On trapped oscillations of a rotating fluid in a thin spherical shell

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ABSTRACT

The structure of the trapped modes conjectured by Stern (1963) and Bretherton (1964) is investigated in detail. Numerical values of the longer periods are found and an asymptotic formula derived for the shorter periods. Each mode is found to have pathological features near the bounding surface and to contain substantial regions where the fluid is at rest. It is suggested that the trapped modes might constitute virtually all the free periods of oscillation of the fluid and their physical relevance is examined.

Introduction

Consider an incompressible inviscid fluid contained between two rigid concentric spherical boundaries of radii a and b, where $(a-b)/(a+b)<1$, the whole rotating with uniform angular velocity Ω about a common diameter of the spheres. The free axially-symmetric small oscillations of the fluid, that are set up when an appropriate disturbance is given to the system, were considered by Stern (1963) with particular reference to the possibility of there being trapped modes, in which the region of disturbance is confined to within a small distance of the equational plane. Although the argument he gave in support of their existence may be criticized on mathematical grounds (Stewartson & Rickard, 1969), and indeed leads to incorrect values for the periods, a more convincing argument in favour of their occurrence was later supplied by Bretherton (1964).

The study of modal oscillations of a fluid in a shell is of interest in connection with problems in oceanography, meteorology (Greenspan, 1968, § 5.3) and geomagnetism (Hide & Stewartson, 1972). Little is known about their properties but the available information suggests that they are unusual. For example the *asymmetric* modes have certain pathological features and such commonly accepted assumptions as the square-integrability of the velocity field have been called into question (Stewartson & Rickard, 1969). It is worth while therefore probing deeper into the properties of trapped symmetric modes and we shall carry out this task in this paper.

Basic equation and formal solutions

Let the distance of any point of the fluid from the common centre be $r=b+(a-b)\,z$, let the latitude be $y[(a-b)/b]^{1/2}$ and let the corresponding components of velocity be $w[(a-b)/b]^{1/2}$ and u. Further let the frequency of the oscillation be $\omega' = 2\Omega\ [(a-b)/b]^{1/2}\omega$ and define the stream function $\psi^{(x,z)} \exp i\omega' t$ of the meridional motion by $\partial\psi/\partial y = w$, $\partial\psi/\partial z = -u$ so that ψ satisfies the boundary conditions

$$\psi(y,0)=\psi(y,1)=0 \quad \text{for all } y, \qquad (2.1)$$

Then Veronis (1963) and Stern (1963) showed that ψ satisfies the differential equation

$$(y^2-\omega^2)\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial y^2}+2y\frac{\partial^2\psi}{\partial y\,\partial z}+\frac{\partial\psi}{\partial z}=0 \qquad (2.2)$$

Given ψ, the azimuthal component of velocity follows at once. The condition for trapped modes can be formally stated as

$$\psi=0 \quad \text{if } |y|>Y_0 \qquad (2.3)$$

for some positive Y_0.

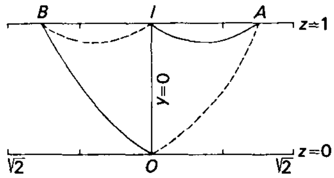

Fig. 1. An exceptional closed ray-pattern. – – –, Γ_+; ——, Γ_-; $\omega = \frac{1}{2}$; $IA = 1$.

The formal solution of (2.2) is

$$\psi = f((y+\omega)^2 - 2z) + g((y-\omega)^2 - 2z) \qquad (2.4)$$

where f, g are arbitrary functions. The characteristics are thus parabolas in this coordinate system: we shall denote by Γ_+ those of the form $(y+\omega)^2 - 2z = \text{const.}$, and by Γ_- those of the form $(y-\omega)^2 - 2z = \text{const.}$ In terms of Cartesian coordinates the characteristics are coaxial and similar cones so that in any meridional section they would appear as straight lines.

