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Extension of complete monotonicity results involving the digamma function

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Abstract. By using some analytical techniques, we prove a complete monotonicity property of a certain function involving the (p, k)-digamma function. Subsequently, we derive some inequalities for the (p, k)-digamma function. As special cases of the established results, we deduce some new results concerning the p-digamma and the k-digamma functions. Our results are extensions of some previous results due to Qiu and Vuorinen, Mortici, and Merovci.
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1. Introduction

The digamma function, which is also known in some texts as the psi function is defined as follows (see [1, p. 258-259], [22, p. 139-140]).

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = -\gamma + \int_0^\infty \frac{e^{-u} - e^{-zu}}{1 - e^{-u}}\, du, \quad z > 0,$$
$$= -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \frac{z}{k(k+z)}, \quad z > 0,$$

where, $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right)$ is the Euler-Mascheroni's constant and

$$\Gamma(z) = \int_0^\infty u^{z-1} e^{-u}\, du,$$

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is the well-known gamma function. The digamma function provides a power tool for computing infinite rational series. Aside this, it also plays an important role in the theory of special functions and has applications in statistics and mathematical physics.

It is well known that the integral

$$\frac{s!}{z^{s+1}} = \int_0^\infty u^s e^{-zu}\, du, \tag{1.1}$$

holds for $z > 0$ and $s \in \mathbb{N}_0$. See for instance [1, p. 255].

The (p,k)-digamma function, which is a two parameter deformation of the ordinary digamma function is defined for $p \in \mathbb{N}$, $k > 0$ as [20]

$$\psi_{p,k}(z) = \frac{d}{dz}\ln\Gamma_{p,k}(z) = \frac{1}{k}\ln(pk) - \int_0^\infty \frac{1-e^{-k(p+1)u}}{1-e^{-ku}} e^{-zu}\, du, \quad z > 0, \tag{1.2}$$

$$= \frac{1}{k}\ln(pk) - \sum_{n=0}^{p} \frac{1}{nk+z}, \quad z > 0, \tag{1.3}$$

where, $\Gamma_{p,k}(z)$ is the (p,k)-gamma function defined as [20]

$$\Gamma_{p,k}(z) = \frac{(p+1)!k^{p+1}(pk)^{\frac{z}{k}-1}}{z(z+k)(z+2k)\dots(z+pk)}, \quad z > 0.$$

The function $\psi_{p,k}(z)$ satisfies the properties

$$\psi_{p,k}(z+k) - \psi_{p,k}(z) = \frac{1}{z} - \frac{1}{z+pk+k} \tag{1.4}$$

$$\psi_{p,k}(k) = \frac{1}{k}\left[\ln(pk) - H(p+1)\right],$$

where, $H(r)$ is the r-th harmonic number. It also satisfies the limit relations [20]

$$\begin{array}{ccc} \psi_{p,k}(z) & \xrightarrow{p\to\infty} & \psi_k(z) \\ {\scriptstyle k\to 1}\downarrow & & \downarrow{\scriptstyle k\to 1} \\ \psi_p(z) & \xrightarrow[p\to\infty]{} & \psi(z) \end{array}$$

where, $\psi_p(z)$ is the p-digamma function and $\psi_k(z)$ is the k-digamma functions defined as follows (see [9], [4], [19], [21], [24]).

$$\psi_p(z) = \ln p - \sum_{n=0}^{p} \frac{1}{n+z}, \quad z > 0$$

$$= \ln p - \int_0^\infty \frac{1-e^{-(p+1)u}}{1-e^{-u}} e^{-zu}\, du, \quad z > 0,$$

$$\psi_k(z) = \frac{\ln k - \gamma}{k} - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{nk(nk+z)}, \quad z > 0,$$

$$= \frac{\ln k - \gamma}{k} + \sum_{n=0}^{\infty}\left(\frac{1}{nk+k} - \frac{1}{nk+z}\right), \quad z > 0,$$

$$= \frac{\ln k - \gamma}{k} + \int_0^\infty \frac{e^{-ku} - e^{-zu}}{1-e^{-ku}}\, du, \quad z > 0,$$

$$= \frac{\ln k - \gamma}{k} + \int_0^1 \frac{u^{k-1} - u^{z-1}}{1-u^k}\, du, \quad z > 0.$$

