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A note on preserving the spark of a matrix

Abstract. Let $\mathcal{M}_{m\times n}(\mathbb{F})$ be the vector space of all $m\times n$ matrices over a field $\mathbb{F}$. In the case where $m\geq n$, $\operatorname{char}(\mathbb{F})\neq 2$ and $\mathbb{F}$ has at least five elements, we give a complete characterization of linear maps $\Phi\colon \mathcal{M}_{m\times n}(\mathbb{F})\to\mathcal{M}_{m\times n}(\mathbb{F})$ such that $\operatorname{spark}(\Phi(A))=\operatorname{spark}(A)$ for any $A\in\mathcal{M}_{m\times n}(\mathbb{F})$.

1. Preliminaries and introduction

Throughout the text, m and n stand for positive integers, and $\mathbb{F}$ denotes a field. We define $\mathcal{M}_{m\times n}(\mathbb{F})$ to be the vector space of all $m\times n$ matrices over $\mathbb{F}$. The $m\times n$ zero matrix will be denoted by $O_{m\times n}$ and the nth full linear group over $\mathbb{F}$ by $\mathcal{GL}_n(\mathbb{F})$ (i.e., $\mathcal{GL}_n(\mathbb{F})=\{V\in\mathcal{M}_{n\times n}(\mathbb{F}):\ \det(V)\neq 0\}$). Finally, if $x_1,\dots,x_n$ are the components of a (row or column) vector $x\in\mathbb{F}^n$, then the *Hamming weight* of x is defined by

$$\|x\|_0=\#\big\{j\in\{1,\dots,n\}:\ x_j\neq 0\big\}.$$

In [3], Donoho and Elad introduced the concept of spark of a matrix into the mathematical theory of compressed sensing. Let us recall the definition.

Definition 1.1
Suppose that $C_1,\dots,C_n\in\mathcal{M}_{m\times 1}(\mathbb{F})$ are the columns of a matrix $A\in\mathcal{M}_{m\times n}(\mathbb{F})$. The spark of A is defined to be the infimum of the set of all positive integers ℓ with the property that

$$\exists\, j_1,\dots,j_\ell\in\{1,\dots,n\}:\begin{cases} j_1<\ldots<j_\ell,\\ C_{j_1},\dots,C_{j_\ell}\ \textit{are linearly dependent.}\end{cases}$$

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The following facts about the spark are well known and easy to prove.

Proposition 1.2
Let $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ and $U \in \mathcal{GL}_m(\mathbb{F})$. Then

(i) $\operatorname{spark}(A) \in \{1, \dots, n\} \cup \{+\infty\}$,

(ii) $\operatorname{spark}(A) = +\infty$ *if and only if* $\operatorname{rank}(A) = n$,

(iii) $\operatorname{spark}(A) = 1$ *if and only if A has a zero column,*

(iv) $\operatorname{spark}(A) \leq \operatorname{rank}(A) + 1$ *whenever* $\operatorname{spark}(A) \neq +\infty$,

(v) $\operatorname{spark}(UA) = \operatorname{spark}(A)$.

When dealing with a reasonable map f defined on $\mathcal{M}_{m\times n}(\mathbb{F})$, it is always of interest to know what linear endomorphisms $\Phi\colon \mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{m\times n}(\mathbb{F})$ have the property that $f(\Phi(A)) = f(A)$ for any $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. Such endomorphisms are called *linear preservers* of the map f. The theory of linear preserver problems dates back to 1890s (Frobenius' theorem on linear preservers of the determinant function) and still attracts the attention of many mathematicians. We refer to [2] for a nice overview of results.

This note provides some remarks on linear preservers of the function

$$\mathcal{M}_{m\times n}(\mathbb{F}) \ni A \mapsto \operatorname{spark}(A) \in \{1, \dots, n\} \cup \{+\infty\}.$$

We will need the following technical definition.

Definition 1.3
Let $U \in \mathcal{GL}_m(\mathbb{F})$ and $V \in \mathcal{GL}_n(\mathbb{F})$. A map $\Phi\colon \mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{m\times n}(\mathbb{F})$ is said to be (U, V)-standard, if either $\Phi(A) = UAV$ for all $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, or $m = n$ and $\Phi(A) = UA^{\mathrm{T}}V$ for all $A \in \mathcal{M}_{m\times n}(\mathbb{F})$.

Notice that the (U, V)-standard map is a linear automorphism of $\mathcal{M}_{m\times n}(\mathbb{F})$.

The note is based on the characterization of rank k preservers given by Beasley and Laffey (see [1]), which we recall below.

Theorem 1.4
Let k be a positive integer such that $k \leq \min\{m, n\}$. Suppose that the field $\mathbb{F}$ has at least four elements. If a linear automorphism $\Phi\colon \mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{m\times n}(\mathbb{F})$ satisfies the condition

$$\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{F}):\ \operatorname{rank}(A) = k \implies \operatorname{rank}(\Phi(A)) = k,$$

then it is a (U, V)-standard map, for some $U \in \mathcal{GL}_m(\mathbb{F})$ and some $V \in \mathcal{GL}_n(\mathbb{F})$.

