Oceanological and Hydrobiological Studies

International Journal of Oceanography and Hydrobiology

Volume 44, Issue 4, December 2015

pages (563-568)

ISSN 1730-413X

eISSN 1897-3191

A note on the vertical distribution of momentum transport in water waves[a]

by

Jerzy Kołodko[1], Gabriela Gic-Grusza[2,*]

DOI: **10.1515/ohs-2015-0053**
Category: **Short communication**
Received: **May 16, 2014**
Accepted: **June 18, 2015**

[1]*Institute of Hydroengineering PAS (IBW PAN), ul. Kościerska 7, 80–328 Gdańsk, Poland*

[2]*Institute of Oceanography, Faculty of Oceanography and Geography, University of Gdańsk, Al. M. Piłsudskiego 46, 81–378 Gdynia, Poland*

* Corresponding author: *ocegg@ug.edu.pl*

[a] Paper presented at the PROZA Conference "*Modelling of hydrodynamic phenomena in the Baltic Sea — PROZA project in view of modern research*", Gdynia, Poland, 16th October 2012. The project PROZA, entitled „*Operational decision-making based on atmospheric conditions*" is a part of the Polish Innovative Economy Programme.

Abstract

In this paper, the classical problem of horizontal wave-induced momentum transport is analyzed once again. A new analytical approach has been employed to reveal the vertical variation of this transport in the Eulerian description.

In mathematical terms, this variation is shown to have (after "smoothing out" the surface corrugation) the character of a generalized function (distribution) and is described by a classical function in the water depths and by an additional Dirac-delta-function component on the averaged free surface.

In terms of physics, the considered variation consists of two entities: (i) a continuous distribution of the mean momentum transport flux density (tensorial radiation pressure) over the entire water column, and (ii) an additional momentum transport flux concentrated on the mean free surface level (tensorial radiation surface pressure). Simple analytical formulae describing this variation have been derived.

This allowed a conventional expression to be derived, describing the depth-integrated excess of horizontal momentum flux due to the presence of waves (the so-called "radiation stress"), confirming to some extent the correctness of the whole analysis carried out.

The results obtained may be important to the ocean dynamics, especially in view of their possible application in the field of hydrodynamics of wave-dominated coastal zones.

Key words: radiation stress, coastal zone, wave-induced currents, momentum transport, three-dimensional modeling

DE GRUYTER

The *Oceanological and Hydrobiological Studies* is online at oandhs.ocean.ug.edu.pl

Introduction

The proper description of many problems related to the ocean dynamics (in particular those connected with wave-dominated coastal zones) requires reliable information on the forces responsible for the generation of wave-induced currents.

The first step in this direction has been done by Longuet-Higgins and Stewart in a series of papers published in the early sixties (1960, 1962, 1964). It was the famous idea of "radiation stress" – a quantity defined as a depth-integrated excess flow of momentum due to the presence of waves. Despite the fact that even the name "radiation stress" seems not to be very accurate (since the dimension of this quantity is that of the surface tension and the name "wave thrust" proposed by Lundgren (1962, 1963) could be more suitable from the physical as well as the engineering point of view), its mathematical expression has been effectively used to calculate the mean horizontally two-dimensional (vertically integrated) translatory water motions (currents) in wave-dominated coastal zones. It has also been used to develop some theoretical models of such phenomena as wave set-up, rip currents, wave-current interactions, infragravity waves and many others.

Despite its undisputable importance, the classical concept of radiation stress is known to have a rather fundamental limitation: only two-dimensional coastal-zone models can be constructed with its use. For three-dimensional modelling of coastal-zone hydrodynamics, some knowledge concerning the vertical distribution of the wave driving forces is required.

Stive and Wind (1986) are known to have been the first who succeeded (at least partly) in this challenge. In their pioneer paper, they presented a solution to this problem restricted to the cross-shore direction. Despite its limitations, this accomplishment resulted – starting from the paper of De Vriend and Stive (1987) – in numerous contributions devoted to the so-called quasi-three-dimensional models of wave-induced currents in coastal zones.

In recent years, a number of attempts to solve the fully 3-dimensional problem have been made (see e.g. Deigaard 1993; Nobuoka et al. 1998; Mellor 2003, 2011, 2013; Lin et al. 2004; Webb et al. 2004; Xia et al. 2004; Herman 2006). However, none of them has the simplicity and compactness achieved by using the generalized functions. In some of them (e.g. in those of Mellor), the so-called sigma coordinate system is used, which makes any comparison very difficult. Therefore (and because of their enormous complexity), a more extensive analysis of these attempts seems to be beyond the scope of the present paper. As the only exception, the interesting attempt of Herman (2006) connected with some kind of non-Airy waves should be noticed.