Differentiating s number of times of (1.2) and (1.3), gives

$$\psi_{p,k}^{(s)}(z) = (-1)^{s+1} \int_0^\infty \frac{1-e^{-k(p+1)u}}{1-e^{-ku}} u^s e^{-zu}\, du, \qquad (1.5)$$
$$= (-1)^{s+1} s! \sum_{n=0}^{p} \frac{1}{(nk+z)^{s+1}},$$

where $s \in \mathbb{N}$.

A real function h is said to be completely monotonic on D if it is infinitely differentiable on D and

$$(-1)^s h^{(s)}(z) \geq 0, \qquad (1.6)$$

for all $z \in D$ and $s \in \mathbb{N}_0$. It is said to be strictly completely monotonic if inequality (1.6) is strict. The concept of completely monotonic functions was introduced by Hausdorff [7] in 1921 and has since attracted the attension of several researchers. There is a very rich literature on this subject. See for example [2], [3], [5], [6], [8], [10], [12], [15] and [27].

These functions have some remarkable applications in various aspects of mathematics. In particular, they play a pivotal role in the theory of inequalities and special functions.

Qiu and Vuorinen [23] established among other things that, the function

$$\theta(z) = \psi\left(z + \frac{1}{2}\right) - \psi(z) - \frac{1}{2z}, \qquad (1.7)$$

is strictly decreasing and convex on $(0, \infty)$. Motivated by this result, Mortici [14] proved a more generalized and deeper result which states that, the function

$$\theta_c(z) = \psi(z+c) - \psi(z) - \frac{c}{z}, \quad c \in (0,1), \qquad (1.8)$$

is strictly completely monotonic on $(0, \infty)$. Consequently, he obtained a sharp inequality bounding the function $\psi(z+c) - \psi(z)$. Also inspired by Mortici's results, Merovci [13] proved that for $p \in \mathbb{N}$, the function

$$\theta_{c,p}(z) = \psi_p(z+c) - \psi_p(z), \quad c \in (0,1), \qquad (1.9)$$

is strictly completely monotonic on $(0, \infty)$ and eventually obtained a sharp inequality bounding the function $\psi_p(z+c) - \psi_p(z)$. In this paper, the goal is to extend these results to the (p,k)-digamma function. We present our results in the following section.

2. Results and Discussion

Lemma 2.1. *The function $h(u) = \frac{1-e^{-u}}{u}$ is strictly decreasing on $(0, \infty)$.*

Proof. By direct computation, we obtain

$$h'(u) = \frac{(u+1)e^{-u}}{u^2} - \frac{1}{u^2} < 0,$$

since $e^u > u + 1$ for all $u \in (0, \infty)$. This gives the desired result.

Theorem 2.1. *For $c \in (0,1)$, $p \in \mathbb{N}$ and $k > 0$, the function*

$$\theta_{c,p,k}(z) = \psi_{p,k}(z+ck) - \psi_{p,k}(z) - \frac{c}{z} + \frac{c}{z+pk+k}, \qquad (2.1)$$

is strictly completely monotonic on $(0, \infty)$.

