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ASPECTS OF THE CYCLING PHENOMENON IN THE LINEAR PROGRAMMING PROBLEM (LPP) THROUGH THE EXAMPLE OF MARSHALL AND SUURBALLE

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Abstract: *A complete analysis of the cycling phenomenon in the case of the linear programming problem (LPP) is far from being achieved. Even if* [5] *states that the answer to the fundamental question of this problem is found, the proposed solution is very difficult to apply, being necessary to find a solution of a complex system of inequalities. Additionally, it is difficult to recognize a problem that, by applying the primal simplex algorithm, leads us to the occurrence of this phenomenon. The example given by Marshall and Suurballe, but also the example given by Danzig, lead us to draw some useful conclusions about this phenomenon, whether the given problem admits the optimal solution or has an infinite optimal solution.*

Keywords: linear programming problem (LPP), simplex algorithm, cycling, basic solution, degenerated base solution

1. Introduction

As also mentioned in [1], since the emergence of the algorithm proposed by Danzig [2], there have been searched for situations in which it does not work, the first example being given by Hoffman [3].

Until recently, in the literature, almost all cycling examples to be found have only two constraints, or are reducible to a problem with two constraints, Marshall and Suurballe being the only exception.

In [5], there is given a way of constructing some examples of some linear programming problems that admit the optimal solution or have infinite optimal solution, in which the number of restrictions is greater than two, the construction being made on the basis of solving a system of inequalities.

For the time being, from the analysis of the following example

$$\begin{cases} max(14\cdot x_1 - 25\cdot x_2 + 720\cdot x_3 - 20\cdot x_4) \\ x_1 - 2\cdot x_2 - 110\cdot x_3 + 5\cdot x_4 \le 0 \\ 710\cdot x_1 - 310\cdot x_2 - 1100\cdot x_3 + 1950\cdot x_4 \le 0 \\ x_1 + x_2 + x_3 + x_4 \le 5 \\ x_1 + 2\cdot x_2 + 3\cdot x_3 + x_4 \le 10 \\ x_1, x_2, x_3, x_4 \ge 0 \end{cases}, (1)$$

which has the standard form

$$\begin{cases} -min(-14\cdot x_1 + 25\cdot x_2 - 720\cdot x_3 + 20\cdot x_4) \\ x_1 - 2\cdot x_2 - 110\cdot x_3 + 5\cdot x_4 + x_5 = 0 \\ 710\cdot x_1 - 310\cdot x_2 - 1100\cdot x_3 + 1950\cdot x_4 + x_6 = 0 \\ x_1 + x_2 + x_3 + x_4 + x_7 = 5 \\ x_1 + 2\cdot x_2 + 3\cdot x_3 + x_4 + x_8 = 10 \\ x_i \ge 0, (\forall) i \in \overline{1,8} \end{cases}$$

and is given in [5], it was not possible to reach a cycle, with the input and output criteria from the base set for the primal simplex algorithm reaching the optimal solution ($x_1 = 0$, $x_2 = 0$, $x_3 = 10/3$ and $x_4 = 0$, and $z_{max} = 2400$), the solution being obtained even from the first stage.

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Table no. 1 Simplex table for Zornig's problem

B	c_B	$\bar{x}^B$	-14	25	-720	20	0	0	0	0	θ_i
			a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	
a^5	0	0	1	−2	−110	5	1	0	0	0	-
a^6	0	0	710	−310	−1100	1950	0	1	0	0	-
a^7	0	5	1	1	1	1	0	0	1	0	5
a^8	0	10	1	2	3	1	0	0	0	1	3 1/3
z_j		0	0	0	0	0	0	0	0	0	–
z_j-c_j		–	14	−25	720	−20	0	0	0	0	–
a5	0	366 2/3	37 2/3	71 1/3	0	41 2/3	1	0	0	36 2/3	
a6	0	3666 2/3	1076 2/3	423 1/3	0	2316 2/3	0	1	0	366 2/3	
a7	0	1 2/3	2/3	1/3	0	2/3	0	0	1	− 1/3	
a3	−720	3 1/3	1/3	2/3	1	1/3	0	0	0	1/3	
z_j		−2400	−240	−480	−720	−240	0	0	0	−240	–
z_j-c_j		–	−226	−505	0	−260	0	0	0	−240	–

As it can be seen from the table above, if the entry and exit criteria are met, no cycle is reached, but the optimal solution to the problem is obtained very quickly.

