

On the largest part size and its multiplicity of a random integer partition

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Abstract. Let λ be a partition of the positive integer n chosen uniformly at random among all such partitions. Let $L_n = L_n(\lambda)$ and $M_n = M_n(\lambda)$ be the largest part size and its multiplicity, respectively. For large n , we focus on a comparison between the partition statistics L_n and $L_n M_n$. In terms of convergence in distribution, we show that they behave in the same way. However, it turns out that the expectation of $L_n M_n - L_n$ grows as fast as $\frac{1}{2} \log n$. We obtain a precise asymptotic expansion for this expectation and conclude with an open problem arising from this study.

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1 Introduction and statement of the main results

Partitioning integers into summands (parts) is a subject of intensive research in combinatorics, number theory and statistical physics. If n is a positive integer, then, by a partition λ of n , we mean the representation

$$\lambda: \quad n = \lambda_1 + \lambda_2 + \dots + \lambda_k, \quad k \geq 1, \quad (1)$$

where the positive integers λ_j satisfy $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$. Let $\Lambda(n)$ be the set of all partitions of n and let $p(n) = |\Lambda(n)|$ (by definition $p(0) = 1$ assuming that the unique partition of 0 is the empty partition). The number $p(n)$ is determined asymptotically by the famous partition formula of Hardy and Ramanujan [10]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \quad n \rightarrow \infty.$$

A precise asymptotic expansion for $p(n)$ was found by Rademacher [14] (see also [2, Chapter 5]). Further on, we assume that, for fixed integer $n \geq 1$, a partition $\lambda \in \Lambda(n)$ is selected uniformly at random. In other words, we assign the probability $1/p(n)$ to each $\lambda \in \Lambda(n)$. We denote this probability measure on $\Lambda(n)$ by $\mathbb{P}$. Let $\mathbb{E}$ be the expectation with respect to $\mathbb{P}$. In this way, each

numerical characteristic of $\lambda \in \Lambda(n)$ can be regarded as a random variable, or, a statistic produced by the random generation of partitions of n .

In this paper, we focus on two statistics of random integer partitions $\lambda \in \Lambda(n)$: $L_n = L_n(\lambda) = \lambda_1$, which is the largest part size in representation (1) and $M_n = M_n(\lambda)$, equal to the multiplicity of the largest part λ_1 (i.e., $M_n(\lambda) = m, 1 \leq m \leq k-1$, if $\lambda_1 = \dots = \lambda_m > \lambda_{m+1} \geq \dots \geq \lambda_k$ in (1), and $M_n(\lambda) = k$, if $\lambda_1 = \dots = \lambda_k$).

Each partition $\lambda \in \Lambda(n)$ has a unique graphical representation called Ferrers diagram; see, e.g., [2, Chapter 1]. The Ferrers diagram of the partition (1) is an array of dots in the plane, left-justified, with λ_j dots in the j -th row counting from the top ($j = 1, \dots, k$). Ferrers diagrams may be also considered as unions of disjoint blocks (rectangles) of dots whose side lengths represent the part sizes and the multiplicities of parts, respectively. For instance, Figure 1 illustrates the partition $7 + 5 + 5 + 5 + 4 + 2 + 1 + 1 + 1$ of 31 in which $L_{31} = 7$ and $M_{31} = 1$.

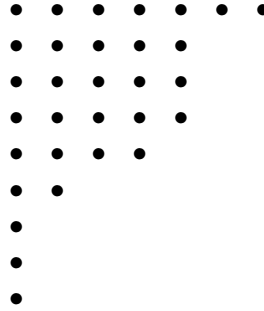


Figure 1

The earliest asymptotic results on random integer partition statistics has been obtained long ago by Husimi [11] and Erdős and Lehner [5]. Husimi has derived an asymptotic expansion for $\mathbb{E}(L_n)$ in the context of a statistical physics model of a Bose gas. Erdős and Lehner were apparently the first who studied random partition statistics in terms of probabilistic limit theorems. In fact, they showed that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{L_n}{\sqrt{n}} - \frac{1}{2c} \log n \leq u \right) = H(u), \quad (2)$$

where

$$H(u) = \exp \left(-\frac{1}{c} e^{-cu} \right), \quad -\infty < u < \infty \quad (3)$$

and

$$c = \frac{\pi}{\sqrt{6}}. \quad (4)$$

Later on, Szekeres has studied in detail the asymptotic behavior of the number of integer partitions of n whose largest part is $\leq k$ and $= k$ in the whole range of values of $k = k(n)$. In particular, he has obtained in [17] Erdős and Lehner's limiting distribution (2) using an entirely different method of proof. Husimi's asymptotic result was subsequently reconfirmed by Kessler and Livingston [12]. Higher moments of L_n were studied in [15]. A general method providing asymptotic expansions of

