Hydromagnetic symmetrical terms for the zonal kinetic energy equation

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ABSTRACT

The symmetrical formulation of the mean zonal kinetic energy equation, derived by Starr & Gaut, is extended to include magnetic effects that give rise to additional energy integrals in the balance equations of a hydromagnetic fluid, confined to the polar cap of a rotating spherical shell. The form of these additional magnetic energy integrals is mathematically and physically analogous to the form of the energy integrals that appear in the purely hydrodynamic part of the energy equation. Various aspects of the equation are discussed when magnetic effects are included.

I. Introduction

Starr & Gaut (1969) derived a symmetrical form of the mean zonal kinetic energy equation which illustrates conveniently the role of the basic rotation Ω, the relative mean motion $[u]$, and the eddy motion u' in the generation and dissipation of kinetic energy of the mean flow in a hydrodynamic fluid confined to the polar cap of a rotating spherical shell. A symmetrical formulation of the equation is also possible for stellar atmospheres in the presence of a magnetic field. In addition to purely dynamic actions, magnetic effects will then give rise to Maxwell stress-work transport terms that generally act to destroy mean zonal kinetic energy.

This paper is concerned with the reformulation of the zonal kinetic energy equation to include these effects. The treatment represents a generalization to spherical coordinates of this equation which appears among others in the more complete system of hydromagnetic energy balance equations derived by Starr & Gilman (1966). At the same time, the present treatment is an extension to hydromagnetic fluids of the analysis by Starr & Gaut (1969), henceforth identified as S & G. The system treated, the rotation, and the general methods are identical to those of S & G for the purely hydrodynamic part of the balance equation.

II. Basic equations

The fundamental angular momentum balance equation in terms of the symbolism introduced by S & G becomes

$$\varrho \frac{dM}{dt} = -\frac{\partial P}{\partial \lambda} + r\mathfrak{F}_m + r\mathfrak{F} \tag{1}$$

where $r = R \cos \phi$ is the torque arm, R being the radius, ϕ latitude and λ longitude. The equation expresses the fact that the time rate of change of absolute angular momentum M of a unit mass of density ϱ is equal to the sum of the torques due to the pressure P, the magnetic force $\mathfrak{F}_m$, and the friction force $\mathfrak{F}$. Our primary concern here is the modification introduced by the magnetic torque, $r\mathfrak{F}_m$, that gives rise to additional energy integrals in equation (17) of the S & G paper. In this case, $\mathfrak{F}_m$ represents in the spherical coordinates of eq. (1) the zonal component per unit volume of the magnetic force vector

$$\mathbf{F} = \frac{1}{4\pi} (\nabla \times \mathbf{H}) \times \mathbf{H} \tag{2}$$

where $\mathbf{H}$ is the magnetic field vector. Here the standard magnetohydrodynamic approximations have been made (see Shercliff, 1965 for example), which also imply that

$$\nabla \cdot \mathbf{H} = 0 \tag{3}$$

Our immediate task is to write eq. (2) in spherical coordinates and to extract the zonal component. Before doing so, we note from vector algebra and by virtue of (3) that

$$\frac{1}{4\pi}(\nabla\times\mathbf{H})\times\mathbf{H} = -\frac{1}{8\pi}\nabla(\mathbf{H}\cdot\mathbf{H}) + \frac{1}{4\pi}\nabla\cdot(\mathbf{H}\,\mathbf{H}) \quad (4)$$

where the first term on the right side represents the gradient of the magnetohydrostatic pressure Conversion of this term into spherical coordinates is effected by merely replacing the ∇-operator by its spherically equivalent form.

The last term in equation (4) denotes the divergence of a dyad. Physically it represents a tension along the magnetic lines of force. Expansion of this term by the product rule of differential calculus yields derivatives (with respect to the curvilinear distances) of the Maxwell stresses and of the cartesian unit vectors. These latter derivatives are due to the converging meridians and the diverging radii in the spherical shell geometry and may be converted into trigonometric vector quantities by standard procedures (see, e.g., Haltiner & Martin, 1957, p. 167). Having done this, the zonal component of equation (4) yields after some rearrangement, the magnetic torque per unit volume

$$r\mathfrak{F}_m = \frac{1}{4\pi}\left\{\frac{\partial}{\partial\lambda}(H_\lambda^2) + \cos\phi\frac{\partial}{\partial\phi}(H_\lambda H_\phi) - 2\sin\phi \times H_\lambda H_\phi + 3\cos\phi\, H_\lambda H_R + R\cos\phi \times\frac{\partial}{\partial R}(H_\lambda H_R) - \frac{\partial}{\partial\lambda}\left(\frac{1}{2}\mathbf{H}^2\right)\right\}, \quad (5)$$

where H_λ, H_ϕ, and H_R are the magnitudes of the magnetic field vector in the zonal, meridional and vertical (zenith) directions, respectively.

