrence-based approach and reproven algebraically in [11]. In this paper, we consider an extension of the problem from ternary to k-ary words over the alphabet $\Sigma_k = \{\mathtt{0}, \mathtt{1}, \dots, \mathtt{k-1}\}$ and let $b_n^{(k)}$ denote the sequence enumerating these k-ary words. We find relationships between such k-ary words and a generalization of the Jacobsthal numbers by making use of finite automata and generating functions, employing the former to find a system of equations satisfied by the latter which we can subsequently solve explicitly. As is shown, the number of closed walks of length n on the complete graph K_m is then given by $b_{n+1}^{(m)}$ for $n \geq 0$, as is the cardinality of the class of colored tilings of the same length starting with a domino in which squares come in $m-2$ colors and dominos in $m-1$ colors. Since when $k=3$ we have $\frac{b_{n+2}^{(3)}}{2} = J_n$, the numbers $\frac{b_{n+2}^{(k)}}{k-1}$ may be regarded as a generalized Jacobsthal sequence. Finally, we consider a polynomial extension of $b_n^{(k)}$ in terms of statistics defined on the class of k-ary words enumerated by $b_n^{(k)}$. Explicit formulas are found for the total value and the sign balance of these statistics from which some binomial identities may be deduced.

2 Automata theory and generating functions

We first recall some terminology and notation, mostly from Sakarovitch [12]. Let Σ be an alphabet whose elements are called *symbols*. A *word* over Σ is a finite sequence of symbols from Σ. The set of all words over Σ, i.e., the free monoid generated by Σ, is denoted by Σ^*. Each subset of Σ^* is called a *formal language* over Σ. The identity element ϵ of Σ^* is called the *empty word*. For any word $w \in \Sigma^*$, $|w|$ denotes its *length*, i.e., the number of symbols occurring in w. The length of ϵ is taken to be equal to 0.

An *automaton* $\mathcal{M}$ is a 5-tuple $\mathcal{M} = (\Sigma, Q, q_0, F, E)$, where Σ is a nonempty input alphabet, Q is a nonempty set of so-called *states* that are disjoint from Σ, $q_0 \in Q$ is a distinguished member of Q known as the *initial* state, $\emptyset \neq F \subseteq Q$ is the set of final states of $\mathcal{M}$ and $E \subseteq Q \times \Sigma \times Q$ is the set of transitions of $\mathcal{M}$. If Q, Σ and E are finite sets, then we say that $\mathcal{M}$ is a *finite automaton*.

An automaton has a natural graphical representation as a directed graph, called a *transition diagram*. In this graph, the states are represented by circles and final states by double circles, where the initial state is labelled by an arrow entering that state and each transition is represented by an arrow between its states, labelled with a symbol.

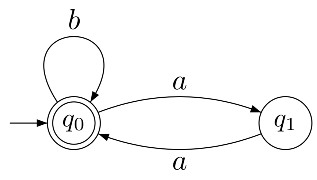


Figure 1: Transition diagram of $\mathcal{M}$, Example 2.1.

Example 2.1 Consider the finite automaton $\mathcal{M} = (\Sigma, Q, q_0, F, E)$, where $\Sigma = \{a, b\}$, $Q = \{q_0, q_1\}$, $F = \{q_0\}$ and $E = \{(q_0, a, q_1), (q_0, b, q_0), (q_1, a, q_0)\}$. The transition diagram of $\mathcal{M}$ is as shown in Figure 1.

Let $e = (p, a, q)$ denote a transition of an automaton $\mathcal{M}$. We say that a is the *label* of e and will write $p \xrightarrow{a} q$. The state p is called the *source* and q the *destination* of the transition e. A *computation* c in an automaton $\mathcal{M}$ is a sequence of transitions where the source of each transition is the destination of the previous. A computation may be represented by

$$c := p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \cdots \xrightarrow{a_n} p_n.$$

The state p_0 is the source and p_n the destination of the computation c. A computation in $\mathcal{M}$ is said to be *successful* if its source is an initial and its destination a final state of $\mathcal{M}$. The *label* of c is the concatenation of all the labels of the transitions of c. A word in Σ^* is said to be *accepted*, or *recognized*, by $\mathcal{M}$ if it corresponds to the label of a successful computation in $\mathcal{M}$. The language recognized by an automaton $\mathcal{M}$ is denoted by $L(\mathcal{M})$.

For example, the word $w = bbaab$ is accepted by the automaton in Figure 1. Indeed, this word is the label of the computation:

$$q_0 \xrightarrow{b} q_0 \xrightarrow{b} q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_0 \xrightarrow{b} q_0.$$

It is easy to verify that the language recognized by the automaton in Figure 1 is

$$L(\mathcal{M}) = \{aa, b\}^* = \{\epsilon, b, aa, bb, bbb, baa, aab, bbbb, aaaa, bbaa, baab, aabb, \dots\}.$$

An automaton $\mathcal{M} = (\Sigma, Q, q_0, F, E)$ is *deterministic* if for all states p in Q and all symbols a in Σ, there exists at most one transition in E with source p and label a. It is clear that the automaton of Example 2.1 is deterministic. The language recognized by a deterministic finite automaton is referred to as *regular*.

An ordinary generating function $H = \sum_{n=0}^{\infty} h_n z^n$ corresponds to a formal language L if $h_n = |\{w \in L : |w| = n\}|$, i.e., if the n-th coefficient h_n gives the number of words in L with length n. How to find the ordinary generating function (GF) corresponding to a regular language (and, more generally, to a context-free language) is known as the *Chomsky-Schützenberger methodology*. See, for example, [7, 8] for further details. The idea behind this procedure is to obtain a system of equations using a deterministic finite automaton, where the unknowns are the ordinary generating functions associated with each state of the automaton.

Let $\mathcal{M}$ be a deterministic finite automaton with set of states $Q = \{q_0, q_1, \dots, q_m\}$, initial state q_0 and final states $F = \{q_{i_1}, q_{i_2}, \dots, q_{i_f}\}$. The generating function of the language $L(\mathcal{M})$ is a rational function that is determined by the matrix form

$$L(z) = \mathbf{u}(I - zT)^{-1}\mathbf{v},$$

where T is the transition matrix of the diagram of $\mathcal{M}$, the row vector $\mathbf{u}$ is the vector $(1, 0, \dots, 0)$ and the column vector $\mathbf{v} = (v_0, \dots, v_m)$ is such that $v_j = [q_j \in F]$ (using Iverson's notation where $[P] = 1$ if the statement P is true and $[P] = 0$ if P is false). Note that if L is a regular language, then the generating function corresponding to L is rational (see, e.g., [5,8]).