expectations of integer partition statistics was recently proposed by Grabner et al. [9]. Among the numerous examples, they derived a rather complete asymptotic expansion for $\mathbb{E}(L_n)$, namely,

$$\begin{aligned} \mathbb{E}(L_n) = & \frac{\sqrt{n}}{2c}(\log n + 2\gamma - 2\log c) + \frac{\log n}{2c^2} + \frac{1}{4} \\ & + \frac{1 + 2\gamma - 2\log c}{4c^2} + O\left(\frac{\log n}{n}\right), \quad n \rightarrow \infty, \end{aligned} \quad (5)$$

where c is given by (4) and $\gamma = 0.5772\dots$ denotes the Euler's constant (see [9, Proposition 4.2]). Notice that, by conjugation of Ferrers diagrams, the largest part and the total number of parts in a random partition of n are identically distributed for any n . The sequence $\{p(n)\mathbb{E}(L_n)\}_{n \geq 1}$ is given in [16] as A006128.

There are serious reasons to believe that the multiplicity M_n of the largest part of a random partition of n behaves asymptotically in a much simpler way than many other partition statistics. Grabner and Knopfmacher [8] used Erdős-Lehner limit theorem (2) to establish that

$$\lim_{n \rightarrow \infty} \mathbb{E}(M_n) = 1. \quad (6)$$

In addition, among many other important asymptotic results, Fristedt, in his remarkable paper [7], showed that, with probability tending to 1 as $n \rightarrow \infty$, the first m_n largest parts in a random partition of n are distinct if $m_n = o(n^{1/4})$. Hence it may be that L_n does not constitute the main contribution to n by a single part size and some smaller part sizes may occur with sufficient multiplicity so that the products of these part sizes with their multiplicities could be much larger than L_n . In terms of the Ferrers diagram this means that its first block has typically smaller area than several next block areas with larger heights (multiplicities of parts).

Our aim in this paper is to study the asymptotic behavior of the area $L_n M_n$ of the first block in the Ferrers diagram of a random partition of n . We show some similarities and differences between the single part size L_n and its corresponding block area $L_n M_n$. As a first step, we obtain a distributional result for M_n that confirms the limit in (6).

THEOREM 1.1 *For any $n \geq 1$, we have*

$$\mathbb{P}(M_n = 1) = \frac{p(n-1)}{p(n)}. \quad (7)$$

In addition, if $n \rightarrow \infty$, then

$$\mathbb{P}(M_n = 1) = 1 - \frac{c}{\sqrt{n}} + \frac{1 + c^2/2}{n} + O(n^{-3/2}), \quad (8)$$

where the constant c is given by (4).

Combining the Erdős-Lehner limit theorem (2) with (8), one can easily observe that the limiting distributions of L_n and $L_n M_n$ coincide under the same normalization.

COROLLARY 1.2 *For any real u , we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{L_n M_n}{\sqrt{n}} - \frac{1}{2c} \log n \leq u\right) = H(u),$$

where $H(u)$ and c are given by (3) and (4), respectively.

Although L_n and $L_n M_n$ follow the same limiting distribution, the difference in means $\mathbb{E}(L_n M_n) - \mathbb{E}(L_n)$ grows as fast as $\frac{1}{2} \log n$. A complete estimate is given in the following

THEOREM 1.3 *If $n \rightarrow \infty$, then, as $n \rightarrow \infty$,*

$$\mathbb{E}(L_n M_n) = \mathbb{E}(L_n) + \frac{1}{2} \log n - C + O(1/\log n),$$

where $C = \log c + 1 - \gamma = 0.67165\dots$ and $\mathbb{E}(L_n)$ and c are given by (5) and (4), respectively.

REMARK 1.4 Sequence $\{p(n)\mathbb{E}(L_n M_n)\}_{n \geq 1}$ is A092321 in [16].

The proofs of Theorems 1.1 and 1.3 are based on generating function identities established in [3] that involve products of the form $P(x)G(x)$, where $P(x)$ is the Euler partition generating function

$$P(x) := \sum_{n=0}^{\infty} p(n)x^n = \prod_{j=1}^{\infty} (1 - x^j)^{-1} \quad (9)$$

and $G(x)$ is a function which is analytic in the open unit disk and does not grow too fast as $x \rightarrow 1$. Our asymptotic expansions in (8) and Theorem 1.3 are obtained using a general asymptotic result of Grabner et al. [9] for the n th coefficient $x^n[P(x)G(x)]$. In the proof of Theorem 1.3 we also apply a classical way of estimating the growth of a power series around its main singularity.