Our description of the processes taking place in the region of space above the wave troughs and below the crests is thus carried out in a very simplified manner, i.e. using the Dirac-delta function. From the perspective of three-dimensional coastal-zone modelling, such a method seems to be most suitable. It is, however, by no means the only way. For instance, the structure of the Eulerian wave-induced mass transport is known in this region to be describable by the use of a continuous, tongue-like convexity (see e.g. Nielsen 1992; Cieślikiewicz and Gudmestad 1993, 1994).

The analysis presented below is based upon the following additional simplifying assumptions:

- The waves are assumed to be monochromatic, unidirectional, of a small, constant amplitude a, and moving over a very gently inclined bottom (mild-slope approximation) of the local depth D (the so-called Airy waves). Neither viscosity effects nor capillary forces are taken into account.
- The problem is treated in terms of the second-order interaction theory, i.e. all resulting quantities of the order higher than $O(a^2)$ are neglected.
- Only so-called wave-dominated coastal zones are taken under consideration, which means that the mean free-surface elevation as well as the mean current velocities are assumed to be of $O(a^2)$ order.
- All short-time averages, denoted by $<...>$, are understood to be taken over the wave period; all other averaged quantities (in particular the mean free-surface level) are assumed to change over time intervals much longer than the wave period and thus treated as practically stationary.
- The whole analysis is carried out in a rectangular, three-dimensional, right-handed coordinate system (x, y, z) where the x-coordinate is directed offshore, the y-coordinate – alongshore and the

DE GRUYTER

z-coordinate – vertically upwards (with $z = 0$ denoting the mean water level). Partial derivatives are written in an abbreviated form: $\partial/\partial t = \partial_t$, $\partial/\partial x = \partial_x$, $\partial/\partial y = \partial_y$, $\partial/\partial z = \partial_z$. Moreover, $\nabla = (\partial_x, \partial_y)$ is the horizontal nabla operator.

The momentum flux density is known to have a tensorial character. It appears as a rule in all momentum-transport equations when they are written in the so-called divergence form. In the case of an inviscid fluid, it is usually understood as the following 3x3 tensor (cf. e.g. Krauss 1973):

$$\mathbf{M} = p\,\mathbf{I} + \rho\,\mathbf{v}\,\mathbf{v}\,, \qquad (1)$$

where p denotes the local pressure, $\mathbf{I}$ – a 3x3 unit tensor, ρ – the water density, and $\mathbf{v} = (u, v, w)$ – the 3-component velocity vector.

In the absence of waves ($\mathbf{v} = \mathbf{0}$), the pressure is purely hydrostatic (provided the mean free-surface level remains unchanged): $p = -\rho g z$ (g – gravity acceleration). Therefore, the momentum flux density in this case reads:

$$\mathbf{M}_0 = -\rho g z \mathbf{I}\,. \qquad (2)$$

The difference between these two quantities:

$$\Delta\mathbf{M} = \mathbf{M} - \mathbf{M}_0 = (p + \rho g z)\mathbf{I} + \rho\,\mathbf{v}\,\mathbf{v} \qquad (3)$$

determines the instantaneous excess of the momentum flux density due to the presence of waves.

From the point of view of larger-scale ocean dynamics (wave-dominated coastal zones in particular), the most important is the mean value of $\Delta\mathbf{M}$ (with the wave motion regarded as some kind of turbulence). This may be obtained by means of the phase averaging procedure. However, in the case of Airy waves: $<uw> = <wu> = <vw> = <wv> = 0$. Therefore:

$$<\Delta\mathbf{M}> = (<p> + \rho g z)\mathbf{I} + \rho(<\mathbf{u}\,\mathbf{u}> + <w^2>\mathbf{n}\,\mathbf{n}), \qquad (4)$$

where $\mathbf{u} = (u, v, 0)$ denotes the horizontal part of the orbital velocity, $\mathbf{n} = (0, 0, 1)$ – the vertical unit vector. For simplicity, this quantity may be split into the following two parts:

(1) the horizontal one:

$$\boldsymbol{\Sigma} = (<p> + \rho g z)\boldsymbol{I} + \rho<\boldsymbol{u}\,\boldsymbol{u}>, \qquad (5)$$

where $\boldsymbol{u} = (u, v)$ denotes the horizontal 2-component orbital velocity vector, $\boldsymbol{I}$ – the 2x2-unit tensor, and

(2) its vertical counterpart $\boldsymbol{T} = T\,\mathbf{n}\,\mathbf{n}$, where:

$$T = <p> + \rho g z + \rho<w^2>. \qquad (6)$$

The orbital velocities are assumed to be known, however the mean pressure $<p>$ is a priori unknown as well as the way of determining the influence of free-surface averaging.