An exceptional closed ray-pattern

Bretherton showed that it is illuminating to consider the trapped modes in terms of the characteristics of (2.1) and pointed out that if a closed pattern can be built up from the characteristics, together with their reflections from the boundaries $z = 0,1$, then there is a possibility of a trapped mode. He gave two examples of which the first is shown in Fig. 1 and corresponds to $\omega = 1/2$. Suppose $f(\theta) = \alpha$ when $\theta = 1/4$. Then $g(\theta) = -\alpha$ at $\theta = 1/4$ from the boundary condition on ψ at (0,0), and $g(\theta) = -\alpha$ at $\theta = -7/4$ from the condition on ψ at $A(1,1)$. It follows that $f(\theta) = \alpha$ at $\theta = -7/4$, using the condition at I and finally $g(\theta) = -\alpha$ at $\theta = -1/4$ from the condition at $B(-1,1)$. Thus the ray pattern is consistent.

Nevertheless it is contended that this example is physically unacceptable. For if $\psi \neq 0$ at any point P inside the pattern it must be non-zero on at least one of the two characteristics through P, both of which penetrate outside the closed pattern of Fig. 1. Further, neighboring rays to OIABD do not recycle. Calculation shows that a ray passing through (ε, 0) where $\varepsilon < 1$ returns after appropriate reflections to

$$\varepsilon \pm \tfrac{16}{9}\,\varepsilon^2 + O(\varepsilon^3),\ 0]$$

Finally continuity of ψ precludes a wave restricted to the boundary of the pattern.

The first trapped mode

The second example given by Bretherton is shown in Fig. 2 and corresponds to $\omega = \frac{2}{3}\sqrt{2}$.

Only the part $y \geq 0$ of the plane is shown in the diagram, the remainder being obtained by reflection about OI and exchange of Γ_+ with Γ_-. Without any loss of generality we can suppose that $\psi = 0$ on $y = 0$. Solutions with $\partial\psi/\partial y = 0$ at $y = 0$ follow in a similar way and any trapped mode nust be a linear combination of the two. The closed ray pattern is given by $DABCD$ where the Γ_- characteristic touches $z = 0$ at A. It is easy to satisfy oneself that it is possible for $\psi = 0$ in IDC and beyond AB, while at the same time $\psi \neq 0$ in $DOABCD$, because any characteristic starting from any point in DOA remains inside $DOABCD$ even after reflections. Further inside $DABCD$ there are characteristics (Γ_+) which extend outside the $DOABCD$ but it is only necessary to ensure that $g = 0$ on these characteristics to preserve the trapped mode.

Only the pattern $DABCD$ of rays is closed and any other will, immediately or after sufficient number of reflections, depending on the sense in which the pattern is taken, tend to $DABCD$ in a monotonic way. Thus the successive points z_n, in which the pattern intersects OI, increase monotonically to $z_D = 4/9$ as a limit and similarly for the intersection of the pattern with $z = 0,1$. As a result $\partial\psi/\partial y$ is arbitrary on all points of $y = 0$ in the interval OM, where M is connected to $K(\omega, 1)$ by characteristics with one reflection only and $z_M = 0.2407$. For any characteristic, starting out from a point in this interval, next intersects $y = 0$ in the interval $M\,M_2$, which is in turn a subinterval of MD and includes the point H, where the characteristic from O next meets the line $y = 0$. Thus the value (z) of $\partial\psi/\partial y$ on OI can be regarded as a generalized periodic

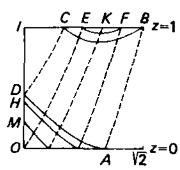

Fig. 2. One half of the closed ray-pattern for the first trapped mode. – – –, Γ_+; ——, Γ_-; $\omega = \frac{2}{3}\sqrt{2}$; $OA = \omega$.

function in that any particular value of the function is taken an infinite number of times but the distance between the corresponding and successive values of z is a function of z and tends rapidly to zero as $z \to 4/9$.