We organize the paper as follows. Section 2 contains some auxiliary facts related to generating functions and the asymptotic analysis of their coefficients. In Section 3 we present the proofs of the main results. Finally, in Section 4 we conclude with an open problem on the position of $L_n M_n$ in the sequence of ordered block areas of a random Ferrers diagram.

REMARK 1.5 An extended abstract of this paper was published in [13].

2 Preliminaries: generating functions and an asymptotic scheme

We start with two generating function identities. In the next two lemmas, $P(x)$ is the generating function given by (9). Furthermore, we assume that every product of real or complex numbers $\{a_j\}$ of the form $\prod_{j \in J} a_j$ equals 1 if $J = \emptyset$.

LEMMA 2.1 (i) *For any positive integer m , we have*

$$\sum_{n=1}^{\infty} p(n) \mathbb{P}(M_n = m) x^n = x^m \prod_{j \geq m} (1 - x^j)^{-1} = P(x) x^m \prod_{j=1}^{m-1} (1 - x^j).$$

(ii) *We have*

$$\sum_{n=1}^{\infty} p(n) \mathbb{E}(L_n M_n) x^n = \sum_{k=1}^{\infty} \frac{k x^k}{1 - x^k} \prod_{j=1}^k (1 - x^j)^{-1} = P(x) F(x),$$

where

$$F(x) = \sum_{k=1}^{\infty} \frac{k x^k}{1 - x^k} \prod_{j=k+1}^{\infty} (1 - x^j). \quad (10)$$

Proof. (sketch) Part (i) is the last assertion of Theorem 2.3 from [3]. Part (ii) is given in A092321, see [16]. It also follows from Proposition 4.1 in [3]. $\square$

We shall essentially use the main result from [9, Theorem 2.3]. We present here only slight modifications of those parts of this theorem that we will need for our further asymptotic analysis. Furthermore, by $\log x$ we denote the main branch of the logarithmic function that satisfies the inequality $\log x < 0$ if $0 < x < 1$.

LEMMA 2.2 *Suppose that, for some constants $K > 0$ and $\eta < 1$, the function $G(x)$ satisfies*

$$G(x) = O(e^{K/(1-|x|)^\eta}), \quad |x| \rightarrow 1. \quad (11)$$

(i) *Suppose that, for $t \in \mathbb{C}$, $G(e^{-t}) = at^b + O(|f(t)|)$ as $t \rightarrow 0$, with $\Re t > 0$, where $b \geq 0$ is an integer and a is a real number. Then, we have*

$$\begin{aligned} \frac{1}{p(n)} x^n [P(x)G(x)] &= a \left(\frac{2\pi}{\sqrt{24n-1}} \right)^b \frac{s}{s-1} \sum_{j=0}^{b+1} \frac{(b+j+1)!}{j!(b+j-1)!} \left(-\frac{1}{2s} \right)^j \\ &+ O(e^{-2s}) + O\left(e^{-n^{1/2-\epsilon}} + f(c/\sqrt{n} + O(n^{-1/2-\epsilon}))\right) \end{aligned}$$

for any $\epsilon \in (0, (1-\eta)/2)$, where

$$s = \sqrt{\frac{2\pi^2}{s} \left(n - \frac{1}{24} \right)} = 2c\sqrt{n - \frac{1}{24}} \quad (12)$$

and c is given by (4).

(ii) *Suppose that $G(x)$ satisfies condition (11) and, for $t = u + iv$, let $G(e^{-t}) = a \log \frac{1}{t} + O(f(|t|))$ as $t \rightarrow 0$, where $u > 0$, $v = O(u^{1+\epsilon})$ as $u \rightarrow 0^+$, where ϵ and a are as in part (i). Then, we have*

$$\frac{1}{p(n)} x^n [P(x)G(x)] = a \log \left(\frac{\sqrt{24n-1}}{2\pi} \right) + O(n^{-1/2} + f(c/\sqrt{n} + O(n^{-1/2-\epsilon})))$$

with c given by (4).

As in [9], we remark that parts (i) and (ii) can be combined so that Lemma 2.2 generalizes to mixed asymptotic expansions involving sums of powers of t and logarithms of $1/t$. The proof of Lemma 2.2, based on the saddle point method, is presented in [9, Section 3].