In the example given by Marshall and Suurballe, which can be found for example in [4], things are different, meaning that there are more input alternatives and especially exit from the base (equal values are obtained).

And Hoffman's example [3] is interesting, and even if it can be easily shown that it has an optimal solution, the algorithm is applied in one case to a cycle, and in all other alternatives (there are also more values equal to the application of the exit criterion from the base) concludes that the problem has an infinite optimal solution [1].

We will then look at the example given by Marshall and Suurballe.

2. Analysis of the Marshall and Suurballe example

The linear programming problem proposed by Marshall and Suurballe [4] is the following:

$$\begin{cases} \min(-0.4 \cdot x_1 - 0.4 \cdot x_2 + 1.8 \cdot x_3) \\ \quad 0.6 \cdot x_1 - 6.4 \cdot x_2 + 4.8 \cdot x_3 \le 0 \\ \quad 0.2 \cdot x_1 - 1.8 \cdot x_2 + 0.6 \cdot x_3 \le 0 \\ \quad 0.4 \cdot x_1 - 1.6 \cdot x_2 + 0.2 \cdot x_3 \le 0 \\ \qquad\qquad\quad x_2 \qquad\qquad \le 1 \\ x_1, x_2, x_3 \ge 0 \end{cases}, \quad (2)$$

The standard form of the above problem is

$$\begin{cases} \min(-0.4 \cdot x_1 - 0.4 \cdot x_2 + 1.8 \cdot x_3) \\ 0.6 \cdot x_1 - 6.4 \cdot x_2 + 4.8 \cdot x_3 + x_4 = 0 \\ 0.2 \cdot x_1 - 1.8 \cdot x_2 + 0.6 \cdot x_3 + x_5 = 0 \\ 0.4 \cdot x_1 - 1.6 \cdot x_2 + 0.2 \cdot x_3 + x_6 = 0 \\ \qquad\qquad x_2 + \qquad\qquad x_7 = 1 \\ x_i \ge 0, (\forall) i \in \overline{1,7} \end{cases}$$

Further on, we will present how to determine the optimal solution, but also the way that leads to the cycle, depending on the choice of the values for the entry and exit criteria from the base.

Table no. 2 Simplex table for Marshall and Suurballe problem

B	c_B	$\bar{x}^B$	$-2/5$	$-2/5$	$9/5$	0	0	0	0	θ_i
			a^1	a^2	a^3	a^4	a^5	a^6	a^7	
a^4	0	0	3/5	−32/5	24/5	1	0	0	0	–
a^5	0	0	1/5	−9/5	3/5	0	1	0	0	–
a^6	0	0	2/5	−8/5	1/5	0	0	1	0	–
a^7	0	1	0	1	0	0	0	0	1	0*
z_j		0	0	0	0	0	0	0	0	–
z_j-c_j		–	2/5	2/5*	−9/5	0	0	0	0	–

a^4	0	32/5	3/5	0	24/5	1	0	0	32/5	32/3
a^5	0	9/5	1/5	0	3/5	0	1	0	9/5	9
a^6	0	8/5	2/5	0	1/5	0	0	1	8/5	8*
a^2	–2/5	1	0	1	0	0	0	0	1	–
z_j		–2/5	0	–2/5	0	0	0	0	–2/5	–
$z_j - c_j$		–	2/5*	0	–9/5	0	0	0	–2/5	–
a^4	0	4	0	0	9/2	1	0	–3/2	4	
a^5	0	1	0	0	1/2	0	1	–1/2	1	
a^1	–2/5	4	1	0	1/2	0	0	5/2	4	
a^2	–2/5	1	0	1	0	0	0	0	1	
z_j		–2	–2/5	–2/5	–1/5	0	0	–1	–2	–
$z_j - c_j$		–	0	0	–2	0	0	–1	–2	–

On this choice of input and output criteria from the base, the optimal solution is $x_1 = 4$, $x_2 = 1$ and $x_3 = 0$, and the objective function value is $z_{min} = -2$.

