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On spectral multiplication as viewed in physical space

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ABSTRACT

The properties of spectral multiplication as viewed in physical space are investigated. A multiplication matrix is defined which permits the unaliased multiplication of two grid-point fields. Use of the multiplication matrix is less efficient than the transform method but does permit the investigation of the non-localness of spectral multiplication.

1. Introduction

With the discovery of the transform technique (Orszag, 1970; Eliassen et al., 1970) and subsequent successful implementation of spectral models in an operational context (Daley et al., 1976) the debate between so-called "grid pointers" and "spectral modellers" has cooled especially since the latter group must do much of the necessary calculation in physical space on a transform grid for the sake of efficiency. The success of the pseudospectral method where only derivatives are evaluated spectrally has also led to a meeting of minds on the issue. In fact, as has been pointed out by Fornberg (1974), the spectral definitions of derivatives can be interpreted as a weighted linear combination of neighbouring grid points in physical space. The weights just happen to be different from those of traditional finite differences. The purpose of this article is to point out that spectral multiplication can also be defined as a grid-point operation, not requiring any transformation to spectral space, and to investigate some of the properties of this nonlinear operation when viewed in physical space.

2. Formulation

Consider the multiplication of two variables $a(\lambda)$ and $b(\lambda)$ which are 2π periodic function of λ and represented by a *truncated* Fourier series of the form,

$$a(\lambda) = \sum_{m=-M}^{M} A(m) \exp\{im\lambda\} \tag{1}$$

$$b(\lambda) = \sum_{m=-M}^{M} B(m) \exp\{im\lambda\} \tag{2}$$

The result of the spectral multiplication of a and b, denoted by c, is defined as,

$$c(\lambda) = \sum_{m=-M}^{M} C(m) \exp\{im\lambda\} \tag{3}$$

where

$$C(m) = \sum_{n=-M}^{M} \sum_{p=-M}^{M} A(n)\, B(p)\, I(m,n,p) \tag{4}$$

and $I(m,n,p) = 1$ if $m = n + p$; and is zero otherwise.

Now there is a one-to-one correspondence between values of a, b and c at the set of equally spaced grid points $\lambda_j = j2\pi N^{-1}$ ($N = 2M + 1$) and A, B and C based on relations of the form,

$$a_j = a(\lambda_j) = \sum_{m=-M}^{M} A(m) \exp\{im\lambda_j\}$$

$$A(m) = N^{-1} \sum_{j=1}^{N} a_j \exp\{-im\lambda_j\} \tag{5}$$

and therefore there must be a relationship between the grid-point values of c and the products of grid-point values of a and b.

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By substitution of relations like (5) into (4) and (3), and after changing the order of summation we find that

$$c_l = \sum_{j=1}^{N} \sum_{k=1}^{N} P_{ljk}\, a_j\, b_k \tag{6}$$

where the multiplication matrix is defined as

$$P_{ljk} \equiv \frac{1}{N^2} \sum_{m=-M}^{M} \sum_{n=-M}^{M} \sum_{p=-M}^{M} \times \exp\{im\lambda_l - in\lambda_j - ip\lambda_k\}\, I(m,n,p) \tag{7}$$

The most simple grid-point definition of multiplication would be $P_{ljk} = 1$ if $l = j = k$ and zero otherwise. However, grid-point modellers often employ local averaging so that $P_{ljk} \neq 0$ for additional values of the indices when their differences are small. *In any case we see that spectral multiplication can be viewed as a particular definition of multiplication on the grid in physical space.*

3. Some properties of the multiplication matrix

Some of the properties of P_{ljk} can be determined without explicitly evaluating the summation (7). We simply state the properties we have found and their significance. Those properties which are not easily proven are demonstrated in an appendix.

(A) P_{ljk} depends only on the difference between indices

(B) $P_{ljk} = P_{lkj}$

(C) $\sum_{k=1}^{N} P_{ljk} = \delta_{lj}$ and therefore $\sum_{j=1}^{N}\sum_{k=1}^{N} P_{ljk} = 1$

(D) The spectral multiplication is not in general associative

(E) A combination of spectral multiplication and spectral differentiation ensure that $\frac{\partial}{\partial\lambda} a^2 = 2a \frac{\partial a}{\partial \lambda}$ at grid points.

(F) It is not possible to express P_{ljk} as the product of two-dimensional matrices, i.e. $P_{ljk} \neq Q_{lj} Q_{lk}$.