3 Proofs

Proof. (of Theorem 1.1). First, we set $m = 1$ in Lemma 2.1(i). We have

$$\sum_{n=1}^{\infty} \mathbb{P}(M_n = 1) x^n = xP(x). \quad (13)$$

This implies at once (7). The asymptotic behavior of the quotient in (7) may be found using Rademacher's "exact-asymptotic" formula [14] (see also [2, Chapter 5]). It seems that a quicker

way is to apply the result of Lemma 2.2(i). Here we have $G(x) = x$, which obviously satisfies (11). Setting $x = e^{-t}$ and expanding e^{-t} by Taylor formula, we can write there as many powers of t as we wish, which will be transferred into powers of $n^{-1/2}$ in the asymptotic expansion of $\mathbb{P}(M_n = 1)$. We decide to bound the error of estimation up to a term of order $O(n^{-3/2})$ and write

$$e^{-t} = 1 - t + \frac{1}{2}t^2 + f(t), \quad (14)$$

with

$$f(t) = \sum_{j=3}^{\infty} \frac{t^j}{j!}. \quad (15)$$

The representation (14) requires to apply Lemma 2.2(i) twice: first, for the term $-t$ with $a = -1$ and $b = 1$, and second, for the term $\frac{1}{2}t^2$ with $a = 1/2$ and $b = 2$. Furthermore, (15) implies that $f(c/\sqrt{n} + O(n^{-1/2-\epsilon})) = O(n^{-3/2})$. Thus, from (13) it follows that

$$\mathbb{P}(M_n = 1) = \frac{x^n[xP(x)]}{p(n)} = 1 - A_1(n) + A_2(n) + O(n^{-3/2}). \quad (16)$$

The computation of $A_1(n)$ and $A_2(n)$ is based on Lemma 2.2(i), with s given by (12). We have

$$\begin{aligned} A_1(n) &= \frac{2\pi}{\sqrt{24n-1}} \frac{s}{s-1} \sum_{j=0}^2 \frac{(2+j)!}{j!(2-j)!} \left(-\frac{1}{2s}\right)^j \\ &= \frac{c}{\sqrt{n}} \left(1 - \frac{1}{24n}\right)^{-1/2} \left(1 - \frac{2}{s} + \frac{1}{s^2} + O(s^{-3})\right) O(e^{-2s}) \\ &= \frac{c}{\sqrt{n}} \left(1 + \frac{1}{48n} + O(n^{-2})\right) \left(1 - \frac{1}{c\sqrt{n}} \left(1 - \frac{1}{24n}\right)^{-1/2} + O(n^{-1})\right) \\ &= -\frac{c}{\sqrt{n}}(1 + O(n^{-1})) \left(1 - \frac{1}{c\sqrt{n}} + O(n^{-1})\right) = -\frac{c}{\sqrt{n}} - \frac{1}{n} + O(n^{-3/2}). \end{aligned} \quad (17)$$

In the same way, for $A_2(n)$, we obtain

$$A_2(n) = \frac{c^2}{2n} + O(n^{-3/2}). \quad (18)$$

The proof is completed by substituting (17) and (18) into (16). $\square$

Proof. (of Corollary 1.2). The total probability formula and the asymptotic estimate given in Theorem 1.1 imply that

$$\begin{aligned} \mathbb{P}\left(\frac{L_n M_n}{\sqrt{n}} - \frac{1}{2c} \log n \leq u\right) &= \mathbb{P}\left(\frac{L_n M_n}{\sqrt{n}} - \frac{1}{2c} \log n \leq u \mid M_n = 1\right) \mathbb{P}(M_n = 1) \\ &+ \mathbb{P}\left(\frac{L_n M_n}{\sqrt{n}} - \frac{1}{2c} \log n \leq u \mid M_n \neq 1\right) \mathbb{P}(M_n \neq 1) \\ &= \mathbb{P}\left(\frac{L_n}{\sqrt{n}} - \frac{1}{2} \log n \leq u\right) (1 + O(1/\sqrt{n})) + O(1/\sqrt{n}). \end{aligned}$$

Hence Corollary 1.2 follows easily from Erdős and Lehner's result (2). $\square$

Proof. (of Theorem 1.3). First, we represent the function $F(x)$ given by (10) as

$$F(x) = F_1(x) + F_2(x), \quad (19)$$

where

$$F_1(x) = \sum_{k=1}^{\infty} kx^k \prod_{j=k+1}^{\infty} (1 - x^j), \quad (20)$$

$$\begin{aligned} F_2(x) &= \sum_{k=1}^{\infty} k \left(\sum_{l=2}^{\infty} x^{kl} \right) \prod_{j=k+1}^{\infty} (1 - x^j) \\ &= \sum_{k=1}^{\infty} \frac{kx^{2k}}{1 - x^k} \prod_{j=k+1}^{\infty} (1 - x^j). \end{aligned} \quad (21)$$