Let us consider the structure of the ray pattern near the limit cycle $DABCD$. Fix attention on the ray which passes through $(0, z_n = (4/9) - \varepsilon)$ where $0 < \varepsilon < 1$ following it in a clockwise direction. It passes successively through

$$(\tfrac{1}{3}\sqrt{2} + \tfrac{1}{2}\varepsilon\sqrt{2}, 1), \quad (\sqrt{2} - \tfrac{1}{2}\varepsilon\sqrt{2}, 1)$$
$$\times (\tfrac{2}{3}\sqrt{2} - \tfrac{5}{8}\varepsilon\sqrt{2}, 0 \tag{4.1}$$

terms $O(\varepsilon^2)$ being neglected. The next step carries the ray back to OI and we have

$$z_{n+1} = \frac{4}{9} - \frac{25\,\varepsilon^2}{64} + O(\varepsilon^3) \tag{4.2}$$

The period is asymptotically proportional to $(z - 4/9)^2$ and we may express this fact by defining

$$X = \log\log \frac{1}{\frac{4}{9} - z} \tag{5.3}$$

and $k(X)$ to be an arbitrary periodic function of X of period log 2. Then

$$h(z) \sim k(X) \quad \text{as } z \to \tfrac{4}{9}^- \tag{4.4}$$

An immediate consequence is that if s denotes the distance from one of the rays of the limit cycle $DABCD$, then the velocity component parallel to the ray, of the fluid in the oscillation,

$$\sim \frac{1}{s \log (1/s)} \quad \text{as } s \to 0^+ \tag{4.5}$$

The structure of the trapped wave is consequently highly pathological in the neighborhood of the limit cycle, the fluid velocity varying ever more rapidly as $s \to 0$ while its amplitude develops an unintegrable singularity. The pathology is in fact much more pronounced than that found by Stewartson & Rickard (1969) for asymmetric modes and their physical relevance is even more doubtful. We shall return to this question in the discussion.

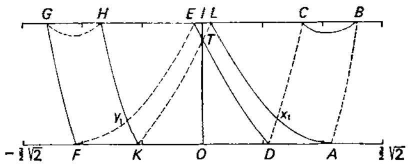

Fig. 3. The closed ray-pattern for the second trapped mode. – – –, Γ_+; ——, Γ_-; $\omega = 1.5005$; $OA = \omega$.

General properties of trapped oscillations

There are an infinite set of trapped modes of the kind exemplified in §3. We merely have to choose an ω such that the Γ_- characteristic through $(\omega, 0)$, which touches the line $z = 0$ at that point, after an appropriate number of reflections alternately from the boundaries $z = 0,1$ finally passes through the point $(-3\omega + (4\omega^2 + 2)^{1/2}, 1)$. The ray reflected from this point is a Γ_+ which does not intersect $z = 0$ but meets $z = 1$ again at $(\omega - \sqrt{4\omega^2 + 2}, 1)$. The pattern is half completed by a Γ_- ray from $(-\omega, 0)$ which must intersect the point $(\omega - \sqrt{4\omega^2 + 2}, 1)$. The remaining half of the cycle follows by symmetry. As already stated an infinite number of ω can be chosen for which the pattern is closed. The second trapped mode is shown in Fig. 3.

The cycle in this mode in $ABCDEFGHKLA$: ψ may be zero in the curvilinear triangles HY_1E, KTD, LX_1C and beyond FG and AB; in the remainder of the region $0 < z < 1$ it is non zero. In the nth trapped mode there are $2n + 1$ curvilinear triangles in which $\psi = 0$. Also the interval of the z axis (OI) on which ψ or $\partial\psi/\partial y$ is non-zero includes I and O alternately. If its length is Δ_n the range of values of z it $1 - \Delta_n < z < 1$ if n is even and $0 < z < \Delta_n$ if n is odd. Further the ray-pattern ABC – – – LA acts as a limit cycle in a similar way to §3, the other rays converging on it and the fluid speed piling up to a strong singularity. The first five trapped modes have been computed and in Table 1 the values of ω_n and Δ_n are displayed.

