# Oscillations in a Viscous Liquid with an Application to Tidal Motion

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## Abstract

Oscillatory laminar flow is considered in a liquid consisting of two superposed layers of different densities and viscosities, and the steady second order drift velocity outside the boundary layer is calculated for various density and viscosity ratios and heights of interface above the boundary.

When the layer next to the boundary is the more viscous, and the density difference negligible, then the drift velocity exceeds Schlichting's value for a uniform liquid. The model is related to the motion of sediment layers in tidal estuaries, and a general landward drift of sediment is deduced.


## 1. Introduction

A first order oscillatory motion in a viscous liquid is known to generate a steady second order velocity just outside the laminar boundary layer. The earliest investigations were those of RAYLEIGH (1883) and SCHLICHTING (1932) who showed that this velocity is of magnitude $-3\,UU'/4\sigma$, where the oscillatory velocity parallel to the boundary is $U(x)\exp(i\sigma t)$. An important application of this is in the propagation of gravity waves in a fluid of finite depth. LONGUET-HIGGINS (1953) has generalised the result to complex values of $U(x)$, and, deriving the corresponding particle drift velocity, has used this as the required boundary condition near the bed for the second order drift currents under gravity waves of small amplitude. A knowledge of these drift currents is required in order to estimate the ability of gravity waves to cause erosion or silting. In shallow water however the bed material is frequently sand whose velocity is related to the local fluid velocity in a manner as yet not fully understood.

A more promising application concerns tidal motion in estuaries. The bed material is, on occasions, a thick liquid mud which behaves in many respects as a viscous liquid and whose motion necessitates subsequent dredging. We may reasonably expect the motion of such a material to follow more closely the results of simple fluid mechanics than sand. The first order tidal motion can be represented as a sum of simple harmonic constituents and the corresponding drift velocities calculated throughout the boundary layer. Schlichting's analysis considers a liquid of uniform viscosity and the resulting steady drift just outside the boundary layer is found to be independent of that viscosity. It does not follow that the same result will be found in a liquid of variable viscosity.

The purpose of the present paper is to consider oscillations in a liquid consisting of two layers of unequal viscosities and densities, and to determine the resulting drift velocity just outside the boundary layer. In the case of tidal estuaries the viscosity of the bottom layer of mud will be considerably greater than that of the water above, while its density is only slightly greater. Denoting the density and kinematic viscosity of the upper and lower layers by $\varrho$, $\nu$ and $\varrho_1$, $\nu_1$ respectively, typical ratios for heavy mud concentrations would be $\varrho_1/\varrho=2$ and $\nu_1/\nu=10^3$, while lower concentrations where $\varrho_1/\varrho\simeq 1{\cdot}2$ and $\nu_1/\nu\simeq 5$—10 are common. It is found that for large viscosity ratios, and when the density difference is neglected, the calculated drift current can greatly exceed Schlichting's value, depending on the depth of the more viscous layer adjacent to the rigid boundary. Conversely, a lower viscosity at the boundary leads to a reduction in the drift current. The generalisation to a number of harmonic constituents is given, and the application of the results to the distribution of bed material in an estuary is discussed.

## 2. Oscillations in a Stratified Fluid

We consider two superposed layers of fluid, their interface being parallel to a horizontal rigid boundary. The density and kinematic viscosity in the upper layer are $\varrho$, $\nu$, and $\varrho_1$, $\nu_1$ in the lower layer. The depth of the lower layer is $y_0=h\ (\nu/\sigma)^{\frac{1}{2}}$, where $\sigma$ is the frequency of an oscillatory motion introduced in the upper fluid

$$U(x,t)=U_0(x)e^{i\sigma t}, \tag{1}$$

where $U_0$ is real. If the horizontal component of velocity is written as

$$u(x,y,t)=u_0(x,y,t)+u_1(x,y,t) \tag{2}$$

where $u_0$ and $u_1$ are respectively the first and second order terms in ascending powers of the amplitude of the motion, then the boundary layer equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\varrho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \tag{3}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{4}$$