The choice of the other alternative for the differences $z_j - c_j$ leads us to the following solution:

Table no. 3 Simplex table for Marshall and Suurballe problem – variant 1

Step	B	c_B	$\overline{x}^B$	–2/5	–2/5	9/5	0	0	0	0	θ_i
				a^1	a^2	a^3	a^4	a^5	a^6	a^7	
0	a^4	0	0	3/5	–32/5	24/5	1	0	0	0	0*
	a^5	0	0	1/5	–9/5	3/5	0	1	0	0	0
	a^6	0	0	2/5	–8/5	1/5	0	0	1	0	0
	a^7	0	1	0	1	0	0	0	0	1	–
	z_j		0	0	0	0	0	0	0	0	–
	$z_j - c_j$		–	2/5*	2/5	–9/5	0	0	0	0	–
1	a^1	–2/5	0	1	–32/3	8	5/3	0	0	0	–
	a^5	0	0	0	1/3	–1	–1/3	1	0	0	0*
	a^6	0	0	0	8/3	–3	–2/3	0	1	0	0
	a^7	0	1	0	1	0	0	0	0	1	1
	z_j		0	–2/5	64/15	–16/5	–2/3	0	0	0	–
	$z_j - c_j$		–	0	14/3*	–5	–2/3	0	0	0	–
2	a^1	–2/5	0	1	0	–24	–9	32	0	0	–
	a^2	–2/5	0	0	1	–3	–1	3	0	0	–
	a^6	0	0	0	0	5	2	–8	1	0	0*
	a^7	0	1	0	0	3	1	–3	0	1	1/3
	z_j		0	–2/5	–2/5	54/5	4	–14	0	0	–
	$z_j - c_j$		–	0	0	9*	4	–14	0	0	–
3	a^1	–2/5	0	1	0	0	3/5	–32/5	24/5	0	–
	a^2	–2/5	0	0	1	0	1/5	–9/5	3/5	0	–
	a^3	9/5	0	0	0	1	2/5	–8/5	1/5	0	–
	a^7	0	1	0	0	0	–1/5	9/5	–3/5	1	5/9 *
	z_j		0	–2/5	–2/5	9/5	2/5	2/5	–9/5	0	–
	$z_j - c_j$		–	0	0	0	2/5	2/5*	–9/5	0	–
4	a^1	–2/5	32/9	1	0	0	–1/9	0	24/9	32/9	–
	a^2	–2/5	1	0	1	0	0	0	0	1	–
	a^3	9/5	8/9	0	0	1	2/9	0	–3/9	8/9	4*
	a^5	0	5/9	0	0	0	–1/9	1	–1/3	5/9	–
	z_j		0	–2/5	–2/5	9/5	4/9	0	–15/9	–2/9	–
	$z_j - c_j$		–	0	0	0	4/9*	0	–15/9	–2/9	–

5	a^1	−2/5	4	1	0	1/2	0	0	5/2	4	
	a^2	−2/5	1	0	1	0	0	0	0	1	
	a^4	0	4	0	0	9/2	1	0	−3/2	4	
	a^5	0	1	0	0	1/2	0	1	−1/2	1	
	z_j		−2	−2/5	−2/5	−1/5	0	0	−1	−2	–
	$z_j - c_j$		–	0	0	−2	0	0	−1	−2	–

And this choice of entry and exit criteria from the base leads us to the same optimal solution.

Next we will choose the other alternative for the differences $z_j - c_j$ in the third stage of the previous table, which leads us to the occurrence of the cycling phenomenon.