III. Formation of the Maxwell stress energy integrals

Reasoning in exactly similar fashion as in S & G, we obtain, after multiplying (5) by the zonally averaged angular velocity $[u]/r$ and integrating over the volume, τ, of the spherical cap, the contribution of the magnetic torque to the total kinetic energy in the system as defined by eq. (12) in S & G. This contribution is

$$\int_\tau \frac{[u]}{r} r\mathfrak{F}_m\, d\tau = \frac{1}{4\pi}\iint\oint\frac{[u]}{r}\left\{\frac{\partial}{\partial\lambda}(H_\lambda^2) + \cos\phi\frac{\partial}{\partial\phi} \times (H_\lambda H_\phi) - 2\sin\phi\, H_\lambda H_\phi + R\cos\phi \times\frac{\partial}{\partial R}(H_\lambda H_R) + 3\cos\phi\, H_\lambda H_R - \frac{1}{2}\frac{\partial}{\partial\lambda}(\mathbf{H}^2)\right\} R^2\cos\phi\, dR\, d\phi\, d\lambda \quad (6)$$

where the volume element is $d\tau = R^2\cos\phi dR d\phi d\lambda$ and the square brackets define the zonal average $[(\ \ )] \equiv 1/(2\pi)\oint(\ \ )\,d\lambda$. The first and last terms on the right side of the equation vanish upon evaluating the inner cyclic integral.

The remaining integrals in (6) may be manipulated to yield a variety of different forms. For our purpose, we may use the results of S & G as a guide to obtain a symmetrical form of the integrals, each of which should consist of a product of three factors: the moment of inertia, the Maxwell stress, and the gradient of the mean relative angular velocity.

It is immediately apparent that with the aid of the identity

$$R^2\cos\phi\sin\phi \equiv -\frac{1}{2}\frac{\partial}{\partial\phi}(R^2\cos^2\phi), \quad (7)$$

the second and third integrals in (6) may be combined and progressively rearranged by integration by parts to give

$$\frac{1}{4\pi}\iint\oint\frac{[u]}{r}\left\{\cos\phi\frac{\partial}{\partial\phi}(H_\lambda H_\phi) - 2\sin\phi\, H_\lambda H_\phi\right\} \times R^2\cos\phi\, dR\, d\phi\, d\lambda$$
$$= \frac{1}{4\pi}\iint\oint\frac{[u]}{r}\frac{\partial}{\partial\phi}\{R^2\cos^2\phi\, H_\lambda H_\phi\}\, dR\, d\phi\, d\lambda$$
$$= \frac{1}{4\pi}\iint_{\phi_1}^{\pi/2}\frac{\partial}{\partial\phi}\left\{\frac{[u]}{r}R^2\cos^2\phi\left(\oint H_\lambda H_\phi\, d\lambda\right)\right\} \times dR\, d\phi$$
$$-\frac{1}{4\pi}\iint R^2\cos^2\phi\left(\oint H_\lambda H_\phi\, d\lambda\right) \times\frac{\partial}{\partial\phi}\left(\frac{[u]}{r}\right) dR\, d\phi \quad (8)$$

The fourth and fifth integrals of (6) may be similarly combined. Here we note that straight forward expansion of the fourth integral by successive integration by parts yields an integral that exactly cancels the fifth integral in the form given in eq. (6). We may write

$$\frac{1}{4\pi}\iint\oint\frac{[u]}{r}R^3\cos^2\phi\frac{\partial}{\partial R}(H_\lambda H_R)\,dR\,d\phi\,d\lambda$$

$$+\frac{3}{4\pi}\iint\oint\frac{[u]}{r}R^2\cos^2\phi\,H_\lambda H_R\,dR\,d\phi\,d\lambda$$

$$=\frac{1}{4\pi}\iint_{R_0}^{R_1}\frac{\partial}{\partial R}\left\{R^3\cos^2\phi\left(\oint H_\lambda H_R\,d\lambda\right)\frac{[u]}{r}\right\}$$

$$\times\,dR\,d\phi$$

$$-\frac{1}{4\pi}\iint R^3\cos^2\phi\left(\oint H_\lambda H_R\,d\lambda\right)$$

$$\times\frac{\partial}{\partial R}\left(\frac{[u]}{r}\right)dR\,d\phi \tag{9}$$

The boundary integrals implied in (8) and (9) are evaluated, respectively, at $\phi=\phi_1$ (representing the southern wall of the polar cap) and at $R=R_0$ and $R=R_1$ (representing the inner and outer surface of the spherical shell). It is assumed here for generality that the magnetic field penetrates all three bounding surfaces. This assumption is the magnetic analog to the bounding surfaces of an open region in hydrodynamics and implies that the magnetic field is continuous across the interface with the adjoining, electrically non-conducting medium (Chandrasekhar, 1961, p. 162). It is desirable in some applications to treat the adjoining medium as a perfect conductor. In that event the boundary integrals in (8) and (9) are zero since the magnetic components normal to the boundaries must vanish.