2. Results

Our main purpose is to prove

Theorem 2.1
If $\mathbb{F}$ has at least five elements, $\operatorname{char}(\mathbb{F}) \neq 2$, *and* $m \geq n$, *then for a linear endomorphism* $\Phi\colon \mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{m\times n}(\mathbb{F})$, *the following conditions are equivalent:*

(1) $\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{F}):\ \operatorname{spark}(\Phi(A)) = \operatorname{spark}(A)$,

(2) *there exist a matrix* $U \in \mathcal{GL}_m(\mathbb{F})$, *a diagonal matrix* $D \in \mathcal{GL}_n(\mathbb{F})$, *and an* $n \times n$ *permutation matrix* P *such that* $\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{F}):\ \Phi(A) = UADP$.

The proof will use two simple propositions and a lemma. The propositions are of independent interest.

Proposition 2.2
Let $\Phi\colon \mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{m\times n}(\mathbb{F})$ *be a linear map. Suppose that* $\operatorname{spark}(\Phi(A)) = \operatorname{spark}(A)$ *for any* $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. *Then* Φ *is bijective.*

Proof. Pick a matrix $B \in \mathcal{M}_{m\times n}(\mathbb{F})$ such that $\Phi(B) = O_{m\times n}$. It is enough to show that $B = O_{m\times n}$. Since $\operatorname{spark}(B) = \operatorname{spark}(\Phi(B)) = 1$, the matrix B has a zero column. Assume that B has a nonzero column as well (and hence $n \geq 2$). Let $S \in \mathcal{M}_{m\times n}(\mathbb{F})$ be the matrix consisting of n copies of this nonzero column. Then $\operatorname{spark}(S) = 2$ and

$$\operatorname{spark}(\Phi(S)) = \operatorname{spark}(\Phi(S - B)) = \operatorname{spark}(S - B) = 1,$$

a contradiction. Consequently, $B = O_{m\times n}$.

Proposition 2.3
Suppose that $\operatorname{char}(\mathbb{F}) \neq 2$. *Then, for a matrix* $V \in \mathcal{GL}_n(\mathbb{F})$, *the following conditions are equivalent:*

(1) $\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{F}):\ \operatorname{spark}(AV) = \operatorname{spark}(A)$,

(2) *there exist a diagonal matrix* $D \in \mathcal{GL}_n(\mathbb{F})$ *and an* $n \times n$ *permutation matrix* P *such that* $V = DP$.

Proof. Let $W = [w_{ij}] \in \mathcal{M}_{n\times n}(\mathbb{F})$ be such that

$$\exists\, \ell, p, q \in \{1, \ldots, n\} : \begin{cases} p \neq q, \\ w_{p\ell} \neq 0,\ w_{q\ell} \neq 0. \end{cases}$$

We will show that there is a matrix $B = [b_{kj}] \in \mathcal{M}_{m\times n}(\mathbb{F})$ with $\operatorname{spark}(B) = 2$ and $\operatorname{spark}(BW) = 1$.

Let $\lambda = w_{1\ell} + \ldots + w_{n\ell}$. Assume that $w_{r\ell} - \lambda \neq 0$ for some $r \in \{p, q\}$. Then it suffices to define

$$b_{kj} = \begin{cases} 1, & \text{if } k = 1 \text{ and } j \neq r, \\ 1 - \lambda w_{r\ell}^{-1}, & \text{if } k = 1 \text{ and } j = r, \\ 0, & \text{otherwise.} \end{cases}$$

Assume, therefore, that $w_{p\ell} - \lambda = 0 = w_{q\ell} - \lambda$. Then $\lambda + (\lambda - w_{p\ell} - w_{q\ell}) = 0$, and hence it suffices to define

$$b_{kj} = \begin{cases} 2, & \text{if } k = 1 \text{ and } j \notin \{p, q\}, \\ 1, & \text{if } k = 1 \text{ and } j \in \{p, q\}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if a matrix $V \in \mathcal{GL}_n(\mathbb{F})$ satisfies condition (1), then each column of V (and hence each row) has exactly one element different from 0. Condition (2) follows. Implication (2) $\Rightarrow$ (1) is obvious and holds true over an arbitrary field.

We denote by Σ_n the set of all permutations of $\{1, \dots, n\}$.