If A were false, the definition of multiplication would have preferred locations in space. If B were false the definition of a square would be strange and if C were false so would the multiplication of a field by a constant. Property D results from the fact that spectral multiplication is unaliased. Property E is reflective of the quadratic invariant properties of the spectral method. Property F was determined by Platzman (1961) and implies that it is not possible to split the spectral multiplication into two linear transformations of the grid-point fields followed by a point-by-point multiplication. *However, it does not mean that a spectral mutliplication requires that the fields be transformed to spectral space.*

4. Explicit evaluation of the multiplication matrix

The mulitplication matrix P_{ljk} can be evaluated explicitly by careful consideration of the *finite* nature of the summation (7). The result is

$$P_{ljk} = \frac{1}{N^2} \frac{(-)^{j-k}}{\sin\theta} [G(\theta') - G(\theta'')] \tag{8}$$

where $\theta = (j-k)\,\pi N^{-1}$, $\theta' = (l-j)\,\pi N^{-1}$, $\theta'' = (l-k)\,\pi N^{-1}$, and

$$G(X) \equiv \frac{\sin (M+1)\, X \sin MX}{\sin X}.$$

The formula fails for $j = k$ and $j = k = l$. Taking appropriate limits we find that

$$P_{ljj} = \frac{1}{N^2} \left(\frac{\sin M\theta'}{\sin \theta'}\right)^2 \tag{9}$$

and

$$P_{lll} = 1 - \frac{M(M+1)}{N^2}$$

The detailed structure of the multiplication matrix is presented in Tables 1 and 2 for $M = 4$ and 8 respectively. Only half the values are presented because of symmetry. It is noteworthy that $P_{lll} \geqslant 3/4$ for any resolution, giving a substantial weight to the grid-point product at the point in question. Otherwise the elements of the matrix generally get smaller as the points combined are increasingly separated from the point in question. It can be shown that for a *fixed distance* of separation of points that all elements (except P_{lll}) approach zero with the square of the resolution. Also, if we investigate the limiting values of the matrix elements as M gets large, for a *fixed grid-point number separation*

Table 1. *The elements of the multiplication matrix P_{0jk} for $M = 4$. The index j varies horizontally, k varies vertically*

	−4	−3	−2	−1	0	1	2	3	4
−4	.0053								
−3	−.0464	.0123							
−2	.0046	.0378	.0035						
−1	−.0464	.0378	−.1089	.1024					
0	.0053	.0123	.0035	.1024	.7531				
1	.0303	−.0464	.0378	−.1089	.1024	.1024			
2	.0086	.0086	.0046	.0378	.0035	−.1089	.0035		
3	.0086	−.0247	.0086	−.0464	.0123	.0378	.0378	.0123	
4	.0303	.0086	.0086	.0303	.0053	−.0464	.0046	−.0464	.0053

Table 2. *Same as Table 1 except for $M = 8$*

	−8	−7	−6	−5	−4	−3	−2	−1	0
−8	.0016								
−7	−.0211	.0024							
−6	.0014	.0183	.0012						
−5	−.0096	.0033	−.0247	.0044					
−4	.0013	.0056	.0011	.0226	.0010				
−3	−.0096	.0056	−.0136	.0072	−.0367	.0116			
−2	.0014	.0033	.0011	.0072	.0010	.0349	.0009		
−1	−.0211	.0183	−.0247	.0226	−.0367	.0349	−.1034	.1016	
0	.0016	.0024	.0012	.0044	.0010	.0116	.0009	.1016	.7509
1	.0172	−.0211	.0183	−.0247	.0226	−.0367	.0349	−.1034	.1016
2	.0020	.0020	.0014	.0033	.0011	.0072	.0010	.0349	.0009
3	.0050	−.0087	.0050	−.0096	.0056	−.0136	.0072	−.1367	.0116
4	.0028	.0018	.0018	.0028	.0013	.0056	.0011	.0226	.0010
5	.0028	−.0072	.0027	−.0072	.0028	−.0096	.0033	−.0247	.0044
6	.0050	.0018	.0027	.0027	.0018	.0050	.0014	.0183	.0012
7	.0020	−.0087	.0018	−.0072	.0018	−.0087	.0020	−.0211	.0024
8	.0172	.0020	.0050	.0028	.0028	.0050	.0020	.0172	.0016

	1	2	3	4	5	6	7	8
1	.1016							
2	−.1034	.0009						
3	.0349	.0349	.0116					
4	−.0367	.0010	−.0367	.0010				
5	.0226	.0072	.0072	.0226	.0044			
6	−.0247	.0011	−.0136	.0011	−.0247	.0012		
7	.0183	.0033	.0056	.0056	.0033	.0183	.0024	
8	−.0211	.0014	−.0096	.0013	−.0096	.0014	−.0211	.0016

it is found that the magnitude of the elements fall off with the square of the separation. These results imply that the extent of nonlocalness of spectral multiplication should be dependent on the square of the resolution.