Example 2.2 Consider the deterministic finite automaton from Example 2.1. Then we obtain the set of equations

$$\begin{cases} \mathcal{L}_0 &= (\{b\} \times \mathcal{L}_0) \cup (\{a\} \times \mathcal{L}_1) \cup \{\epsilon\}, \\ \mathcal{L}_1 &= \{a\} \times \mathcal{L}_0, \end{cases}$$

where $\mathcal{L}_0$ and $\mathcal{L}_1$ denote the languages associated with the states q_0 and q_1, respectively. This gives rise to a system of equations for the associated GFs:

$$\begin{cases} L_0(z) &= zL_0(z) + zL_1(z) + 1, \\ L_1(z) &= zL_0(z). \end{cases}$$

Solving this system, we obtain the generating function corresponding to $L(\mathcal{M})$. It is $L_0(z)$ since the initial state of the automaton is q_0, where

$$L_0(z) = \frac{1}{1 - z - z^2} = \sum_{n=0}^{\infty} F_{n+1} z^n$$

and F_n is the n-th Fibonacci number; see, e.g., sequence A000045 in the OEIS [15].

3 The automaton associated with the counting problem

From the description of the language, it is not difficult to show that if

$$L_n^{(3)} := \{w \in \Sigma_3 : w = w_1 w_2 \cdots w_n, w_1 = \mathsf{0} = w_n \text{ and } w_i \neq w_{i+1} \text{ for } 1 \leq i \leq n-1\},$$

then $L_n^{(3)} = L(\mathcal{M}_3)$, where $\mathcal{M}_3$ is the automaton in Figure 2.

In Table 1 below, we list the first few words accepted by this automaton. The automaton $\mathcal{M}_3$ gives rise to the following set of equations for the associated GFs:

$$\begin{cases} L_I(z) &= zL_0(z), \\ L_0(z) &= zL_1(z) + zL_2(z) + 1, \\ L_1(z) &= zL_0(z) + zL_2(z), \\ L_2(z) &= zL_0(z) + zL_1(z). \end{cases}$$

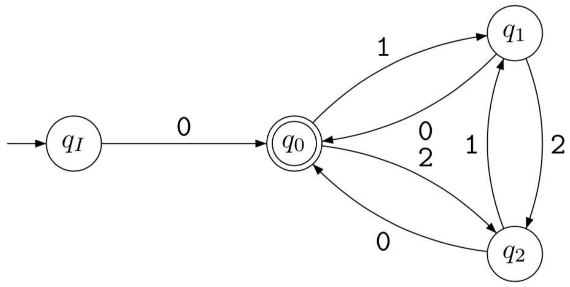


Figure 2: Transition diagram of $\mathcal{M}_3$.

Length of the word	Words	Number of Words
1	0	1
2	-	0
3	010, 020	2
4	0120, 0210	2
5	01210, 02120, 01010, 02010, 01020, 02020	6
6	010120, 010210, 020120, 020210, 012120, 021210, 012010, 021010, 012020, 021020	10

Table 1: First few words accepted by the automaton $\mathcal{M}_3$.

The augmented matrix for this system is

$$\left[\begin{array}{cccc|c} 1 & -z & 0 & 0 & 0 \\ 0 & 1 & -z & -z & 1 \\ 0 & -z & 1 & -z & 0 \\ 0 & -z & -z & 1 & 0 \end{array}\right].$$

Using Gaussian elimination, we obtain the generating functions

$$\begin{bmatrix} L_I(z), & L_0(z), & L_1(z), & L_2(z) \end{bmatrix} = \left[\frac{(1-z)z}{(1+z)(1-2z)}, \frac{1-z}{(1+z)(1-2z)}, \frac{z}{(1+z)(1-2z)}, \frac{z}{(1+z)(1-2z)}\right].$$

From the preceding equalities and (1), we have the following result.

Theorem 3.1 *[10, Theorem 2.1] The ordinary generating function of the sequence b_n is*

$$B_3(z) := \sum_{n=0}^{\infty} b_n z^n = \frac{(1-z)z}{(1+z)(1-2z)} = z + 2z^3 + 2z^4 + 6z^5 + 10z^6 + 22z^7 + \cdots.$$

Moreover, $b_n = 2J_{n-2}$ for $n \geq 1$.

Note that the generating functions

$$z^2 L_1(z) = \frac{z^3}{(1+z)(1-2z)} = z^2 L_2(z)$$

count the words in $L(\mathcal{M}_3)$ that begin with 01 or 02. Moreover, it is clear that if $w \in L(\mathcal{M}_3)$, then the reversal of w is also an accepted word by $\mathcal{M}_3$. Therefore, we obtain the following result.

THEOREM 3.2 *The generating function for the number of words in* $L_n^{(3)}$ *that begin with 01 or 02 (or end with 10 or 20) is*

$$\frac{z^3}{(1+z)(1-2z)} = z^2 J(z).$$

Moreover, the number of these words is the Jacobsthal number J_{n-2} *for* $n \geq 2$.

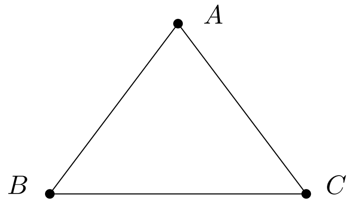
On the other hand, the transition diagram of the sub-automaton $\mathcal{M}'_3$ of $\mathcal{M}_3$ restricted to the states q_0, q_1, q_2 is a directed complete graph on three vertices. Therefore, the generating function for the number of closed walks of length n, which we denote by k_n, on the complete graph K_3 is given by $L_0(z) = \frac{1}{z} B_3(z)$, which implies the following.

THEOREM 3.3 *The generating function for the sequence* k_n *is*

$$\frac{1-z}{(1+z)(1-2z)}.$$

Moreover, $k_n = b_{n+1} = 2J_{n-1}$ *for* $n \geq 0$.