lead to the equations

$$\left.\begin{aligned}\frac{\partial u_0}{\partial t}-\nu\frac{\partial^2 u_0}{\partial y^2}&=\frac{\partial U}{\partial t}\\ \frac{\partial u_1}{\partial t}-\nu\frac{\partial^2 u_1}{\partial y^2}&=U\frac{\partial U}{\partial x}-u_0\frac{\partial u_0}{\partial x}-v_0\frac{\partial u_0}{\partial y}\end{aligned}\right\} \tag{5}$$

in the upper layer [Schlichting (1955), p. 181], and

$$\left.\begin{aligned}\frac{\partial u_0}{\partial t}-\nu_1\frac{\partial^2 u_0}{\partial y^2}&=\left(\frac{\varrho}{\varrho_1}\right)\frac{\partial U}{\partial t}\\ \frac{\partial u_1}{\partial t}-\nu_1\frac{\partial^2 u_1}{\partial y^2}&=\left(\frac{\varrho}{\varrho_1}\right)U\frac{\partial U}{\partial x}-u_0\frac{\partial u_0}{\partial x}-v_0\frac{\partial u_0}{\partial y}\end{aligned}\right\} \tag{6}$$

in the lower layer adjacent to the rigid boundary. The continuity equation (4) is to be satisfied by both the first and second order terms in each layer. The boundary conditions are

$$u_0=u_1=v_0=0 \qquad \text{on } y=0$$

and

$$u_0=U(x,t) \qquad \text{at } y=\infty$$

If we write

$$u_0=\partial\psi_0/\partial y,\quad v_0=-\partial\psi_0/\partial x,$$

where

$$\psi_0=(\nu/\sigma)^{\frac{1}{2}}\,U_0(x)F_0(\eta)e^{i\sigma t} \tag{7}$$

and

$$\eta=y(\sigma/\nu)^{\frac{1}{2}}$$

then in the upper fluid, region (a),

$$iF'_{0a}-F'''_{0a}=i \tag{8}$$

and in the lower fluid, region (b)

$$iF'_{0b}-(\nu_1/\nu)F'''_{0b}=i(\varrho/\varrho_1) \tag{9}$$

The solutions of (8) and (9) are

$$F_{0a}=\eta+Ae^{-(1+i)\eta/\sqrt{2}}+C \tag{10}$$

$$F_{0b}=d\{\eta+B(e^{-(1+i)\eta\gamma/\sqrt{2}}-1)+ + D(e^{(1+i)\eta\gamma/\sqrt{2}}-1)\} \tag{11}$$

$$\text{where } \gamma^2=\nu/\nu_1 \text{ and } d=\varrho/\varrho_1. \tag{12}$$

The boundary conditions are

$$F'_{0a}=1 \qquad \text{as } \eta\to\infty \tag{13}$$

$$F_{0b}=F'_{0b}=0 \qquad \text{on } \eta=0 \tag{14}$$

and

$$\left.\begin{aligned}F_{0a}&=F_{0b}\\ F'_{0a}&=F'_{0b}\\ \nu F''_{0a}&=\nu_1F''_{0b}\end{aligned}\right\}\text{ on } \eta=h \tag{15}$$

The three conditions (15) express the continuity of both components of velocity and the horizontal component of stress at the interface. The first two conditions in (13), (14) are already satisfied by (10) and (11). The remaining conditions give simple relations for the determination of the constants A, B, C, D, namely

$$\left.\begin{aligned} h + C + d(B + D - h) &= 0 \\ B - D - \sqrt{2}/\gamma(1 + i) &= 0 \\ Ae^{-(1+i)h/\sqrt{2}} - dB\gamma e^{-(1+i)h\gamma/\sqrt{2}} &+ \\ + dD\gamma e^{(1+i)h\gamma/\sqrt{2}} - (1 - d)\sqrt{2}/(1 + i) &= 0 \\ Ae^{-(1+i)h/\sqrt{2}} - dBe^{-(1+i)h\gamma/\sqrt{2}} &- \\ - dDe^{(1+i)h\gamma/\sqrt{2}} &= 0 \end{aligned}\right\} \quad (16)$$

To find the second order velocity we write

$$u_1 = \partial\psi_1/\partial y \qquad v_1 = -\partial\psi_1/\partial x$$

where

$$\psi_1 = (\nu/\sigma)^{\frac{1}{2}} U_0(x) U_0'(x) \frac{1}{\sigma} \{ F_{11}(\eta) e^{2i\sigma t} + F_{12}(\eta) \} \quad (17)$$