There are, therefore, the following two problems to be solved: (1) the problem of flow averaging inside the water column, and (2) the problem of free-surface averaging. In both cases, different problems of determining the pressure appear. It was, therefore, decided to analyze the problem of wave-induced momentum transport separately in two regions:

- the region fairly far from the mean free surface (the internal transport), and
- the region close to the mean free surface (the surface transport).

The internal transport

The first problem appears to be easy to solve with the use of the equation describing the mean vertical momentum transport. Because of the negligible influence of forces associated with turbulence and molecular viscosity (the main forces in this direction are due to gravity), this eq. may be written in the following simplified form:

$$\partial_z T = 0. \qquad (7)$$

After simple integration T = const. In other words, the vertical momentum transport is constant over the entire water column. This statement is also valid in the air, but since its density is very small (in comparison to that of water) and the air pressure is assumed to be zero, thus const. = 0 (physically it means that at the mean free surface, the air pressure is in equilibrium with the vertical wave-induced "Reynolds stresses"). The following simple equality may be, therefore, finally written:

$$<p> = -\rho g z - \rho<w^2>. \qquad (8)$$

As a consequence, the mean pressure under the waves is known to be somewhat less than the hydrostatic pressure in the absence of waves (cf. e.g. Krauss 1973).

Eventually, the mean internal excess of the horizontal momentum flux density, due to the presence of waves, may be written in the following form:

$$\boldsymbol{\Sigma} = \rho\langle \boldsymbol{uu} - w^2\boldsymbol{I}\rangle. \quad (9)$$

Both components of its right-hand side resemble the well-known turbulent "Reynolds stresses", although the waves are assumed to be deterministic. In the case of Airy waves:

$$\boldsymbol{\Sigma} = (2\beta - 1)(E/D)[\boldsymbol{I} - (\boldsymbol{I} - \boldsymbol{k}^0\boldsymbol{k}^0)\cosh^2 k(z + D)], \quad (10)$$

where $\beta = c_g/c = ½(1 + kD\sinh^{-1}2kD)$ (c_g – group velocity, c – phase velocity, both in the sense of absolute values), $E = ½\rho g a^2$ denotes the local wave energy density (per unit horizontal area), $\boldsymbol{k}^0 = \boldsymbol{k}/k$ – the unit vector in the direction of wave propagation ($\boldsymbol{k}$ – wave vector, $k = |\boldsymbol{k}|$ – its modulus).

In the case of deep-water waves $\beta = ½$ and consequently $\boldsymbol{\Sigma} = 0$ (there is no excess of mean internal in this case; the whole horizontal momentum flux is concentrated on the mean free surface!).

In the case of shallow water waves, this does not apply. It seems, however, worth noting that the mean internal excess of the horizontal momentum flux density in the direction of wave propagation:

$$\boldsymbol{k}^0 \cdot \boldsymbol{\Sigma} = (2\beta - 1)(E/D)\,\boldsymbol{k}^0 \quad (11)$$

is always non-negative and z-independent in this case.

The surface transport

To determine this transport, the wavy surface must be averaged. The only reasonable way seems to be the integration of the transport considered between the instantaneous level ($z = \zeta$) and the mean one ($z = 0$). For this reason, the pressure fluctuations in the vicinity of the free surface should be determined first. They are ζ-induced and therefore certainly O(a). Any higher-order components are therefore not needed and simple linearized equations may be used.