EXAMPLE 3.4 In Figure 3, we illustrate the $k_4 = 2J_3 = 6$ closed walks of length 4 on the complete graph K_3.



$A \to B \to A \to B \to A \qquad A \to C \to A \to C \to A$

$A \to B \to A \to C \to A \qquad A \to C \to A \to B \to A$

$A \to B \to C \to B \to A \qquad A \to C \to B \to C \to A$

Figure 3: Closed walks of length 4 in K_3.

4 A generalization

In this section, we introduce a generalization of the previous counting problem to an alphabet of size k. Let us consider the following language

$$L_n^{(k)} := \{w \in \Sigma_k : w = w_1 w_2 \cdots w_n, w_1 = 0 = w_n \text{ and } w_i \neq w_{i+1} \text{ for } 1 \leq i \leq n-1\}.$$

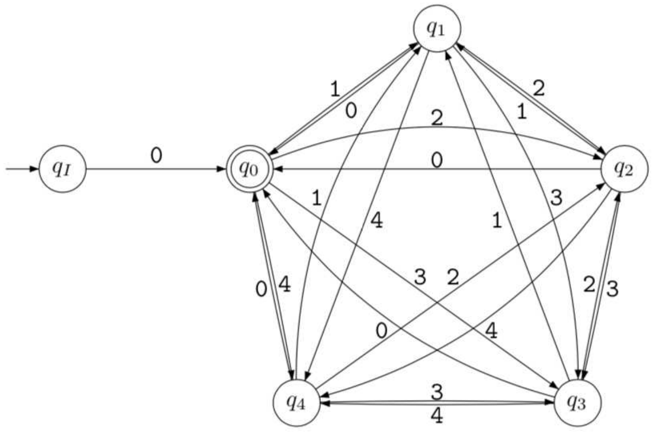


Figure 4: Transition diagram of $\mathcal{M}_5$.

We denote by $b_n^{(k)}$ the number of words in $L_n^{(k)}$. In particular, if $k = 3$, we have $b_n^{(3)} = b_n$. This language is accepted by the automaton $\mathcal{M}_k$ defined by the 5-tuple $\mathcal{M} = (\Sigma_k, Q, q_0, F, E)$, where $Q = \{q_I, q_0, q_1, \dots, q_{k-1}\}$, $F = \{q_0\}$ and $E = \{(q_i, j, q_j) : \text{for all } 0 \le i, j \le k-1,\ i \ne j\} \cup \{(q_I, \mathtt{0}, q_0)\}$. In Figure 4 above, we show the transition diagram for the automaton $\mathcal{M}_5$.

In Table 2, we list the first few words accepted by this automaton.

Length of the word	Words	Number of Words
1	0	1
2	-	0
3	010, 020, 030, 040	4
4	0120, 0130, 0140, 0210, 0230, 0240, 0310, 0320, 0340, 0410, 0420, 0430	12

Table 2: First few words accepted by the automaton $\mathcal{M}_5$.

The automaton $\mathcal{M}_k$ gives rise to a set of equations for the associated GFs:

$$\begin{cases} L_I(z) = zL_0(z), \\ L_0(z) = zL_1(z) + zL_2(z) + \cdots + zL_{k-1}(z) + 1, \\ L_1(z) = zL_0(z) + zL_2(z) + \cdots + zL_{k-1}(z), \\ \quad\vdots \\ L_{k-1}(z) = zL_0(z) + zL_2(z) + \cdots + zL_{k-2}(z). \end{cases}$$

By solving the above system of equations, we obtain the following result.

THEOREM 4.1 *The ordinary generating function of the sequence* $b_n^{(k)}$ *is*

$$B_k(z) := \sum_{n=0}^{\infty} b_n^{(k)} z^n = \frac{(1-(k-2)z)z}{(1+z)(1-(k-1)z)}. \tag{3}$$

Moreover,

$$b_n^{(k)} = \frac{(k-1)^{n-1} + (k-1)(-1)^{n-1}}{k}, \qquad n \geq 1. \tag{4}$$

Since $\frac{b_{n+2}^{(3)}}{2} = J_n$, the sequence $\frac{b_{n+2}^{(k)}}{k-1}$ may be regarded as a kind of generalized Jacobsthal number. The first few values ($k = 3, 4, 5, 6$) of $B_k(z)$ are as follows:

$$B_3(z) = \frac{(1-z)z}{(1+z)(1-2z)} = z + 2z^3 + 2z^4 + 6z^5 + 10z^6 + 22z^7 + 42z^8 + 86z^9 + \cdots \text{ (A078008)}.$$
$$B_4(z) = \frac{(1-2z)z}{(1+z)(1-3z)} = z + 3z^3 + 6z^4 + 21z^5 + 60z^6 + 183z^7 + 546z^8 + \cdots \text{ (A054878)}.$$
$$B_5(z) = \frac{(1-3z)z}{(1+z)(1-4z)} = z + 4z^3 + 12z^4 + 52z^5 + 204z^6 + 820z^7 + 3276z^8 + \cdots \text{ (A109499)}.$$
$$B_6(z) = \frac{(1-4z)z}{(1+z)(1-5z)} = z + 5z^3 + 20z^4 + 105z^5 + 520z^6 + 2605z^7 + 13020z^8 + \cdots \text{ (A109500)}.$$

For a different generalization of the Jacobsthal number arising in conjunction with an extension of the Collatz problem, see [6].

In the next result, we consider the shortest non-trivial prefix (or suffix) in our language and count the number of words in $L_n^{(k)}$ that begin (or end) with it.

THEOREM 4.2 *The generating function for the number of words in* $L_n^{(k)}$ *that begin with* `0`ℓ *(or end with* ℓ`0`*) for* $\ell =$ `1,2,...,k-1` *is*

$$\frac{z}{(1+z)(1-(k-1)z)}.$$

Extending the proof of Theorem 3.3 above leads to the following generalization.

THEOREM 4.3 *The generating function for the sequence* $k_n^{(m)}$ *counting the closed walks of length* n *on the complete graph* K_m *is*

$$\frac{1}{z} B_m(z) = \frac{1-(m-2)z}{(1+z)(1-(m-1)z)}.$$

Moreover, $k_n^{(m)} = b_{n+1}^{(m)}$ *for* $n \geq 0$.