Grabner and Knopfmacher [8, formula 6.2] found a simple alternative representation for the series expansion (20) of $F_1(x)$:

$$F_1(x) = \sum_{k=1}^{\infty} \frac{x^k}{1 - x^k}.$$

It is also known that

$$\sum_{n=1}^{\infty} p(n) \mathbb{E}(L_n) x^n = P(x) F_1(x)$$

(see, e.g., [9, p. 1059]). In addition, Grabner et al. [9, p. 1084] used Mellin transform technique to show that

$$F_1(e^{-t}) = \frac{\log(1/t) + \gamma}{t} + \frac{1}{4} - \frac{t}{144} + O(|t|^3), \quad t \rightarrow 0.$$

From this expansion and their main result (see also both parts of Lemma 2.2), they derived the asymptotic formula (5) for $\mathbb{E}(L_n)$. From (19), it follows that $x^n[F(x)] = x^n[F_1(x)] + x^n[F_2(x)]$, which in turn implies that

$$\mathbb{E}(L_n M_n) = \mathbb{E}(L_n) + \frac{x^n[F_2(x)]}{p(n)}. \quad (22)$$

The asymptotic analysis of the second summand in the right-hand side of (22) will be based on Lemma 2.2(ii). So, we need a suitable expansion for $F_2(e^{-t})$. We set $t = u + iv$, with u and v satisfying the conditions of Lemma 2.2(ii), and focus on an asymptotic estimate for $F_2(e^{-u})$ as $u \rightarrow 0^+$. For the sake of convenience, we also set

$$g_k(u) = \prod_{j=k+1}^{\infty} (1 - e^{-ju}), \quad u > 0. \quad (23)$$

Representing $g_k(u)$ as a Riemann sum with step size $u \rightarrow 0^+$ and replacing it by the corresponding integral, we obtain the following basic estimate:

$$g_k(u) = \exp\left(\sum_{j>k} \log(1 - e^{-ju})\right) = \exp\left(\frac{1}{u} \int_{ku}^{\infty} \log(1 - e^{-w}) dw + o(1)\right). \quad (24)$$

Next, we proceed with the representation:

$$F_2(e^{-u}) = \sum_{j=1}^4 S_j(u), \quad (25)$$

where, by (21) and (23)

$$S_j(u) = \sum_{k \in I_j} \frac{ke^{-2ku}}{1 - e^{-ku}} g_k(u), \quad j = 1, 2, 3, 4. \quad (26)$$

The subintervals I_j are defined as follows. Let $\alpha > 0$ be fixed. We set

$$I_1 = [1, \alpha/u], \quad (27)$$

$$I_2 = \left(\frac{\alpha}{u}, \frac{1}{u} \left(\log \frac{1}{u} - \log \log \left(\frac{1}{u} \right)^3 \right) \right], \quad (28)$$

$$I_3 = \left(\frac{1}{u} \left(\log \frac{1}{u} - \log \log \frac{1}{u} - \log 3 \right), \frac{1}{u} \left(\log \frac{1}{u} + \log \log \frac{1}{u} + \log 2 \right) \right], \quad (29)$$

$$I_4 = \left(\frac{1}{u} \left(\log \frac{1}{u} + \log \log \left(\frac{1}{u} \right) \right) + \log 2, \infty \right). \quad (30)$$

We start with an estimate for $S_1(u)$. Since, for $k \in I_1$,

$$\int_{ku}^{\infty} \log(1 - e^{-w}) dw \leq \int_{\alpha}^{\infty} \log(1 - e^{-w}) dw = -c_{\alpha} < 0,$$

from (24) we get

$$g_k(u) = O(e^{-c_{\alpha}/u}).$$

Hence, using (27), we obtain

$$\begin{aligned} S_1(u) &= O(e^{-c_{\alpha}/u}) \sum_{u \leq ku \leq \alpha} \frac{ke^{-2ku}}{1 - e^{-ku}} \\ &= O(u^{-2} e^{-c_{\alpha}/u}) \int_0^{\alpha} \frac{we^{-2w}}{1 - e^{-w}} dw = O(u^{-2} e^{-c_{\alpha}/u}). \end{aligned} \quad (31)$$