We are only concerned with the term $F_{12}(\eta)$ and from equation (6) it must satisfy the equations

$$-F_{12a}''' = \frac{1}{2} - \frac{1}{2} F_{0a}' F_{0a}'^{*} + \frac{1}{4}(F_{0a} F_{0a}''^{*} + F_{0a}^{*} F_{0a}'') \quad (18)$$

in region (a), and

$$-(\nu_1/\nu) F_{12b}''' = \frac{d}{2} - \frac{1}{2} F_{0b}' F_{0b}'^{*} + \\ + \frac{1}{4}(F_{0b} F_{0b}''^{*} + F_{0b}^{*} F_{0b}'') \quad (19)$$

in region (b). Substituting for $F_{0a}$ and $F_{0b}$ from (10) and (11), and integrating equations (18) and (19) twice gives the following solutions

$$F_{12a}' = -\frac{A}{4}\{\eta + C^* + 2\sqrt{2}(1 - i)\} e^{-(1+i)\eta/\sqrt{2}} + \frac{1}{4} AA^* e^{-\eta\sqrt{2}} - \\ - \frac{A^*}{4}\{\eta + C + 2\sqrt{2}(1 + i)\} e^{-(1-i)\eta/\sqrt{2}} + J\eta + K, \quad (20)$$

$$\frac{\nu_1}{\nu} F_{12b}' = \frac{d^2 B^*}{4\gamma^3}\{-(\gamma^2 + 1)(1 + i)\sqrt{2} + \gamma(B + D - \eta)\} e^{-(1-i)\eta\gamma/\sqrt{2}} + \\ + \frac{d^2 B}{4\gamma^3}\{-(\gamma^2 + 1)(1 - i)\sqrt{2} + \gamma(B^* + D^* - \eta)\} e^{-(1+i)\eta\gamma/\sqrt{2}} + \\ + \frac{d^2 D^*}{4\gamma^3}\{(\gamma^2 + 1)(1 + i)\sqrt{2} + \gamma(B + D - \eta)\} e^{(1-i)\eta\gamma/\sqrt{2}} + \\ + \frac{d^2 D}{4\gamma^3}\{(\gamma^2 + 1)(1 - i)\sqrt{2} + \gamma(B^* + D^* - \eta)\} e^{(1+i)\eta\gamma/\sqrt{2}} + \\ + \frac{d^2}{4}\{BB^* e^{-\eta\gamma\sqrt{2}} + DD^* e^{\eta\gamma\sqrt{2}} + DB^* e^{i\eta\gamma\sqrt{2}} + BD^* e^{-i\eta\gamma\sqrt{2}}\} + \\ + \frac{1}{4}\eta^2 d(1 - d) + E\eta + F, \quad (21)$$

where $E$, $F$, $J$ and $K$ are constants. The boundary conditions are

$$F_{12a}' = \text{finite} \qquad \text{as } \eta \to \infty \qquad (22)$$

$$F_{12b}' = 0 \qquad \text{on } \eta = 0 \qquad (23)$$

and

$$\left.\begin{aligned} F_{12a}' &= F_{12b}' \\ \nu F_{12a}'' &= \nu_1 F_{12b}'' \end{aligned}\right\} \quad \text{on } \eta = h \qquad (24)$$

Condition (22) requires that $J = 0$, while (23) and (24) determine $K$, $E$ and $F$.

The steady second order velocity just outside the boundary layer is, from (17) and (20)

$$\bar{u}_1 = \sigma^{-1} U_0 U_0' F_{12a}'(\infty) = \sigma^{-1} U_0 U_0' K. \quad (25)$$

$$Q_4 = \sin\Gamma\,(\gamma \sinh\Gamma + \cosh\Gamma)$$
$$Q_5 = \cos\Gamma\,(\gamma \sin\Gamma + \cos\Gamma)$$
$$Q_6 = \cos\Gamma\,(\gamma \sin\Gamma - \cos\Gamma)$$