We now make a connection between $L_n^{(k)}$ and colored tilings. Recall that a square-and-domino tiling (see [3, Chapter 1]) of length n is a linear sequence of (indistinguishable) 1×1 squares and 1×2 dominos where n equals the sum of the number of squares and twice the number of dominos. Let $T_n^{(k)}$ denote the set of square-and-domino tilings of length n in which squares come in one of $k-2$ colors and dominos in one of $k-1$ colors where $k \geq 3$, and let $R_n^{(k)}$ denote the subset of $T_n^{(k)}$ whose members start with a domino (of any color). The following result shows that members of $L_n^{(k)}$ may be regarded in terms of colored tilings.

PROPOSITION 4.4 *If* $n, k \geq 3$, *then* $|L_n^{(k)}| = |R_{n-1}^{(k)}|$.

Proof. To show this equality, we define a bijection between $L_n^{(k)}$ and $R_{n-1}^{(k)}$. Let $[m] = \{1, 2, \ldots, m\}$. Assume that the sets comprising the colors of the squares and of the dominos within a member of $T_n^{(k)}$ are given by $[k-2]$ and $[k-1]$, respectively. Let $w = 0w_2w_3 \cdots w_{n-1}0 \in L_n^{(k)}$ be expressed as $w = 0\alpha_1 0\alpha_2 \cdots 0\alpha_\ell 0$ for some $\ell \geq 1$, where each α_i subword contains no zeros. To define our mapping, first let $\alpha_1 = y_1 y_2 \cdots y_m$. Assign to the letter y_1 a domino having color y_1. Since $y_2 \neq y_1$, there are $k-2$ possibilities for y_2 and we assign to y_2 a square of color i, where i is such that y_2 is the i-th smallest member of the set $[k-1] - \{y_1\}$. Similarly, assign y_3 a square of color j, where y_3 is the j-th smallest member of $[k-1] - \{y_2\}$. Continue through the letter y_m. This results in a subtiling $T_1 = ds^{m-1}$, where the color of the individual tiles are as described. Repeat process for all other α_i subwords to obtain comparable subtilings $T_2, \ldots, T_\ell$. Let $\lambda \in R_{n-1}^{(k)}$ be obtained by concatenating the T_i, i.e., $\lambda = T_1 T_2 \cdots T_\ell$. One may verify that the mapping $w \mapsto \lambda$ is a bijection from $L_n^{(k)}$ to $R_{n-1}^{(k)}$. For example, if $n = 9$, $k = 4$ and $w = 030231020 \in L_9^{(4)}$, then $\lambda = d^2 s^2 d \in R_8^{(4)}$, where the sequence of colors for the tiles of λ is 32212. $\square$

One may also view a member of $L_n^{(k)}$ as the base-k representation of some positive integer. For example, if $n = 8$, $k = 5$ and $w = 02031040 \in L_8^{(5)}$, then w has decimal value given by $2 \cdot 5^6 + 3 \cdot 5^4 + 1 \cdot 5^3 + 4 \cdot 5 = 33,270$. We conclude this section by finding a formula for the sum of the base-k representations of all members of $L_n^{(k)}$. This generalizes a result from [10] corresponding to the $k = 3$ case where it was found the sum of ternary numbers of a fixed length. Note that the interactions between numerical systems and automata theory have been an object of previous study (cf. [4]).

Given $k \geq 3$, let $S_n = S_n^{(k)}$ denote the sum of the decimal values of all members of $L_n^{(k)}$ read as base-k numbers. To find a recurrence for S_n where $n \geq 4$, let $w = 0w_2 \cdots w_{n-1}0 \in L_n^{(k)}$ and first suppose $w_{n-2} \neq 0$. Then there are $k-2$ choices for w_{n-1}, and the contribution from the digits $0w_2 \cdots w_{n-2}$ within all such words is seen to be $k(k-2)S_{n-1}$ as each digit is shifted to the left by one place in going from $0w_2 \cdots w_{n-2}0$ to $0w_2 \cdots w_{n-1}0$. On the other hand, the w_{n-1} digits within such words contribute

$$k(1 + \cdots + (k-1))\frac{k-2}{k-1}b_{n-1}^{(k)} = \frac{k^2(k-2)}{2}b_{n-1}^{(k)},$$

since for each choice of w_{n-1}, there are $\frac{k-2}{k-1}b_{n-1}^{(k)}$ possibilities concerning the initial sequence of digits $0w_2 \cdots w_{n-2}$. Combining the two cases implies that the sum of all members of $L_n^{(k)}$ for which $w_{n-2} \neq 0$ is given by $k(k-2)S_{n-1} + \frac{k^2(k-2)}{2}b_{n-1}^{(k)}$. By similar reasoning, the sum of members of $L_n^{(k)}$ for which $w_{n-2} = 0$ is given by $k^2(k-1)S_{n-2} + \frac{k^2(k-1)}{2}b_{n-2}^{(k)}$. Combining the previous cases and verifying directly for $n = 3$ yields the recurrence

$$S_n = k(k-2)S_{n-1} + k^2(k-1)S_{n-2} + \frac{k^2(k-2)}{2}b_{n-1}^{(k)} + \frac{k^2(k-1)}{2}b_{n-2}^{(k)}, \qquad n \geq 3, \tag{5}$$

with $S_1 = S_2 = 0$.

Given $n \geq 0$ and $k \geq 3$, let $J_n^{(k)} = \frac{b_{n+2}^{(k)}}{k-1}$ denote the aforementioned generalized Jacobsthal number. It is possible to express the sum $S_n^{(k)}$ in terms of $J_n^{(k)}$.