When n is large and there are a large number of reflections in the closed ray-patterns the value of ω may be estimated. Let three successive reflecting points be $(y, 0)$ $(y_1, 1)$, $(y_2, 0)$ and suppose that $|\omega \pm y| > 1$. We have

Table 1

n	ω_n	ω_n^2	$\frac{3n}{2} - 0.81$	Δ_n	$\frac{1}{2\omega_n^2}$
1	0.9428	0.8889	0.69	0.444	0.619
2	1.5005	2.2515	2.19	0.133	0.199
3	1.9274	3.715	3.69	0.118	0.135
4	2.2807	5.202	5.19	0.071	0.096
5	2.5877	6.696	6.69	0.066	0.075

$$(y_1+\omega)^2 - 2 = (y+\omega)^2,$$

so that

$$y_1 - y \doteqdot \frac{1}{\omega + y}.$$

Similarly

$$y_2 - y_1 \doteqdot \frac{1}{\omega - y}$$

and so, after two reflections y has increased by

$$\delta_y = \frac{2\omega}{\omega^2 - y^2} = \frac{\omega\,\delta m}{\omega^2 - y^2} \tag{5.1}$$

where m is the number of reflections. Consequently the total number of reflections needed to get from $(-\omega, 0)$ to $(+\omega, 0)$ is

$$m \doteqdot \int \delta m \doteqdot \int_{-\omega}^{\omega} \frac{\omega^2 - y^2}{\omega}\,dy \doteqdot \tfrac{4}{3}\omega^2 \tag{5.2}$$

For $n = 2$, $m = 3$; for $n = 3$, $m = 5$ and generally $m = 2n - 1$. Hence

$$\omega \approx \sqrt{\frac{3n}{2}} \tag{5.3}$$

The numerical values given in Table 1 suggest that a closer approximation is $\omega^2 = \frac{3}{2}n - 0.81$. Stern's approximate method gave $\omega^2 = n/4$ for all n.

The length of the interval Δ can also be estimated when $n < 1$. We begin the cycle at $A = (\omega, 0)$ and consider the difference in the y coordinates of L and E where L is $(y_L^{(1)}, 1)$ and is reached directly from A (i.e. without reflections) while E is $(y_E^{(1)}, 1)$ and is reached after three reflections. It may be verified that

$$y_L^{(1)} = \omega - \sqrt{2},\ y_E^{(1)} = \omega - \sqrt{2} - \frac{\sqrt{2}}{2\omega^2} + O(\omega^{-3})$$

After two more reflections each the corresponding points are

$$y_L^{(2)} = \omega - \sqrt{4} + O(\omega^{-1},)\ y_E^{(2)} = y_L^{(2)} - \frac{\sqrt{2}}{4\omega^2}\frac{1}{\sqrt{2}} + O(\omega^{-4})$$

and generally

$$\frac{y_L^{(i+1)} - y_E^{(i+1)}}{y_L^{(i)} - y_E^{(i)}} = \frac{\omega - y_L^{(i)}}{\omega - y_L^{(i+1)}}\left[1 + \frac{2\omega}{(\omega + y_L^{(i)})^2(\omega - y_L^{(i)})} + 0(\omega^{-3})\right] = \frac{\omega - y_L^{(i)}}{\omega - y_L^{(i+1)}}\,\frac{\omega + y_L^{(i)}}{\omega + y_L^{(i+1)}}[1 - 0(\omega^{-3})]. \tag{5.4}$$

Hence multiplying up, the difference between the two y coordinates, when $y_L^{(i)} \approx 0$, is

$$\approx (y_L^{(1)} - y_E^{(1)})(\omega - y_L^{(1)})(\omega + y_L^{(1)})/\omega^2 \approx \omega^{-3} \tag{5.5}$$

There are now two possibilities. Either one of the rays reflects off the line $z = 0$ and they both meet on the line $y = 0$ near $z = 0$ or both reflect off the line $z = 0$, one reflects off the line $z = 1$, and they both meet on $y = 0$ near $z = 1$. In either case

$$\Delta \sim \frac{1}{2\omega^2} \tag{5.6}$$

which is in fairly good agreement with the tabulated values of Δ.