As the first of such equations, the following linearized Euler equation describing the instantaneous excess of the mean vertical momentum flux in the region considered is applied (the influence of turbulence and molecular viscosity is neglected again):

$$\partial_t w + \partial_z(p + \rho g z) = 0. \quad (12)$$

However $-a \le \zeta \le +a$ and after vertical integration from 0 to ζ, the first term will be O(a^2), so it may also be omitted, since the remaining terms will be O(a). Finally, the following simplified vertical momentum transport equation:

$$\partial_z(p + \rho g z) = 0 \quad (13)$$

with the boundary condition:

$$z = \zeta: \; p = 0 \quad (14)$$

may be used to obtain the desired pressure distribution:

$$p = \rho g(\zeta - z). \quad (15)$$

As the second one, a linearized version of relation (1) describing the instantaneous momentum flux, due to the presence of waves, is used. Neglecting the last (nonlinear) term of this relation and changing the 3x3 unit tensor $\mathbf{I}$ into its 2x2 counterpart $\boldsymbol{I}$, the following relation describing the linearized instantaneous horizontal momentum flux density, due to the presence of waves (close to the mean free surface), may be obtained:

$$\boldsymbol{M} = p\,\boldsymbol{I} = \rho g(\zeta - z)\,\boldsymbol{I}. \quad (16)$$

Now the surface corrugation may be completely "smoothed out" by: (i) integrating the expression (16) between 0 and ζ, (ii) averaging the result of it, and (iii) making use of the well-known Dirac-delta function $\delta(z)$. The final result may be written as follows (with the last equality valid for Airy waves only):

$$\left\langle \int_0^{\zeta} \boldsymbol{M}\,\mathrm{d}z \right\rangle = ½\rho g\langle\zeta^2\rangle\,\boldsymbol{I} = \int_{0^-}^{0^+} \boldsymbol{\sigma}\delta(z)\mathrm{d}z = \boldsymbol{\sigma} = ½E\boldsymbol{I}, \quad (17)$$

where 0^-, 0^+ means obviously $0 - \varepsilon$ and $0 + \varepsilon$, respectively, with $\varepsilon > 0$ tending to 0. Relation (17) describes effectively the mean, horizontal, momentum transport connected with the mean free surface ($z = 0$). Since the respective momentum transport in the absence of waves is equal to zero, this relation describes at the same time the excess of the horizontal wave-induced momentum flux in this place.

The effective, wave-induced shear stress acting on the mean free surface may be thus written as follows:

$$\boldsymbol{\tau} = -\nabla\cdot\boldsymbol{\sigma} = -\tfrac{1}{2}\nabla E. \tag{18}$$

This formula represents the generalized form of the well-known one-direction expression for non-breaking waves ($\tau_x = -\tfrac{1}{2}\partial_x E$) given by Stive and Wind (1986).

The resulting transport

The resulting momentum transport density may now be written in the following, very elegant form of the generalized function:

$$\{\boldsymbol{\Sigma}\} = \boldsymbol{\Sigma} + \boldsymbol{\sigma}\,\delta(z). \tag{19}$$

The expression for the classical "radiation stress" tensor $\boldsymbol{S}$ may be easily obtained by integrating (19):

$$\boldsymbol{S} = \int_{-D}^{0^+} \{\boldsymbol{\Sigma}\}\mathrm{d}z = \int_{-D}^{0^-} \boldsymbol{\Sigma}\mathrm{d}z + \int_{0^-}^{0^+} \boldsymbol{\sigma}\,\delta(z)\mathrm{d}z\,, \tag{20}$$

or shortly:

$$\boldsymbol{S} = \int_{-D}^{0} \boldsymbol{\Sigma}\mathrm{d}z + \boldsymbol{\sigma}.$$

In the case of Airy waves, the following, very compact formula may be obtained:

$$\boldsymbol{S} = E[\beta\,\boldsymbol{k}^0\boldsymbol{k}^0 + (\beta - \tfrac{1}{2})\boldsymbol{I}]. \tag{21}$$

This result turned out to be consistent with the equivalent expressions given by other authors (cf. e.g. Phillips 1977, Mei 1989, Dingemans 1997), proving, to some extent, the present analysis to be correct.

Conclusions

By making use of the well-known Dirac-delta function (and, in fact, of the powerful apparatus of generalized functions), some very simple and compact formulae describing the vertical variation of the mean value of the excess of the horizontal momentum flux, due to the presence of waves, have been obtained (at least for the case of Airy waves). In such a way, a detailed (and, at the same time, consistent) picture of forces, with which the waves act on the currents, could be presented.

The results obtained in this work are expected to be important to the ocean dynamics. Their special application field seems, however, to be the hydrodynamics of wave-dominated coastal zones, since they help to develop the complete 3-dimensional mathematical models of wave-induced currents in such zones (for an attempt in this direction see Grusza 2007).

Further research should certainly be devoted to the investigation of numerous, still unsolved issues of 3-dimensional coastal-zone phenomena, and to non-Airy waves, for instance – waves over uneven bottoms (see e.g. Herman 2006).