THEOREM 4.5 *If $n \geq 1$ and $k \geq 3$, then the cumulative sum of the decimal values of all members of $L_n^{(k)}$ when read base-k is given by*

$$\frac{k^{n+1}(k^3-3k^2+4k-1)-k^2(2k^2-4k+3)}{2(k-1)(2k-1)(k^2-k+1)}J_n^{(k)} - \frac{k^2(k-2)(k^n-1)}{2(k-1)(2k-1)(k^2-k+1)}J_{n+1}^{(k)}.$$

Proof. Let the generating function $S(x) = S_k(x)$ be defined by $S(x) = \sum_{n\geq 1} S_n x^n$. Multiplying both sides of (5) by x^n, and summing over $n \geq 3$, implies

$$S(x) = k(k-2)xS(x) + k^2(k-1)x^2S(x) + \frac{k^2(k-2)}{2}x(B_k(x)-x) + \frac{k^2(k-1)}{2}x^2B_k(x),$$

where $B_k(x)$ is given by (3). Solving for $S(x)$ in the last equation yields

$$S(x) = \frac{k^2(k-2)x^2+k^2(k-1)x^3}{2}g(x) - \frac{k^2(k-2)x^2}{2}\left(\frac{1}{(1+kx)(1-k(k-1)x)}\right), \tag{6}$$

where

$$g(x) = \frac{1-(k-2)x}{(1+x)(1+kx)(1-(k-1)x)(1-k(k-1)x)}.$$

By partial fractions, we have

$$g(x) = \frac{2/(2k-1)}{1+kx} + \frac{(k^2-2k+2)/(k^2-k+1)}{1-k(k-1)x} - \frac{1/(k^2-k+1)}{k(1+x)} - \frac{1/(2k-1)}{k(1-(k-1)x)}.$$

Grouping together the first two terms as well as the last two gives

$$\begin{aligned} g(x) = {} & \frac{1}{(2k-1)(k^2-k+1)}\left(\frac{k(1-k(k-2)x)}{(1+kx)(1-k(k-1)x)} + \frac{k(2k^2-3k+3)}{(1+kx)(1-k(k-1)x)}\right) \\ & - \frac{1}{(2k-1)(k^2-k+1)}\left(\frac{1-(k-2)x}{(1+x)(1-(k-1)x)} + \frac{k}{(1+x)(1-(k-1)x)}\right). \end{aligned} \tag{7}$$

Also, by partial fractions, we have

$$\frac{1}{(1+x)(1-(k-1)x)} = \frac{1}{k(1+x)} + \frac{k-1}{k(1-(k-1)x)},$$

which implies

$$\begin{aligned} [x^n]\left(\frac{1}{(1+x)(1-(k-1)x)}\right) &= \frac{1}{k}\left((k-1)^{n+1}+(-1)^n\right) \\ &= \frac{1}{k-1}\left(\frac{(k-1)^{n+2}+(k-1)(-1)^{n+2}}{k}\right) = \frac{b_{n+3}^{(k)}}{k-1}, \end{aligned}$$

by (4).

Extracting the coefficient of x^n in (6), and using (7) and (3), leads to the following formula for $n \geq 3$:

$$\begin{aligned} S_n^{(k)} = & \frac{k^n}{2(2k-1)(k^2-k+1)} \Big(k(k-2)b_{n-1}^{(k)} + (k-1)b_{n-2}^{(k)} \\ & \qquad + (2k^2-3k+3) \left(\frac{k(k-2)}{k-1} b_{n+1}^{(k)} + b_n^{(k)} \right) \Big) \\ & - \frac{k^2}{2(2k-1)(k^2-k+1)} \left((k-2)b_{n-1}^{(k)} + (k-1)b_{n-2}^{(k)} + \frac{k(k-2)}{k-1} b_{n+1}^{(k)} + kb_n^{(k)} \right) \\ & - \frac{k^n(k-2)}{2(k-1)} b_{n+1}^{(k)}. \end{aligned}$$

Making repeated use of the recurrence $b_m^{(k)} = (k-2)b_{m-1}^{(k)} + (k-1)b_{m-2}^{(k)}$ for $m \geq 3$, one can express all of the $b_m^{(k)}$ terms in the last formula using only $b_{n+2}^{(k)}$ and $b_{n+3}^{(k)}$. After several algebraic steps, this yields

$$\begin{aligned} S_n^{(k)} = & \frac{k^{n+1}}{2(k-1)^2(2k-1)(k^2-k+1)} \left((k^3-3k^2+4k-1)b_{n+2}^{(k)} - k(k-2)b_{n+3}^{(k)} \right) \\ & - \frac{k^2}{2(k-1)^2(2k-1)(k^2-k+1)} \left((2k^2-4k+3)b_{n+2}^{(k)} - (k-2)b_{n+3}^{(k)} \right). \end{aligned}$$

Grouping together like terms, and noting $b_m^{(k)} = (k-1)J_{m-2}^{(k)}$, completes the proof. □

When $k = 3$, one obtains the following formula which was shown in [10] by a different method.

Corollary 4.6 *[10, Formula 2.7] If $n \geq 1$, then the cumulative sum of the decimal values of the b_n ternary words is given by*

$$\frac{1}{140}(11 \cdot 3^{n+1} - 81)J_n - \frac{1}{140}(3^{n+2} - 9)J_{n+1}.$$

5 A further polynomial extension

In this section, we consider a polynomial generalization of the sequence $b_n^{(k)}$. By an *internal zero* of a word $w = w_1 \cdots w_n \in L_n^{(k)}$, we will mean an index i such that $w_i = 0$ and $i \neq 1, n$. Let S_w denote the set of indices $2 \leq i \leq n-1$ such that $w_i = 0$. We define the following statistics on $L_n^{(k)}$.

Definition 5.1 Given $w \in L_n^{(k)}$, let $\nu(w) = |S(w)|$ be the number of internal zeros of w and $\sigma(w) = \sum_{i \in S_w} i$ be the sum of the corresponding indices.

For example, if $n = 12$, $k = 9$ and $w = 020134102030 \in L_{12}^{(4)}$, then $v(w) = 3$ and $\sigma(w) = 3+8+10 = 21$.

Definition 5.2 For indeterminates p and q, define the distribution polynomial $b_n^{(k)}(p,q)$ by

$$b_n^{(k)}(p,q) = \sum_{w \in L_n^{(k)}} p^{\nu(w)} q^{\sigma(w)}, \qquad n \geq 1, \ k \geq 2.$$

For example, if $1 \leq n \leq 4$, then $\sigma(w) = \nu(w) = 0$ for all $w \in L_n^{(k)}$ and thus $b_n^{(k)}(p,q) = |L_n^{(k)}| = b_n^{(k)}$ for all k. If $n = 5$, then the only possible position for an internal zero is the third, implying $b_5^{(k)}(p,q) = (k-1)^2pq^3 + (k-1)(k-2)^2$. Note that $b_n^{(k)}(p,q)$ reduces to $b_n^{(k)}$ when $p = q = 1$. Equivalently, if one were instead to compute the σ-value after re-indexing the middle $n-2$ letters of $w \in L_n^{(k)}$ so that the first of these letters has index zero, then the corresponding distribution would be $b_n^{(k)}(p/q^2, q)$ in the present notation. For an analogue of the σ-statistic considered on r-mino arrangements, see [14].