Table no. 4 Simplex table for Marshall and Suurballe problem – variant 2

B	c_B	$\bar{x}^B$	−2/5 a^1	−2/5 a^2	9/5 a^3	0 a^4	0 a^5	0 a^6	0 a^7	θ_i
a^4	0	0	3/5	−32/5	24/5	1	0	0	0	0*
a^5	0	0	1/5	−9/5	3/5	0	1	0	0	0
a^6	0	0	2/5	−8/5	1/5	0	0	1	0	0
a^7	0	1	0	1	0	0	0	0	1	–
z_j		0	0	0	0	0	0	0	0	–
$z_j - c_j$		–	2/5*	2/5	−9/5	0	0	0	0	–
a^1	−2/5	0	1	−32/3	8	5/3	0	0	0	–
a^5	0	0	0	1/3	−1	−1/3	1	0	0	0*
a^6	0	0	0	8/3	−3	−2/3	0	1	0	0
a^7	0	1	0	1	0	0	0	0	1	1
z_j		0	−2/5	64/15	−16/5	−2/3	0	0	0	–
$z_j - c_j$		–	0	14/3*	−5	−2/3	0	0	0	–
a^1	−2/5	0	1	0	−24	−9	32	0	0	–
a^2	−2/5	0	0	1	−3	−1	3	0	0	–
a^6	0	0	0	0	5	2	−8	1	0	0*
a^7	0	1	0	0	3	1	−3	0	1	1/3
z_j		0	−2/5	−2/5	54/5	4	−14	0	0	–
$z_j - c_j$		–	0	0	9*	4	−14	0	0	–
a^1	−2/5	0	1	0	0	3/5	−32/5	24/5	0	0*
a^2	−2/5	0	0	1	0	1/5	−9/5	3/5	0	0
a^3	9/5	0	0	0	1	2/5	−8/5	1/5	0	0
a^7	0	1	0	0	0	−1/5	9/5	−3/5	1	–
z_j		0	−2/5	−2/5	9/5	2/5	2/5	−9/5	0	–
$z_j - c_j$		–	0	0	0	2/5*	2/5	−9/5	0	–
a^4	0	0	5/3	0	0	1	−32/3	8	0	–
a^2	−2/5	0	−1/3	1	0	0	1/3	−1	0	0*
a^3	9/5	0	−2/3	0	1	0	8/3	−3	0	0
a^7	0	1	1/3	0	0	0	−1/3	1	1	–
z_j		0	−16/15	−2/5	9/5	0	14/3	−5	0	–
$z_j - c_j$		–	−2/3	0	0	0	14/3*	−5	0	–
a^4	0	0	−9	32	0	1	0	−24	0	–
a^5	0	0	−1	3	0	0	1	−3	0	–
a^3	9/5	0	2	−8	1	0	0	5	0	0*
a^7	0	1	0	1	0	0	0	0	1	–
z_j		0	18/5	−72/5	9/5	0	0	9	0	–
$z_j - c_j$		–	4	−14	0	0	0	9*	0	–
a^4	0	0	3/5	−32/5	24/5	1	0	0	0	
a^5	0	0	1/5	−9/5	3/5	0	1	0	0	
a^6	0	0	2/5	−8/5	1/5	0	0	1	0	
a^7	0	1	0	1	0	0	0	0	1	
z_j		0								–
$z_j - c_j$		–								–

So, in this case, the cycle takes place, but by complying with the entry and exit criteria from the base. If we were to take them in order, namely from left to right for the entry criterion in the base and top down for the exit criterion from the base (obviously at equal values; for a minimal problem the criterion of entry into the base being a^k for which $z_k - c_k = \max\limits_{j \in J_R}(z_j^B - c_j)$, and for the exit from the base a^l for which $\theta_l = \dfrac{x_l}{y_{lk}} = \min\limits_{i \in J_B}\left\{\dfrac{x_i}{y_{ik}} / (\exists) i \in J_B \; cu \; y_{ik} > 0\right\}$), then these choices would lead us directly to the cycle.

Until now, the alternatives available for the entry criterion have been exhausted. Next, we will go over to analyze the other alternatives related to the exit criteria from the base.