Acknowledgements

This research has been partly supported by the Polish National Centre for Research and Development as part of the project "Development of a predictive model of morphodynamic changes in the coastal zone" based on the decision No. DEC-2011/01/D/ST10/07668 and by the European Regional Development Fund, through Innovative Economy grant POIG.01.03.01-00-140/08.

References

Cieślikiewicz W. & Gudmestad T. (1993). Stochastic characteristics of orbital velocities of random water waves. *J. Fluid Mech.* 255: 275-299.

Cieślikiewicz W. & Gudmestad O. T. (1994). Random water wave kinematics. *Archives of Hydro-Engineering and Environmental Mechanics*, 41 (1-2), Part 1. Theory: 3-35; Part 2. Experiment: 37-85.

Deigaard R. (1993). A note on the three-dimensional shear stress distribution in a surf zone, *Coastal Engng.* 20: 157–171.

De Vriend H. J. & Stive M. J. F. (1987). Quasi-3D modelling of nearshore currents. *Coastal Engng.* 11: 565–601.

Dingemans M. A. (1997). *Water wave propagation over uneven bottoms. Part I — Linear wave propagation*. Singapore: World Scientific, 471 pp.

Grusza G. (2007). *Three-dimensional modelling of wave-generated currents in coastal zone.* Ph.D. Diss. (in Polish), Inst. of Oceanogr., University of Gdańsk, Gdynia, Poland, 97 pp.

Herman A. (2006). Three-dimensional structure of wave-induced momentum flux in irrotational waves in combined shoaling-refraction conditions. *Coastal Engng.* 53: 545–555.

Krauss W. (1973). *Dynamics of the homogeneous and the quasihomogeneous ocean*. Methods and Results of Theoretical Oceanography, vol. 1. Berlin: Gebr. Borntraeger.

Lin P. & Zhang D. (2004). The Depth-Dependent Radiation

Stresses and Their Effect on Coastal Currents. The 6 th International Conference on Hydrodynamics, 23-27, November 2004 , West Leederville , Western Australia.

Longuet-Higgins M. S. & Stewart R. W. (1960). Changes in the form of short gravity waves on long waves and tidal currents. *J. Fluid Mech.* 8: 565–583.

Longuet-Higgins M. S. & Stewart R. W. (1962). Radiation stress and mass transport in gravity waves with application to „surf beats". *J. Fluid Mech.* 13: 481–504.

Longuet-Higgins M. S. & Stewart R. W. (1964). Radiation stresses in water waves; a physical discussion with applications *Deep Sea Res.* 11: 529–562.

Lundgren H. (1962). *The concept of the wave thrust.* Coastal Engng. Lab., Tech. Univ. Denmark, Basic Res. - Progr. Rep. 3: 1-5.

Lundgren H. (1963). Wave thrust and energy level. Proc. 10th Congr. Int. Assoc. Hydr. Res., London, Paper 1.20, 1: 147–151.

Mei C. C. (1989). *The Applied Dynamics of Ocean Surface Waves*. Singapore: World Scientific, 740 pp.

Mellor G. L. (2003). The three-dimensional current and surface wave equations. *J. Phys. Oceanogr.*, 33: 1978-1989.

Mellor G. L. (2011). Wave radiation stress. *Ocean Dyn.*, 61 (5), pp. 563–568. DOI: 10.1007/s10236-010-0359-2.

Mellor G. L. (2013). Waves, circulation and vertical dependence. *Ocean Dynamics*, 63, 447–457.

Nielsen P. (1992). *Coastal bottom boundary layers and sediment transport.* Advanced Series in Ocean Engineering, vol. 4, World Scientific Publishing Co., Singapore.

Nobuoka H., Mimura N. & Kato H. (1998). Three-dimensional nearshore currents model based on vertical distribution of radiation stress. Proceedings of the International Conference on Coastal Engineering, ASCE, 1: 829-842.

Phillips O. M. (1977). *The dynamics of the upper ocean.* Cambridge University Press, Cambridge e.a., 336 pp.

Stive M. J. F. & Wind H. G. (1986). Cross-shore mean flow in the surf zone. *Coastal Engng.* 10: 325–340.

Webb B. M. & Slinn D. N. (2004). Vertical distribution of radiation stress for non-linear shoaling waves. AGU Fall Meeting.

Xia H., Xia Z. & Zhu L. (2004). Vertical variation in radiation stress and wave-induced current. *Coastal Engng.* 51: 309–321.