To determine a recurrence for $b_n^{(k)}(p,q)$ where $n \geq 4$, first note that if $w_{n-2} = 0$ in $w = w_1 \cdots w_n \in L_n^{(k)}$, then there are $k-1$ choices regarding the letter w_{n-1} and $b_{n-2}^{(k)}(p,q)$ possibilities for the first $n-2$ letters, by definition. Thus, the weight of all such members of $L_n^{(k)}$ is given by $(k-1)pq^{n-2}b_{n-2}^{(k)}(p,q)$, since the $w_{n-2} = 0$ letter introduces a factor of pq^{n-2}. On the other hand, members $w \in L_n^{(k)}$ such that $w_{n-2} \neq 0$ may be obtained by inserting one of $k-2$ possible letters directly prior to the terminal zero within some member of $L_{n-1}^{(k)}$. Since none of the internal zeros are disturbed by such a move, this yields $(k-2)b_{n-1}^{(k)}(p,q)$ additional possibilities. Thus, we have

$$b_n^{(k)}(p,q) = (k-1)pq^{n-2}b_{n-2}^{(k)}(p,q) + (k-2)b_{n-1}^{(k)}(p,q), \qquad n \geq 4, \tag{8}$$

with $b_1^{(k)}(p,q) = 1$, $b_2^{(k)}(p,q) = 0$ and $b_3^{(k)}(p,q) = k-1$.

In order to study $b_n^{(k)}(p,q)$, we consider the generating function $B(x) = B_{p,q}^{(k)}(x)$ given by

$$B(x) = \sum_{n \geq 1} b_n^{(k)}(p,q)x^n.$$

Multiplying both sides of (8) by x^n, and summing over $n \geq 4$, gives

$$B(x) - x - (k-1)x^3 = (k-1)x^2p(B(xq) - xq) + (k-2)x(B(x) - x),$$

which may be rewritten as

$$B(x) = \frac{x - (k-2)x^2 + (k-1)(1-pq)x^3}{1-(k-2)x} + \frac{(k-1)x^2p}{1-(k-2)x}B(xq). \tag{9}$$

Iterating the functional equation (9) an infinite number of times, where we assume x, p and q are sufficiently close to zero to guarantee convergence, gives

$$\begin{aligned}
B(x) &= \sum_{n\geq 0}\left(\prod_{i=0}^{n-1}\frac{(k-1)(xq^i)^2p}{1-(k-2)xq^i}\right)\frac{xq^n(1-(k-2)xq^n+(k-1)(1-pq)x^2q^{2n})}{1-(k-2)xq^n}\\
&= \sum_{n\geq 0}\frac{(k-1)^n x^{2n+1}p^nq^{n^2}(1-(k-2)xq^n+(k-1)(1-pq)x^2q^{2n})}{\prod_{i=0}^{n}(1-(k-2)xq^i)}\\
&= \sum_{n\geq 0}\frac{(k-1)^n x^{2n+1}p^nq^{n^2}}{\prod_{i=0}^{n-1}(1-(k-2)xq^i)}+\sum_{n\geq 1}\frac{(k-1)^n x^{2n+1}p^{n-1}q^{n^2-1}}{\prod_{i=0}^{n-1}(1-(k-2)xq^i)}\\
&\quad -\sum_{n\geq 1}\frac{(k-1)^n x^{2n+1}p^nq^{n^2}}{\prod_{i=0}^{n-1}(1-(k-2)xq^i)}\\
&= x+\sum_{n\geq 1}\frac{(k-1)^n x^{2n+1}p^{n-1}q^{n^2-1}}{\prod_{i=0}^{n-1}(1-(k-2)xq^i)}.
\end{aligned}$$

Thus, we can state the following result.

THEOREM 5.3 *We have*

$$B_{p,q}^{(k)}(x) = x+\sum_{n\geq 1}\frac{(k-1)^n x^{2n+1}p^{n-1}q^{n^2-1}}{\prod_{i=0}^{n-1}(1-(k-2)xq^i)}. \tag{10}$$

This yields the following explicit formula for $b_n^{(k)}(p,q)$, where $\binom{n}{m}_q$ denotes the q-binomial coefficient.

COROLLARY 5.4 *If $n,k\geq 2$, then*

$$b_n^{(k)}(p,q) = \sum_{m=0}^{\lfloor (n-3)/2\rfloor} p^m q^{m^2+2m}\binom{n-m-3}{m}_q (k-1)^{m+1}(k-2)^{n-2m-3}. \tag{11}$$

Proof. Extracting the coefficient of x^np^m in (10), where $m\geq 0$ and $n\geq 2m+3$, gives

$$\begin{aligned}
[x^np^m](B_{p,q}^{(k)}(x)) &= [x^n]\left(\frac{(k-1)^{m+1}x^{2m+3}q^{(m+1)^2-1}}{\prod_{i=0}^{m}(1-(k-2)xq^i)}\right)\\
&= \frac{(k-1)^{m+1}q^{m^2+2m}}{(k-2)^m}[x^{n-m-3}]\left(\frac{((k-2)x)^m}{\prod_{i=0}^{m}(1-(k-2)xq^i)}\right)\\
&= \frac{(k-1)^{m+1}q^{m^2+2m}}{(k-2)^m}\cdot(k-2)^{n-m-3}\binom{n-m-3}{m}_q,
\end{aligned}$$

where we have applied the formula for the q-binomial generating function (see, e.g., [1]) in the last equality. Considering the weight of all members of $L_n^{(k)}$ according to the number m of internal zeros where $0\leq m\leq\lfloor (n-3)/m\rfloor$ gives (11). $\square$

We have the following formulas for the total value and sign balance of the ν-statistic on $L_n^{(k)}$.