Following S & G, the cyclic integrals in (8) and (9) may be evaluated by expanding the magnetic field components into their zonally averaged mean [( )] and their deviation ( )′ from this mean. Thus

$$\oint H_\lambda H_\phi\,d\lambda=\oint([H_\lambda]+H'_\lambda)([H_\phi]+H'_\phi)\,d\lambda$$

$$=2\pi[H_\lambda][H_\phi]+2\pi[H'_\lambda H'_\phi], \tag{10}$$

and in similar fashion

$$\oint H_\lambda H_R\,d\lambda=2\pi[H_\lambda][H_R]+2\pi[H'_\lambda H'_R]. \tag{11}$$

Evaluation of the boundary integrals and substitution of (8)–(11) into (6) leads, after some slight rearrangement, to the final result.

Internal Horizontal Integrals

$$\iint 2\pi[u][\mathfrak{J}_m]\,R^2\cos\phi\,dR\,d\phi=$$

$$-\iint\tfrac{1}{2}R^2\cos^2\phi\,[H_\lambda][H_\phi]\frac{\partial}{R\partial\phi}$$

$$\times\left\{\frac{[u]}{R\cos\phi}\right\}R\,d\phi\,dR \qquad \text{(a)}$$

$$-\iint\tfrac{1}{2}R^2\cos^2\phi\,[H'_\lambda H'_\phi]\frac{\partial}{R\,\partial\phi}$$

$$\times\left\{\frac{[u]}{R\cos\phi}\right\}R\,d\phi\,dR \qquad \text{(b)}$$

Internal Vertical Integral

$$-\iint\tfrac{1}{2}R^2\cos^2\phi\,[H_\lambda][H_R]\frac{\partial}{\partial R}$$

$$\times\left\{\frac{[u]}{R\cos\phi}\right\}R\,d\phi\,dR \qquad \text{(c)}$$

$$-\iint\tfrac{1}{2}R^2\cos^2\phi\,[H'_\lambda H'_R]\frac{\partial}{\partial R}$$

$$\times\left\{\frac{[u]}{R\cos\phi}\right\}R\,d\phi\,dR \qquad \text{(d)}$$

Free Vertical Boundary Integrals at $\phi=\phi_1$

$$-\int\tfrac{1}{2}R^2\cos^2\phi_1[H_\lambda][H_\phi]\frac{[u]}{R\cos\phi_1}dR \qquad \text{(e)}$$

$$-\int\tfrac{1}{2}R^2\cos^2\phi_1[H'_\lambda H'_\phi]\frac{[u]}{R\cos\phi_1}dR \qquad \text{(f)}$$

Free Horizontal Boundary Integrals at $R = R_0$ and at $R = R_1$

$$+ \int \tfrac{1}{2} R_1^2 \cos^2 \phi \, [H_\lambda] [H_R] \frac{[u]}{R_1 \cos \phi} R_1 \, d\phi \quad \text{(g)}$$

$$+ \int \tfrac{1}{2} R_1^2 \cos^2 \phi \, [H'_\lambda H'_R] \frac{[u]}{R_0 \cos \phi} R_1 \, d\phi \quad \text{(h)}$$

$$- \int \tfrac{1}{2} R_0^2 \cos \phi^2 \, [H_\lambda] [H_R] \frac{[u]}{R_1 \cos \phi} R_0 \, d\phi \quad \text{(i)}$$

$$- \int \tfrac{1}{2} R_0^2 \cos^2 \phi \, [H'_\lambda H'_R] \frac{[u]}{R_0 \cos \phi} R_0 \, d\phi \quad \text{(j)} \quad (12)$$

This is the contribution of the magnetic torque to the mean zonal kinetic energy of fluid motions in the presence of a general magnetic field. The complete mean zonal kinetic energy equation of a hydromagnetic fluid confined to a polar cap of a rotating spherical shell is given simply by adding eq. (12) to eq. (17) in the S & G paper.

IV. Discussion

There are several points to be noted regarding the integrals in eq. (12) and their form upon comparison with their counterparts in the purely hydrodynamic fluid system treated in S & G. In the discussion to follow, we restrict our remarks to the application of this equation to photospheric fluid motions, although the equation is sufficiently general to be applied to other stellar atmospheres as well.

(1) Eq. (12) is identical in form to eq. (17) of the S & G paper when in (17) the basic rotation Ω is neglected. In fact, if the Reynolds stresses are replaced by their corresponding Maxwell stresses, multiplied by the factor $-(1/4\pi)$, the equations are identical. The mathematical similarity of the two equations demonstrates the analogous roles played by these two physically quite different stress work-transport terms in the generation and destruction of kinetic energy of the mean zonal flow $[u]$.