5. Discussion

The trapped oscillations described in this paper form an infinite set of degenerate modes, for in each there is an interval of the line of symmetry $y = 0$ on which ψ and $\partial\psi/\partial y$ may be prescribed arbitrarily. The domain in which they occur may be defined as follows. Two of the characteristic cones touch the inner sphere along small circles, which we define to be b_+ and b_-. Let the member of the other family of characteristics which also passes through b_+ be D_- and define D_+ similarly. Then we see that the domain of the trapped mode lies between D_+ and D_- when the frequency of oscillation is sufficiently small that the semi-vertical angle of the cones is $O((a-b)/(b+a))$. The prescription may however be extended to include characteristic cones of any angle. It is only necessary to ensure that by following the characteristic cones through their reflections we eventually get from b_+ to b_-. Further the concept of trapping can be generalized mathematically by permitting the reflecting cones from b_+ to pass right round the shell any number of times before touching the inner sphere again although such modes could not be regarded as trapped in the physical sense. Such modes one might expect to have the same general properties as those discussed here, particularly the same pathological features near the limit cycle of the closed ray-pattern. Further if the frequency of the oscillation is ω' so that $\omega' = 2\Omega((a-b)/(a+b))^{1/2}\omega$, the area of the plane of symmetry ($\theta = 0$) on which ψ or $\partial\psi/\partial y$ can be prescribed arbitrarily $\sim a(a-b)/\omega^2$ when $\omega \gg 1$ [4.5] and hence $\sim (a-b)^2$ when $\omega' \sim \Omega$. Thus the trapped wave is basically concentrated over a very narrow area when the trapped mode extends a significant distance from the equator. The only other symmetric modes known are those studied by Stewartson & Rickard (1969) who attempted to find oscillations which are smooth in the limit $a \to b$. They found that in that event the velocity must be singular ($\sim 1/\Gamma\cos\theta$) at the poles ($\theta = \pm\pi/2$) and that the pressure must have a logarithmic singularity there, which they found unacceptable, unless $\omega' = 2\Omega$. The solutions investigated here, whatever their other drawbacks, imply a finite pressure since along any ray the pressure variation is proportional to the variation of the stream function.

When $b = 0$ it is known (Malkus, 1967) that there are ∞^2 symmetric modes of free oscillation none of which are pathological. Since the modes we discuss here are degenerate to an arbitrary function and are ∞ in number and cover the shell an arbitrary number of times it may well be that they constitute the limit of the symmetrical modes with $|\omega'| < 2\Omega$ $0 < b < a$ as b tends to a. If that were so one might imagine that for all shells the modes are pathological. It should be borne in mind, however, that the modes are unlike any discussed previously to the literature and it is far from certain whether they are acceptable. In fact there may be *no* acceptable inviscid modes in a shell. Satisfactory ways in which the dilemma posed by these trapped modes can be resolved are (i) to consider an initial value problem with the inviscid equations and examine the structure of the solution as $t \to \infty$, and (ii) to consider the effects of viscosity. It is quite possible that the limit of the viscous solutions, as the viscosity tends to zero, are not the inviscid solutions discussed here. Exploration of either of these suggestions is beyond the scope of this paper.

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О ЗАХВАЧЕННЫХ КОЛЕБАНИЯХ ВО ВРАЩАЮЩЕЙСЯ ЖИДКОСТИ, НАХОДЯЩЕЙСЯ В ТОНКОЙ СФЕРИЧЕСКОЙ ОБОЛОЧКЕ

Детально исследуется структура захваченных мод колебаний, существование которых было предположено Стерном (1963) и Брезертоном (1964). Найдены численные значения длинных периодов, а для коротких периодов получена асимптотическая формула. Найдено, что каждая мода имеет патологические особенности вблизи ограничивающей поверхности и содержит значительные области, где жидкость покоится. Предположено, что захваченные моды могут составлять фактически все периоды свободных колебаний жидкости. Исследуется физическое значение этих мод.