Table no. 5 Simplex table for Marshall and Suurballe problem – variant 3

B	c_B	$\overline{x}^B$	–2/5	–2/5	9/5	0	0	0	0	θ_i
			a^1	a^2	a^3	a^4	a^5	a^6	a^7	
a^4	0	0	3/5	–32/5	24/5	1	0	0	0	0*
a^5	0	0	1/5	–9/5	3/5	0	1	0	0	0
a^6	0	0	2/5	–8/5	1/5	0	0	1	0	0
a^7	0	1	0	1	0	0	0	0	1	–
z_j		0	0	0	0	0	0	0	0	–
$z_j - c_j$		–	2/5*	2/5	–9/5	0	0	0	0	–
a^1	–2/5	0	1	–32/3	8	5/3	0	0	0	–
a^5	0	0	0	1/3	–1	–1/3	1	0	0	0*
a^6	0	0	0	8/3	–3	–2/3	0	1	0	0
a^7	0	1	0	1	0	0	0	0	1	1
z_j		0	–2/5	64/15	–16/5	–2/3	0	0	0	–
$z_j - c_j$		–	0	14/3*	–5	–2/3	0	0	0	–
a^1	–2/5	0	1	0	–24	–9	32	0	0	–
a^2	–2/5	0	0	1	–3	–1	3	0	0	–
a^6	0	0	0	0	5	2	–8	1	0	0*
a^7	0	1	0	0	3	1	–3	0	1	1/3
z_j		0	–2/5	–2/5	54/5	4	–14	0	0	–
$z_j - c_j$		–	0	0	9*	4	–14	0	0	–
a^1	–2/5	0	1	0	0	3/5	–32/5	24/5	0	0*
a^2	–2/5	0	0	1	0	1/5	–9/5	3/5	0	0
a^3	9/5	0	0	0	1	2/5	–8/5	1/5	0	0
a^7	0	1	0	0	0	–1/5	9/5	–3/5	1	–
z_j		0	–2/5	–2/5	9/5	2/5	2/5	–9/5	0	–
$z_j - c_j$		–	0	0	0	2/5*	2/5	–9/5	0	–
a^4	0	0	5/3	0	0	1	–32/3	8	0	–
a^2	–2/5	0	–1/3	1	0	0	1/3	–1	0	0
a^3	9/5	0	–2/3	0	1	0	8/3	–3	0	0*
a^7	0	1	1/3	0	0	0	–1/3	1	1	–
z_j		0	–16/15	–2/5	9/5	0	14/3	–5	0	–
$z_j - c_j$		–	–2/3	0	0	0	14/3*	–5	0	–
a^4	0	0	–1	0	4	1	0	–4	0	–
a^2	–2/5	0	–1/4	1	–1/8	0	0	–5/8	0	–
a^5	0	0	–1/4	0	3/8	0	1	–9/8	0	–
a^7	0	1	1/4	0	1/8	0	0	5/8	1	4*
z_j		0	1/10	–2/5	1/20	0	0	1/4	0	–
$z_j - c_j$		–	1/2*	0	–7/4	0	0	1/4	0	–

a^4	0	4	0	0	9/2	1	0	–3/2	4	
a^2	–2/5	1	0	1	0	0	0	0	1	
a^5	0	1	0	0	1/2	0	1	–1/2	1	
a^1	–2/5	4	1	0	1/2	0	0	5/2	4	
z_j		–2	–2/5	–2/5	–1/5	0	0	–1	–2	–
$z_j – c_j$		–	0	0	–2	0	0	–1	–2	–

Also, with this choice, we come to the same optimal solution as with all the other alternatives, totaling 9.

Of the 9 alternatives that we could choose (equal values for the entry criterion into the base or exit criterion from the base), the cycle phenomenon appeared in one and in all other the optimal solutions, as is normal and not as in the case of Hoffman's example [3], when, except in the case of the cycling phenomenon, in all other cases it is concluded that the problem is an infinite optimal solution, even if it can be easily shown that the problem has actually an optimal solution ($x_1 = x_2 = x_3 = x_4 = 0$ and $z_{max} = 0$) [1].

3. Concluding remarks

Building such examples with computer applications is not always beneficial because we cannot control what is happening and how the solution or message delivered by the application is chosen. For example, such an application will not analyze all the alternatives that may be developed or all the optimal solutions with a linear programming problem.

The Marshall and Suurballe example is relevant in that it provides an answer to the question whether the solution or any of the solutions in the cycle is or are optimal, and here we see clearly that this is not the case.

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