As a preparation for the estimate of $S_2(u)$, we first note that a single integration by parts in the integral from the right-hand side of (24) yields

$$\int_{ku}^{\infty} \log(1 - e^{-w}) dw = -ku \log(1 - e^{-ku}) - \int_{ku}^{\infty} \frac{w}{e^w - 1} dw.$$

On the other hand, from formula 27.1.2 in [1] we have

$$\begin{aligned} \int_y^{\infty} \frac{w}{e^w - 1} dw &= y \sum_{k=1}^{\infty} \frac{e^{-ky}}{k} + \sum_{k=1}^{\infty} \frac{e^{-ky}}{k^2} \\ &= -y \log(1 - e^{-y}) + e^{-y} + O(e^{-2y}), \quad y \rightarrow \infty. \end{aligned}$$

Applying this estimate to (24), we obtain

$$g_k(u) = \exp \left(-\frac{1}{u} e^{-ku} + O(u^{-1} e^{-2ku}) + o(1) \right) \quad (32)$$

for all k that satisfy $ku \rightarrow \infty$ as $u \rightarrow 0^+$. So, from (28) it follows that

$$\begin{aligned} S_2(u) &= \frac{1+o(1)}{u} \sum_{k \in I_2} \frac{k u e^{-2ku}}{1-e^{-ku}} \exp \left(-\frac{1}{u} e^{-ku} + O(u^{-2} e^{-2ku}) \right) \\ &= O \left(\frac{\log \frac{1}{u}}{u} \left(\sum_{k \in I_2} \frac{e^{-2\alpha}}{1-e^{-\alpha}} \right) \exp \left(-\log \left(\frac{1}{u} \right)^3 + O \left(\frac{1}{u} e^{-2 \log \frac{1}{u}} \right) \right) \right) \\ &= O \left(\frac{u^3 \log^2 \frac{1}{u}}{u^2} \right) = O \left(u \log^2 \frac{1}{u} \right), \quad u \rightarrow 0^+. \end{aligned} \quad (33)$$

(In the third line of (33) we have also used the fact that the sum over I_2 contains at most $\frac{1}{u} \log \frac{1}{u}$ summands.)

We proceed now to the estimate for $S_3(u)$, whose contribution to $F_2(e^{-u})$ is the most essential one. First, it is easy to observe that

$$0 \leq \frac{1}{1-e^{-ku}} = O \left(u \log \frac{1}{u} \right), \quad u \rightarrow 0^+$$

uniformly for all $k \in I_3$. Hence, approximating once again a Riemann sum by the corresponding integral, we obtain

$$\begin{aligned} S_3(u) &= \left(1 + O \left(u \log \frac{1}{u} \right) \right) u^{-2} \int_{\log \frac{1}{u} - \log \log \frac{1}{u} - \log 3}^{\log \frac{1}{u} + \log \log \frac{1}{u} + \log 2} w e^{-2w} e^{-\frac{1}{u} e^{-w}} dw \\ &= \left(1 + O \left(u \log \frac{1}{u} \right) \right) \int_{-\log \log \frac{1}{u} - \log 3}^{\log \log \frac{1}{u} + \log 2} \left(z + \log \frac{1}{u} \right) e^{-2z} e^{-e^{-z}} dz. \end{aligned} \quad (34)$$

Changing the variable in (34), we represent $S_3(u)$ as follows:

$$S_3(u) = (J_1(u) + J_2(u)) \left(1 + O \left(u \log \frac{1}{u} \right) \right), \quad (35)$$

where

$$J_1(u) = \int_{2/\log \frac{1}{u}}^{3 \log \frac{1}{u}} (-\log y) y e^{-y} dy, \quad (36)$$

$$J_2(u) = \left(\log \frac{1}{u} \right) \int_{2/\log \frac{1}{u}}^{3 \log \frac{1}{u}} y e^{-y} dy. \quad (37)$$

An easy calculation based on an asymptotic estimate for the incomplete gamma function [1, formula 6.5.32] shows that

$$\begin{aligned} \int_{2/\log \frac{1}{u}}^{3\log \frac{1}{u}} ye^{-y} dy &= \int_0^\infty ye^{-y} dy - \int_0^{2/\log \frac{1}{u}} ye^{-y} dy - \int_{3\log \frac{1}{u}}^\infty ye^{-y} dy \\ &= 1 - O\left(\left(\log \frac{1}{u}\right)^{-2}\right) - O\left(\left(\log \frac{1}{u}\right) e^{-3\log \frac{1}{u}}\right) = 1 - O\left(\left(\log \frac{1}{u}\right)^{-2}\right). \end{aligned}$$