Some of the elements are negative reminding one of the possibility of the square of a field being negative, at least in certain regions. It cannot be proven that the square of a field (under the spectral multiplication) is positive definite under all circumstances. One would expect such a property could be ensured only if aliasing were permitted.

An interesting property which was discovered by summing the computed matrix elements is that $\sum_{j=1}^{N} P_{ljj} = 1$. It can also be proved by examining the definition of the multiplication matrix.

The reason why it is an interesting property can be demonstrated as follows. Suppose $b = a$ and a is a field of random numbers whose ensemble average $\overline{a_j a_k} = a^2 \delta_{jk}$. Then it follows that the ensemble average of $c = a^2$ is given by

$$\bar{c}_l = \sum_{j=1}^{N} \sum_{k=1}^{N} P_{ljk} \overline{a_j a_k}$$

$$= a^2 \sum_{j=1}^{N} \sum_{k=1}^{N} P_{ljk} \delta_{jk}$$

$$= a^2 \sum_{j=1}^{N} P_{ljj} = a^2$$

as one would expect.

5. Final remarks

The above results raise the intriguing possibility that one might construct a spectral model without ever transforming variables into spectral form! However, it is clear that the direct method of spectral multiplication is less efficient than a transform technique since the number of operations required is $O(N^3)$ whereas the transform technique requires $O(N \log N)$. Such a disparity would be even more apparent in two or three dimensions.

6. Appendix

In this appendix we demonstrate properties D and E.

Property D

Given three variables a, b and c defined on an equally spaced grid, is there any difference between $d \equiv (a * b) * c$ and $d' \equiv a * (b * c)$? If we use the convention that repeated indices are to be summed over the appropriate range we have

$$d_j = P_{jkl}(a * b)_k c_l,\; d'_j = P_{jkl}\, a_k (b * c)_l$$

Since $(a * b)_k = P_{krs} a_r b_s = P_{krs} a_s b_r$, then

$$d_j - d'_j = (P_{jrl} P_{rks} - P_{jkr} P_{rls})\, a_k\, b_s\, c_l$$

It follows that $d_j - d'_j$ will be zero for arbitrary a_k, b_s, c_l if and only if

$$P_{jrl} P_{rks} - P_{jkr} P_{rls} \equiv 0$$

Such a property seems unlikely and indeed we can show that it is not generally valid by a sample calculation with $M = 1$.

Property E

Just as one can define a multiplication matrix one can form a spectral differentiation matrix. In fact a spectral (or pseudospectral) definition of a derivative is

$$\left(\frac{\partial a}{\partial \lambda}\right)_j = \sum_{k=1}^{N} a_k\, d_{kj}$$

$$d_{kj} = \frac{1}{N} \sum_{m=-M}^{M} im \exp\{im(\lambda_j - \lambda_k)\}$$

For property E to be valid we should have

$$P_{jkl}\, d_{ql}\, a_k\, a_q = \tfrac{1}{2} d_{lj} P_{lkq}\, a_k\, a_q$$

for all j and where a repeated index implies summation. From the theory of quadratic forms we know that such a condition implies that

$$G_{jkq} \equiv P_{jkl}\, d_{ql} - \tfrac{1}{2} d_{lj} P_{lkq}$$

be antisymmetric in the indices k, q. Further manipulation with dummy indices shows that we require

$$P_{jkl}\, d_{ql} + P_{jql}\, d_{kl} - d_{lj} P_{lkq} = 0$$

By substituting the definitions of P_{jkl} and d_{lj} into the above expression and noting that

$$\sum_l \exp\{i\lambda_l(p - q)\} = N\, \delta_{pq}$$

we find that

$$P_{jkl}\, d_{ql} + P_{jql}\, d_{kl} - d_{lj} P_{lkq}$$

$$= \left(\frac{1}{N}\right)^2 \sum_{m=-M}^{M} \sum_{n=-M}^{M} \sum_{p=-M}^{M} i(n + p - m)$$

$$\times \exp\{im\lambda_l - in\lambda_k - ip\lambda_q\} * I(m, n, p)$$

which proves property E.

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О СЛЕКТРАЛЬНОМ УМНОЖЕНИИ, РАССМАТРИВАЕМОМ В ФИЗИЧЕСКОМ ЛРОСТРАНСТВЕ

Изучаются свойства слектрального умножения, рассматриваемого в физическом лространстве. Определяется матрица умножения, которая позволяет производить умножение двух полей, заданных на сетке, без элиазинга. Использование матрицы умножения, по-видимому, менее эффективно, чем метод преобразования, но оно позволяет исследовать нелокальность спектрального умножения.