Proof. By repeated differentiation with respect to z, we obtain

$$\theta_{c,p,k}^{(s)}(z) = \psi_{p,k}^{(s)}(z+ck) - \psi_{p,k}^{(s)}(z) - \frac{(-1)^s c s!}{z^{s+1}} + \frac{(-1)^s c s!}{(z+pk+k)^{s+1}},$$

for $s \in \mathbb{N}$. Then, by virtue of relations (1.1) and (1.5), we obtain

$$
\begin{aligned}
&(-1)^s \theta^{(s)}_{c,p,k}(z) \\
&= -\int_0^\infty \frac{1-e^{-k(p+1)u}}{1-e^{-ku}} u^s e^{-(z+ck)u}\,du + \int_0^\infty \frac{1-e^{-k(p+1)u}}{1-e^{-ku}} u^s e^{-zu}\,du \\
&\quad - c\int_0^\infty u^s e^{-zu}\,du + c\int_0^\infty u^s e^{-(z+pk+k)u}\,du \\
&= \int_0^\infty \frac{1-e^{-k(p+1)u}}{1-e^{-ku}} u^s e^{-zu}(1-e^{-cku})\,du - c\int_0^\infty u^s e^{-zu}(1-e^{-k(p+1)u})\,du \\
&= \int_0^\infty \left[\frac{1-e^{-cku}}{1-e^{-ku}} - c\right]\left(1-e^{-k(p+1)u}\right) u^s e^{-zu}\,du \\
&= ck\int_0^\infty \left[\frac{1-e^{-cku}}{cku} - \frac{1-e^{-ku}}{ku}\right] \frac{1-e^{-k(p+1)u}}{1-e^{-ku}} u^{s+1} e^{-zu}\,du \\
&> 0,
\end{aligned}
$$

which is as a result of Lemma 2.1.

Corollary 2.1. *Let $c \in (0,1)$, $p \in \mathbb{N}$ and $k > 0$. Then the inequality*

$$
\begin{aligned}
\frac{c}{z} - \frac{c}{z+pk+k} < \psi_{p,k}(z+ck) - \psi_{p,k}(z) \le \psi_{p,k}(ck) - \psi_{p,k}(k)& \\
+ \frac{1}{k}\left(\frac{1}{c} - c\right) + \frac{1}{k}\left(\frac{c}{p+2} - \frac{1}{c+p+1}\right) + \frac{c}{z} - \frac{c}{z+pk+k}&, \qquad (2.2)
\end{aligned}
$$

holds for $z \in [k,\infty)$.

Proof. Since $\theta_{c,p,k}(z)$ is strictly completely monotonic, then it is strictly decreasing. Then for $z \in [k,\infty)$ and by virtue of (1.4), we obtain

$$
\begin{aligned}
0 = \lim\nolimits_{z\to\infty} \theta_{c,p,k}(z) < \theta_{c,p,k}(z) \quad &\le \theta_{c,p,k}(k) \\
&= \psi_{p,k}(k+ck) - \psi_{p,k}(k) - \left(\tfrac{c}{k} - \tfrac{c}{k+pk+k}\right) \\
&= \psi_{p,k}(ck) - \psi_{p,k}(k) + \tfrac{1}{k}\left(\tfrac{1}{c} - c\right) + \tfrac{1}{k}\left(\tfrac{c}{p+2} - \tfrac{1}{c+p+1}\right),
\end{aligned}
$$

$$0 = \lim_{z\to\infty} \theta_{c,p,k}(z) < \theta_{c,p,k}(z) \le \theta_{c,p,k}(k) = \psi_{p,k}(k+ck) - \psi_{p,k}(k) - \left(\frac{c}{k} - \frac{c}{k+pk+k}\right) = \psi_{p,k}(ck) - \psi_{p,k}(k) + \frac{1}{k}\left(\frac{1}{c} - c\right) +$$

which yields inequality (2.2).

Corollary 2.2. *Let $c \in (0,1)$, $p \in \mathbb{N}$ and $k > 0$. Then the inequalities*

$$\psi'_{p,k}(z+ck) - \psi'_{p,k}(z) < \frac{c}{(z+pk+k)^2} - \frac{c}{z^2}, \qquad (2.3)$$

$$\psi''_{p,k}(z+ck) - \psi''_{p,k}(z) > \frac{2c}{z^3} - \frac{2c}{(z+pk+k)^3}, \qquad (2.4)$$

hold for $z \in (0,\infty)$.