Inserting this estimate into (37), we obtain

$$J_2(u) = \log \frac{1}{u} + O\left(1/\log \frac{1}{u}\right). \quad (38)$$

For the estimate of $J_1(u)$, we recall that

$$\int_0^\infty ye^{-y}(\log y) dy = 1 - \gamma$$

(see, e.g., [4, formula 865.902]). Hence we have

$$J_1(u) = \gamma - 1 - \int_0^{2/\log \frac{1}{u}} ye^{-y}(-\log y) dy - \int_{3\log \frac{1}{u}}^\infty ye^{-y}(-\log y) dy.$$

Both integrals in the right-hand side of this representation are negligible. The first one can be estimated bounding e^{-y} by 1 and then integrating by parts. This gives a bound of order $O(\log \log \frac{1}{u} / \log^2 \frac{1}{u})$. For the second one we can use once again [1, formula 6.5.32], which leads to the upper bound $O(u^3 \log^2 \frac{1}{u})$. Thus we have

$$J_1(u) = \gamma - 1 + O\left(\left(\log \log \frac{1}{u}\right) / \log^2 \frac{1}{u}\right). \quad (39)$$

Combining (35) - (39), we finally obtain

$$S_3(u) = \log \frac{1}{u} + \gamma - 1 + O\left(\frac{1}{\log \frac{1}{u}}\right). \quad (40)$$

We end up our analysis with an estimate for $S_4(u)$. First, we bound the last exponent in the right-hand side of (32) by 1 and then, as we did previously, we approximate $S_4(u)$ by an integral, which can be bounded using again [1, formula 6.5.32]. Then, in a similar way, from (30) we obtain

$$\begin{aligned} S_4(u) &= O\left(u^{-2} \int_{\log \frac{1}{u} + \log \log \frac{1}{u} + \log 2}^\infty we^{-2w} dw\right) \\ &= O\left(\int_{\log \log \frac{1}{u} + \log 2}^\infty \left(z + \log \frac{1}{u}\right) e^{-2z} dz\right) \\ &= O\left(\left(\log \log \frac{1}{u}\right) / \log^2 \frac{1}{u}\right) + O\left(1/\log \frac{1}{u}\right) = O\left(1/\log \frac{1}{u}\right). \end{aligned} \quad (41)$$

Now, from (25), (31), (33), (40) and (41) it follows that

$$\begin{aligned} F_2(e^{-u}) &= O(u^{-2}e^{-c_\alpha/u}) + O\left(u^2 \log \frac{1}{u}\right) + \log \frac{1}{u} + \gamma - 1 + O\left(1/\log \frac{1}{u}\right) \\ &= \log \frac{1}{u} + \gamma - 1 + O\left(1/\log \frac{1}{u}\right). \end{aligned} \quad (42)$$

To transfer the variable u in (42) into $t = u + iv$, we recall the relationship between u and v from Lemma 2.2(ii). First, we note that

$$\log \frac{1}{t} = \log \frac{1}{u} + \log \frac{1}{1 + iv/u} = \log \frac{1}{u} + O(|v|/u) = \log \frac{1}{u} + O(u^\epsilon) \quad (43)$$

and

$$e^{-t} = e^{-u}e^{-iv} = e^{-u}(1 + O(|v|)) = e^{-u} + O(u^{1+\epsilon}) \quad (44)$$

as $u \rightarrow 0^+$. Hence, by Taylor formula,

$$F_2(e^{-t}) = F_2(e^{-u}) + O\left(u^{1+\epsilon} \left| \frac{d}{du} F_2(e^{-u}) \right| \right). \quad (45)$$

The last error term can be estimated following the same line of reasoning. We outline briefly the proof. By (21) and (23) we have

$$\begin{aligned} \frac{d}{du} F_2(e^{-u}) &= - \sum_{k=1}^{\infty} \frac{k^2 e^{-ku}}{1 - e^{-ku}} g_k(u) - \sum_{k=1}^{\infty} \frac{k^2 e^{-ku}}{(1 - e^{-ku})^2} g_k(u) \\ &\quad - \sum_{k=1}^{\infty} \frac{k e^{-ku}}{1 - e^{-ku}} g'_k(u), \end{aligned} \quad (46)$$

where

$$g'_k(u) = g_k(u) \sum_{j=k+1}^{\infty} \frac{j e^{-ju}}{1 - e^{-ju}}.$$

The main contribution to the asymptotic of $\left| \frac{d}{du} F_2(e^{-u}) \right|$ as $u \rightarrow 0^+$ is given by the first term of the right-hand side of (46) (since it contains the factor k^2). If we break up the range of summation (as we previously did) into the union $\bigcup_{j=1}^4 I_j$ (see (27) - (30)), we can again conclude that the contribution of the sum over I_3 is the largest one. An approximation by a Riemann integral requires division and multiplication by the cube of the step size, u^3 . Hence, instead of u^{-2} , the factor u^{-3} will multiply the same integral from the right-hand side of (34). An argument similar to (40) and (42) implies the following bound:

$$\frac{d}{du} F_2(e^{-u}) = O\left(\frac{1}{u} \log \frac{1}{u}\right), \quad u \rightarrow 0^+.$$