THEOREM 5.5 *If $n, k \geq 2$, then the total number of internal zeros within all members of $L_n^{(k)}$ is given by*

$$\frac{1}{k^3}\left((-1)^{n-1}(k-1)^2(nk-2k-2)+(k-1)^{n-1}(nk-4k+2)\right).$$

The difference between the number of members of $L_n^{(k)}$ having an even with those having an odd number of internal zeros is given by

$$\frac{(k-2)(\alpha^{n-1}-\beta^{n-1})+\beta^n-\alpha^n}{\alpha-\beta},$$

where α and β are given by $\left(k-2\pm\sqrt{k^2-8k+8}\right)/2$.

Proof. The total number of internal zeros at position i where $2 \leq i \leq n-1$ equals the number of $w = w_1 \cdots w_n \in L_n^{(k)}$ such that $w_i = 0$, which is given by $b_i^{(k)} b_{n-i+1}^{(k)}$. Considering all possible i implies that the number of internal zeros in all members of $L_n^{(k)}$ equals $\sum_{i=2}^{n-1} b_i^{(k)} b_{n-i+1}^{(k)}$. By formula (4), we have

$$\begin{aligned}\sum_{i=2}^{n-1} b_i^{(k)} b_{n-i+1}^{(k)} &= \frac{1}{k^2}\sum_{i=2}^{n-1}\left((k-1)^{i-1}+(-1)^{i-1}(k-1)\right)\left((k-1)^{n-i}+(-1)^{n-i}(k-1)\right)\\ &= \frac{1}{k^2}\sum_{i=2}^{n-1}\left((k-1)^{n-1}+(-1)^n(1-k)^i+(k-1)^n\left(\frac{1}{1-k}\right)^{i-1}+(-1)^{n-1}(k-1)^2\right)\\ &= \frac{1}{k^2}\left((n-2)(k-1)^{n-1}+2(-1)^n\frac{(1-k)^2-(1-k)^n}{k}+(-1)^{n-1}(n-2)(k-1)^2\right)\\ &= \frac{1}{k^3}\left((-1)^{n-1}(k-1)^2(nk-2k-2)+(k-1)^{n-1}(nk-4k+2)\right),\end{aligned}$$

which implies the first statement.

To show the second statement, we find $b_n^{(k)}(1,-1)$. Taking $p=-q=-1$ in (10) gives

$$B_{-1,1}^{(k)}(x) = x + \sum_{n\geq 1}\frac{(-1)^{n-1}(k-1)^n x^{2n+1}}{(1-(k-2)x)^n} = x + \frac{(k-1)x^3}{1-(k-2)x+(k-1)x^2}.$$

Let $e_n = b_n^{(k)}(-1,1)$ for $n \geq 2$. From the form of the generating function, we have the recurrence

$$e_n = (k-2)e_{n-1}-(k-1)e_{n-2}, \qquad n \geq 4, \tag{12}$$

with $e_2 = 0$ and $e_3 = k-1$. Then $e_n = a\alpha^n + b\beta^n$ for $n \geq 2$, where α and β are as stated and a and b are constants to be determined. To aid in finding a and b, we apply (12) in reverse and obtain the initial values $e_0 = \frac{k-2}{1-k}$, $e_1 = -1$. Solving the system $a + b = \frac{k-2}{1-k}$ and $\alpha a + \beta b = -1$, we get

$$a = \frac{1-k+(k-2)\beta}{(k-1)(\alpha-\beta)} \qquad \text{and} \qquad b = \frac{k-1-(k-2)\alpha}{(k-1)(\alpha-\beta)}.$$

Simplifying the resulting expression for e_n, using the fact $\alpha\beta = k-1$, leads to the second statement and completes the proof. □

Remark. The total number of zeros within all members of $L_n^{(k)}$ is given by the first expression in Theorem 5.5 above plus $2b_n^{(k)}$, where $b_n^{(k)}$ is given by (4). In particular, letting $k=3$, the total number of zeros within all ternary words of length n is given by

$$\begin{aligned}
&\frac{1}{27}\big((-1)^{n-1}(12n-32)+2^{n-1}(3n-10)\big)+2b_n^{(3)}\\
&=\frac{1}{27}\left((J_n-2J_{n-1})(12n-32)+(J_n+J_{n-1})(3n-10)\right)+2(J_n-J_{n-1})\\
&=\left(\frac{5n+4}{9}\right)J_n-\left(\frac{7n}{9}\right)J_{n-1}=\left(\frac{17n+8}{18}\right)J_n-\left(\frac{7n}{18}\right)J_{n+1},
\end{aligned}$$

which agrees with [10, Formula 2.2] found using a different method. Finally, by symmetry, the number of occurrences of each non-zero letter $i\in[k-1]$ within all members of $L_n^{(k)}$ is given by

$$\begin{aligned}
&\frac{n-2}{k-1}b_n^{(k)}-\frac{1}{k^3}\big((-1)^{n-1}(k-1)(nk-2k-2)+(k-1)^{n-2}(nk-4k+2)\big)\\
&=\frac{1}{k^3}\big((-1)^{n-1}(nk-2)+(k-1)^{n-1}(nk-2k+2)\big).
\end{aligned}$$

We obtain the following apparently new binomial identities as a consequence of the prior results.

Corollary 5.6 *If $n,k\geq 1$, then*

$$\begin{aligned}
&\sum_{m=2}^{\lfloor n/2\rfloor}(n-m-1)\binom{n-m-2}{m-2}(k-1)^m(k-2)^{n-2m}\\
&\qquad=\frac{1}{k^3}\big((-1)^n(k-1)^2(nk-k-2)+(k-1)^n(nk-3k+2)\big) \qquad (13)
\end{aligned}$$

and

$$\sum_{m=1}^{\lfloor n/2\rfloor}(-1)^{m-1}\binom{n-m-1}{m-1}(k-1)^m(k-2)^{n-2m}=\frac{(k-2)(\alpha^n-\beta^n)+\beta^{n+1}-\alpha^{n+1}}{\alpha-\beta}, \qquad (14)$$

where α and β are as in Theorem 5.5.