Corollary 2.4
Suppose that $\operatorname{char}(\mathbb{F}) \neq 2$. *Then, for a linear endomorphism* $f\colon \mathcal{M}_{1\times n}(\mathbb{F}) \to \mathcal{M}_{1\times n}(\mathbb{F})$*, the following conditions are equivalent:*

(1) $\forall\, x \in \mathcal{M}_{1\times n}(\mathbb{F}):\ \|f(x)\|_0 = \|x\|_0$,

(2) $\exists\, \sigma \in \Sigma_n\, \exists\, a_1, \dots, a_n \in \mathbb{F} \setminus \{0\}\, \forall\, x = (x_1, \dots, x_n) \in \mathcal{M}_{1\times n}(\mathbb{F}):\ f(x) = (a_1 x_{\sigma(1)}, \dots, a_n x_{\sigma(n)})$,

(3) $\forall\, x \in \mathcal{M}_{1\times n}(\mathbb{F}):\ \operatorname{spark}(f(x)) = \operatorname{spark}(x)$.

Proof. If $n = 1$, then there is nothing to do. Assume that $n \geq 2$. Then $\operatorname{spark}(x) \in \{1, 2\}$ for all $x \in \mathcal{M}_{1\times n}(\mathbb{F})$. Moreover, $\operatorname{spark}(x) = 2$ for some $x \in \mathcal{M}_{1\times n}(\mathbb{F})$ if and only if $\|x\|_0 = n$. These two properties yield implication (1) $\Rightarrow$ (3). Let us proceed to (3) $\Rightarrow$ (2). If condition (3) is satisfied, then by Proposition 2.2, the endomorphism f is bijective, and hence

$$\exists\, V \in \mathcal{GL}_n(\mathbb{F})\, \forall\, x \in \mathcal{M}_{1\times n}(\mathbb{F}):\ f(x) = xV.$$

Condition (2) now follows from Proposition 2.3. Implication (2) $\Rightarrow$ (1) is obvious.

The above equivalence (1) $\Leftrightarrow$ (2) is well known and can be easily proved over an arbitrary field, without involving the concept of spark. Implication (1) $\Rightarrow$ (3) also holds true over an arbitrary field.

Example 2.5
The linear endomorphism $g\colon \mathcal{M}_{1\times 3}(\mathbb{Z}_2) \ni (x_1, x_2, x_3) \mapsto (x_1, x_1 + x_2 + x_3, x_3) \in \mathcal{M}_{1\times 3}(\mathbb{Z}_2)$ *satisfies the condition*

$$\forall\, x \in \mathcal{M}_{1\times 3}(\mathbb{Z}_2):\ \operatorname{spark}(g(x)) = \operatorname{spark}(x).$$

However, g *is not a "Hamming isometry".*

Notice that a linear endomorphism $h\colon \mathcal{M}_{n\times 1}(\mathbb{F}) \to \mathcal{M}_{n\times 1}(\mathbb{F})$ is a preserver of the spark if and only if h is bijective.

Let us return to the main purpose of the note.

Lemma 2.6
If $n \geq 2$, then no matrix $V \in \mathcal{GL}_n(\mathbb{F})$ has the property that $\operatorname{spark}(A^{\mathrm{T}}V) = \operatorname{spark}(A)$ *for all $A \in \mathcal{M}_{n\times n}(\mathbb{F})$.*

Proof. Assume that $n \geq 2$ and pick a matrix $V \in \mathcal{GL}_n(\mathbb{F})$. Let $C \in \mathcal{M}_{n\times 1}(\mathbb{F})$ be a nonzero column such that every element of $C^{\mathrm{T}}V$ is different from 0. Define $S \in \mathcal{M}_{n\times n}(\mathbb{F})$ to be the matrix whose first column coincides with C and any other column coincides with $O_{n\times 1}$. Then $\operatorname{spark}(S) = 1$ and $\operatorname{spark}(S^{\mathrm{T}}V) = 2$.

Proof of Theorem 2.1. Implication (2) $\Rightarrow$ (1) is obvious (cf. Proposition 1.2; the implication holds true over an arbitrary field and even if $m < n$). Assume that Φ satisfies condition (1). Then it follows from Proposition 2.2 that Φ is bijective. Moreover, if $\operatorname{rank}(A) = n$ for a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, then $\operatorname{spark}(\Phi(A)) = \operatorname{spark}(A) = +\infty$, and hence $\operatorname{rank}(\Phi(A)) = n$. Theorem 1.4 yields therefore that Φ is a (U,V)-standard map for some $U \in \mathcal{GL}_m(\mathbb{F})$ and some $V \in \mathcal{GL}_n(\mathbb{F})$. Suppose, for a moment, that $m = n \geq 2$ and

$$\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{F}) :\ \Phi(A) = UA^{\mathrm{T}}V.$$

Then $\operatorname{spark}(A^{\mathrm{T}}V) = \operatorname{spark}(\Phi(A)) = \operatorname{spark}(A)$ for any $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, which contradicts Lemma 2.6. Consequently, $\Phi(A) = UAV$ for all $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. This implies that for an arbitrary $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, we have $\operatorname{spark}(AV) = \operatorname{spark}(UAV) = \operatorname{spark}(A)$. Thus, by Proposition 2.3, there exist a diagonal matrix $D \in \mathcal{GL}_n(\mathbb{F})$ and an $n \times n$ permutation matrix P such that $V = DP$. The proof is complete.

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