Consequently, the remainder term in (45) is $O(u^\epsilon \log \frac{1}{u})$. Combining this with (42) - (46), we obtain

$$F_2(e^{-t}) = \log \frac{1}{t} + \gamma - 1 + O\left(1/\log \frac{1}{t}\right), \quad t \rightarrow 0.$$

Therefore, we are ready to apply Lemma 2.2(ii) with $G(x) := F_2(x)$ and $f(t) := 1/\log \frac{1}{t}$. We obtain

$$\frac{1}{p(n)}x^n[F_2(x)] = \log \frac{c}{\sqrt{n}} + \gamma - 1 + O(1/\log n), \quad n \rightarrow \infty. \quad (47)$$

Now, we recall (22), which in combination with (47) completes the proof of Theorem 1.3. $\square$

4 Concluding remarks

The main goal of this study is the comparison between the typical growths of the first block area $L_n M_n$ and its base length L_n in the Ferrers diagram of a random integer partition of n . It turns out that the leading terms in the asymptotic expansions of the expectations of these two statistics are the same for large n : both are equal to $\frac{\sqrt{n}}{2c} \log n$. Erdős and Lehner's limit theorem (2) and Theorem 1.1 show that this leading term controls the weak convergence of L_n and $L_n M_n$. After one and the same normalization, both statistics tend in distribution to a Gumbel distributed random variable. The expectations of L_n and $L_n M_n$ are, however, different for large n . In fact, by Theorem 1.3

$$\lim_{n \rightarrow \infty} \left(\mathbb{E}(L_n M_n) - \mathbb{E}(L_n) - \frac{1}{2} \log n \right) = -C = -0.67165....$$

This phenomenon suggests a question related to the shape of a random Ferrers diagram of n . To state the problem in a more precise way, we denote by $X_n^{(k)}$ the multiplicity of part k ($k = 1, \dots, n$) in a random integer partition of n . Let $Z_n^{(r)}$ be the r th largest member of the sequence $\{k X_n^{(k)}\}_{k=1}^n$. Erdős and Szalay [6] showed that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{c}{\sqrt{n}} Z_n^{(1)} - \frac{1}{2} \log \frac{n}{c^2} - \log \log \log n \leq u \right) = e^{-e^{-u}}, \quad -\infty < u < \infty.$$

Fristedt [7, Theorem 2.7] has also studied this kind of rearrangements of the Ferrers diagrams and generalized Erdős and Szalay's result to $Z_n^{(r)}$, where $r \geq 1$ is fixed. He showed that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{c}{\sqrt{n}} Z_n^{(r)} - \frac{1}{2} \log \frac{n}{c^2} - \log \log \log n \leq u \right) \\ &= \int_{-\infty}^u \frac{\exp(-e^{-w} - rw)}{(r-1)!} dw, \quad -\infty < u < \infty. \end{aligned} \quad (48)$$

So, it might be interesting to determine the typical position of $L_n M_n$ among all ordered areas $Z_n^{(r)}$. If R_n denotes the smallest value of r such that $Z_n^{(r)} = L_n M_n$, we conjecture that

$$\mathbb{E}(R_n) \asymp \log \log n, \quad n \rightarrow \infty. \quad (49)$$

This claim is supported by the following non-rigorous argument. From the result of Theorem 1.3 it follows that

$$\mathbb{E}(L_n M_n) = \frac{\sqrt{n}}{2c} \log n + O(\sqrt{n}), \quad n \rightarrow \infty. \quad (50)$$

A calculation of the expectation of the distribution in the right-hand side of (48) demonstrated in [7, p. 708] shows that if $r = r(n) \rightarrow \infty$ as $n \rightarrow \infty$, then

$$\mathbb{E}(Z_n^{(r)}) = \frac{\sqrt{n}}{2c} (\log n + 2 \log \log \log n - 2 \log r(n)) + O(\sqrt{n}). \quad (51)$$

Combining (50) with (51), one may conclude that $\log r(n) \approx \log \log \log n$, which supports the claim in (49). We hope to return to this question in a future study.

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