Proof. Differentiating both sides of (11) with respect to p, and taking $p=q=1$, gives the total ν-value on $L_n^{(k)}$ of

$$\sum_{m=1}^{\lfloor (n-3)/2\rfloor}(n-m-3)\binom{n-m-4}{m-1}(k-1)^{m+1}(k-2)^{n-2m-3}.$$

Equating this with the expression found in the first part of Theorem 5.5, and replacing n by $n+1$ and m by $m-1$, yields (13). Taking $p=-1$ in (11), and equating with the second formula in Theorem 5.5, leads to (14). □

We have comparable formulas for the σ-statistic on $L_n^{(k)}$.

THEOREM 5.7 *If $n,k\geq 2$, then*

$$\frac{d}{dq}b_n^{(k)}(1,q)\mid_{q=1}=\frac{n+1}{2k^3}\left((-1)^{n-1}(k-1)^2(nk-2k-2)+(k-1)^{n-1}(nk-4k+2)\right) \tag{15}$$

and

$$b_n^{(k)}(1,-1)=\begin{cases}\frac{(k-1)(k-2)(\alpha^t-\beta^t)}{\alpha-\beta}, & \text{if } n \text{ is even,}\\ \frac{(k-1)(\alpha^t-\beta^t)-(k-1)^2(\alpha^{t-1}-\beta^{t-1})}{\alpha-\beta}, & \text{if } n \text{ is odd,}\end{cases} \tag{16}$$

where $t=\lfloor(n-1)/2\rfloor$ and α and β are given by $\left((k-2)^2\pm\sqrt{(k-2)^4+4(k-1)^2}\right)/2$.

Proof. Reasoning as in the proof of Theorem 5.5 above and using (4), we have that the total σ-value taken over all members of $L_n^{(k)}$ is given by

$$\begin{aligned}\sum_{i=2}^{n-1} ib_i^{(k)}b_{n-i+1}^{(k)} &= \frac{1}{k^2}\sum_{i=2}^{n-1}\left((k-1)^{i-1}+(-1)^{i-1}(k-1)\right)\left((k-1)^{n-i}+(-1)^{n-i}(k-1)\right)\\ &=\frac{1}{k^2}\sum_{i=2}^{n-1} i\left((k-1)^{n-1}+(-1)^n(1-k)^i+(k-1)^n\left(\frac{1}{1-k}\right)^{i-1}+(-1)^{n-1}(k-1)^2\right).\end{aligned}$$

Using the identity

$$\sum_{i=2}^{n-1} ir^{i-1}=\frac{2r-r^2-nr^{n-1}+(n-1)r^n}{(1-r)^2},\qquad n\geq 2,$$

and simplifying, leads to (15).

To show (16), we first take $q=-1$ in (10) to get

$$\begin{aligned}\sum_{n\geq 1} b_n^{(k)}(1,-1)x^n &= x+\sum_{n\geq 1}\frac{(-1)^{n-1}(k-1)^n x^{2n+1}}{\prod_{i=0}^{n-1}(1-(k-2)x(-1)^i)}\\ &= x+\sum_{m\geq 1}\frac{(k-1)^{2m-1}x^{4m-1}}{(1-(k-2)^2x^2)^{m-1}(1-(k-2)x)}-\sum_{m\geq 1}\frac{(k-1)^{2m}x^{4m+1}}{(1-(k-2)^2x^2)^m}\\ &= x+\frac{(k-1)x^3(1+(k-2)x)-(k-1)^2x^5}{1-(k-2)^2x^2-(k-1)^2x^4}.\end{aligned}$$

Taking the even and odd parts of the last equality, and replacing x by $\sqrt{x}$, gives

$$\sum_{n\geq 1} b_{2n}^{(k)}(1,-1)x^n=\frac{(k-1)(k-2)x^2}{1-(k-2)^2x-(k-1)^2x^2}$$

and

$$\sum_{n\geq 0} b_{2n+1}^{(k)}(1,-1)x^n=1+\frac{(k-1)x(1-(k-1)x)}{1-(k-2)^2x-(k-1)^2x^2},$$

respectively. Let $e_n=b_{2n}^{(k)}(1,-1)$ for $n\geq 1$. From the form of the generating function, we have

$$e_n=(k-2)^2e_{n-1}+(k-1)^2e_{n-2},\qquad n\geq 3, \tag{17}$$

with $e_1 = 0$ and $e_2 = (k-1)(k-2)$. Then $e_n = c\alpha^n + d\beta^n$, where α and β are as defined above and c and d are constants. The recurrence (17) when $n = 2$ implies $e_0 = \frac{k-2}{k-1}$, which we assume. This gives the system $c + d = \frac{k-2}{k-1}$ and $\alpha c + \beta d = 0$. Solving for c and d, and noting $\alpha\beta = -(k-1)^2$, leads to the even case of (16). A similar proof applies to the odd case, except that now one takes $e_0 = -1$ in computing the values of c and d with $e_1 = k-1$ and $e_2 = (k-1)(k^2-5k+5)$, which completes the proof. □

We conclude with the following further binomial identity.

Corollary 5.8 *If $n, k \geq 1$, then*

$$\sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-m-1}{m} (k-1)^{2m+1}(k-2)^{2n-4m-1} = \frac{(k-1)(k-2)(\alpha^n - \beta^n)}{\alpha - \beta}, \tag{18}$$

where α and β are as in Theorem 5.7.

Proof. Replacing n by $2n+2$ and taking $q = -1$ in (11) implies

$$b_{2n+2}^{(k)}(1,-1) = \sum_{m=0}^{n-1} (-1)^m \binom{2n-m-1}{m}_{-1} (k-1)^{m+1}(k-2)^{2n-2m-1}.$$

It is well-known (see, e.g., [13]) that

$$\binom{n}{k}_{-1} = \begin{cases} 0, & \text{if } n \text{ is even or } k \text{ is odd,} \\ \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor}, & \text{otherwise,} \end{cases}$$

which implies

$$b_{2n+2}^{(k)}(1,-1) = \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-m-1}{m} (k-1)^{2m+1}(k-2)^{2n-4m+1},$$

where we have replaced m with $2m$ in the sum. Equating this with the even case of (16) above implies (18). □

Remark. A similar identity may be obtained using the odd case of (16), though it involves two sums. Evaluating the q-derivative at $q = 1$ of both sides of (11), and equating with (15), gives back (a scalar multiple of) identity (13) after re-indexing.

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