(2) Each of the internal integrals in (12) involves, as in the hydrodynamic system in S & G, a component of relative angular momentum transport multiplied by the spatial gradient of the mean relative angular velocity. In integrals (*b*) and (*d*) this transport is due to the work done by horizontal and vertical eddy Maxwell stresses across surfaces of constant ϕ and R, respectively. In integrals (*a*) and (*c*) this transport is accomplished across the same respective surfaces by mean motions in the fluid. These mean motions stretch and re-orientate the axisymmetric magnetic field lines to give rise to energy conversions involving the symmetric components of the mean Maxwell stress. In particular, the energy conversion represented by integral (*a*) in the solar case (see, e.g., Babcock, 1961 and Starr & Gilman, 1966) is postulated to result from the generation of the toroidal field $[H_\lambda]$ through the stretching of the poloidal field lines into the zonal direction by the differential rotation of the sun.

(3) The horizontal and vertical boundary integrals in (12) likewise involve a component of angular momentum transport, multiplied by the relative mean angular velocity at the respective free boundaries. These integrals again are mathematically the magnetic analog of the corresponding boundary integrals in eq. (17) of the S & G paper.

(4) When eq. (12) and eq. (17) of the S & G paper are combined, the result is the mean zonal kinetic energy equation of a hydromagnetic fluid confined to the rotating system considered in S & G. Except for the work done by Reynolds and Maxwell stresses along the artificially introduced free boundaries, it is seen that physically a change in mean zonal kinetic energy in such a system can occur only through transport of angular momentum along the gradient of relative mean angular velocity. The direction of transport is determined by the sign and relative magnitudes of the Reynolds and Maxwell stress terms.

(5) Some estimates of the angular momentum flux in the photosphere and its adjacent layers were made by Starr & Gilman (1965*a*, *b*). These authors conclude that the bulk of energy resides in quasi-horizontal hydromagnetic disturbances that generate a net amount of mean zonal kinetic energy to maintain the differential rotation of the sun against frictional dissipation. The relevant energy conversion processes are given by integral (*b*) of eq. (12) and by integral {3} of eq. (17) in the S & G paper. Horizontal Reynolds stresses feed eddy kinetic energy into the differential rotation, while horizontal Maxwell stresses convert a portion of this mean zonal kinetic energy into eddy magnetic

energy. Thus, on the basis of this assessment, the net generation of kinetic energy of the mean zonal flow results in the photosphere principally from an up-gradient transport of angular momentum, associated with the zonal eddy motions u' and from a down-gradient transport of angular momentum, associated with the eddy toroidal magnetic field H'_{λ}.

(6) As was stated earlier, there are a number of ways in which eq. (12) could be mathematically manipulated and re-arranged with regard to the form of the boundary terms. In their present form, the boundary integrals in (12) represent physically the total transport of kinetic energy across the bounding surfaces due to the presence of the magnetic field. This is not the case for the boundary integrals in eq. (17) of the purely hydrodynamic system treated in S & G. Although in both equations, the transport of kinetic energy is accomplished through the boundary work done by the stress components, the boundary terms in (17) express mathematically only one of two physical modes of energy transport. The second mode of transport is the advection of kinetic energy by a component of velocity normal to the bounding surfaces. It involves a direct transfer of the specific form of kinetic energy under consideration. Starr & Sims (1970) re-formulated eq. (17) of the S & G paper in terms of this additional boundary transport. Since a similar contribution to the transport of kinetic energy by the magnetic field does not exist, eq. (12) may be combined without further alterations with eq. (26) in the paper by Starr & Sims. In the language of these authors, the resulting hydromagnetic energy equation then contains not only a set of boundary terms which together physically represent the total flux of mean zonal energy across the free boundaries, but also contains the corresponding internal integrals.

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СИММЕТРИЧНЫЕ ГИДРОМАГНИТНЫЕ ЧЛЕНЫ В УРАВНЕНИИ ДЛЯ ЗОНАЛЬНОЙ КИНЕТИЧЕСКОЙ ЭНЕРГИИ

Симметричная форма уравнения для зональной кинетической энергии, полученная Старром и Гаутом, обобщена с целью учета магнитных эффектов, которые приводят к дополнительным интегралам энергии в уравнениях баланса для гидромагнитной жидкости, прилегающей к полярной шапке вращающейся сферической оболочки. Форма этих дополнительных интегралов для магнитной энергии математически и физически аналогична форме интегралов энергии, имеющихся в чисто гидродинамической части уравнения энергии. Обсуждаются различные аспекты полученного уравнения энергии.