Proof. Because $\theta_{c,p,k}(z)$ is strictly decreasing, then $\theta'_{c,p,k}(z) < 0$, which gives (2.3). Also, complete monotonicity of $\theta_{c,p,k}(z)$ implies that it is strictly convex. Thus, $\theta''_{c,p,k}(z) > 0$, which gives (2.4).

From Theorem 2.1 and Corollary 2.1, we deduce the following special cases.

Theorem 2.2. *For $c \in (0,1)$ and $p \in \mathbb{N}$, the function*

$$\theta_{c,p}(z) = \psi_p(z+c) - \psi_p(z) - \frac{c}{z} + \frac{c}{z+p+1}, \tag{2.5}$$

is strictly completely monotonic on $(0,\infty)$ and the inequality

$$\frac{c}{z} - \frac{c}{z+p+1} \ < \ \psi_p(z+c) - \psi_p(z) \ \leq \ \psi_p(c) - \psi_p(1) + \frac{1}{c} - c + \frac{c}{p+2} - \frac{1}{c+p+1} + \frac{c}{z} - \frac{c}{z+p+1}, \tag{2.6}$$

holds for $z \in [1,\infty)$.

Proof. This follows from Theorem 2.1 and Corollary 2.1 by letting $k = 1$.

Theorem 2.3. *For $c \in (0,1)$ and $k > 0$, the function*

$$\theta_{c,k}(z) = \psi_k(z+ck) - \psi_k(z) - \frac{c}{z}, \tag{2.7}$$

is strictly completely monotonic on $(0,\infty)$ and the inequality

$$\frac{c}{z} < \psi_k(z+ck) - \psi_k(z) \leq \psi_k(ck) - \psi_k(k) + \frac{1}{k}\left(\frac{1}{c} - c\right) + \frac{c}{z}, \tag{2.8}$$

holds for $z \in [k,\infty)$.

Proof. This follows from Theorem 2.1 and Corollary 2.1 by letting $p \to \infty$.

Remark 2.1. *By either letting $p \to \infty$ in Theorem 2.2 or letting $k = 1$ in Theorem 2.3, we recover the main results of Mortici* [14].

Remark 2.2. *We noticed the following misprints in the previous works* [13] *and* [14]*. In* [14]*, the inequality*

$$0 < \psi(x+a) - \psi(x) \leq \psi(a) + \gamma + \frac{1}{a} - a,$$

where $x \geq 1$, was presented. However, by the procedure used, the correct form should be

$$\frac{a}{x} < \psi(x+a) - \psi(x) \leq \psi(a) + \gamma + \frac{1}{a} - a + \frac{a}{x}. \tag{2.9}$$

Also, in [13]*, the inequality*

$$0 < \psi_p(x+a) - \psi_p(x) \leq \frac{1}{x} - \frac{1}{x+p+1},$$

where $x \geq 1$ and $p \in \mathbb{N}$, was presented. However, by the procedure used, the correct form should be

$$0 < \psi_p(x+a) - \psi_p(x) \leq \psi_p(a) - \psi_p(1) + \frac{1}{a} - \frac{1}{a+p+1}. \tag{2.10}$$

It follows from

$$g(p) = \frac{1}{2+p} - 1 + \frac{1}{x} - \frac{1}{1+p+x} \leq 0,$$

for $p \geq 1$ and $x \geq 1$ because $g'_p \leq 0$ and $g(p=1) \leq 0$.

Remark 2.3. *Results similar to the results of this paper, and concerning the the polygamma and Nielsen's beta functions, can be found in* [16] *and* [17]*. Other monotonicity properties concerning the (p,k)-digamma function can also be found in the recent works* [11], [18] [25] *and* [26].

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