After considerable simplification we find

$$P = Q_1^2 + Q_3^2 \qquad S = -(1-d)\,(Q_5 + Q_6) - 2dQ_1$$

$$\begin{aligned} K = F_{12a}'(\infty) = {} & \frac{\gamma^2 - 1}{4\gamma P}\left\{dh^2\gamma R + \gamma T - h\sqrt{2}\,T(Q_2 + Q_4)\right\} + \frac{S}{2\gamma^2 P}\{\gamma^2(2-d) - d\} + \\ & + \frac{h(\gamma^2 - d)}{2\sqrt{2}P}(S+R) + \frac{h(1-d)^2}{2\sqrt{2}\gamma}(\cos\Gamma \sinh\Gamma + \sin\Gamma\cosh\Gamma) + \frac{1}{4}h^2 d(1-d)(2-\gamma^2) + \\ & + \frac{h(1-d)}{4\sqrt{2}d\gamma P}\{(S-R)(Q_2 - \gamma Q_3 + dQ_4) + (S+R)(\gamma Q_1 - Q_4 - dQ_2)\} - \\ & - \frac{(\gamma^2+2)}{8\gamma^2 P}\left[\{T + P(1-d)^2 - R(1-d)\}(\cosh 2\Gamma - \cos 2\Gamma) + 2\gamma T \sinh 2\Gamma + \right. \\ & \left. + T\gamma^2(\cosh 2\Gamma + \cos 2\Gamma) - \gamma(1-d)(R\sinh 2\Gamma + S\sin 2\Gamma)\right] \end{aligned} \quad (26)$$

where $h\gamma/\sqrt{2} = \Gamma$, and

$$Q_1 = \sin\Gamma\,(\gamma\cosh\Gamma + \sinh\Gamma)$$
$$Q_2 = \cos\Gamma\,(\gamma\cosh\Gamma + \sinh\Gamma)$$
$$Q_3 = \cos\Gamma\,(\gamma\sinh\Gamma + \cosh\Gamma)$$

$$T = \{\cos\Gamma\cosh\Gamma\,(1-d) + d\}^2 + (1-d)^2\sin^2\Gamma\sinh^2\Gamma$$

$$R = 2dQ_3 + (1-d)\cdot\{Q_5 - Q_6 + 2\,Q_1\sinh\Gamma/\sin\Gamma\}.$$

It may be shown that this result reduces to that of Schlichting, namely

$$F_{12a}'(\infty) = -3/4, \quad \bar{u}_1 = -3\,U_0 U_0'/4\sigma \quad (27)$$

in the two limiting cases (i) $h \to 0$, and (ii) $\gamma = d = 1$.

Values of $-F_{12a}'(\infty)$ are shown in table 1 as a function of the non-dimensional parameters $h$ and $\gamma$ for the case $d = 1$, so that we assume that either $\varrho_1/\varrho \ll \nu_1/\nu$ or $\varrho/\varrho_1 \ll \nu/\nu_1$. When $\gamma < 1$, the viscosity near the boundary exceeds that in the upper region of the fluid, and it can be seen that the drift velocity is greater than the value given by (27) for a uniform fluid, particularly for large values of $h$. When $\gamma > 1$, the fluid near the boundary is less viscous than in the upper region, and the drift velocity is less than in a uniform fluid but still in the same direction.

**Table 1. The function $-F_{12a}'(\infty)$ for values of $\gamma$ and $h/\sqrt{2}$ when $d = 1$.**

| $\dfrac{y_0}{(2\nu/\sigma)^{\frac{1}{2}}}$ | $\gamma = (\nu/\nu_1)^{\frac{1}{2}}$ | | | | |
|---|---|---|---|---|---|
| | 0.01 | 0.1 | 0.5 | 1.0 | 10.0 |
| 10.0 | 5.7361 | 5.7022 | 2.5963 | 0.7500 | 0.2500 |
| 2.0 | 1.7621 | 1.7407 | 1.0756 | 0.7500 | 0.2508 |
| 1.0 | 1.2624 | 1.1904 | 0.8360 | 0.7500 | 0.2559 |
| 0.1 | 0.8000 | 0.7995 | 0.7883 | 0.7500 | 0.4521 |
| 0.0 | 0.7500 | 0.7500 | 0.7500 | 0.7500 | 0.7500 |

Asymptotic forms of the solution (26) are of use in showing the effect of extreme viscosity ratios. As $\gamma \to 0$, the fluid near the boundary being highly viscous, (26) tends to the value

$$F_{12a}'(\infty) = -\frac{3}{4} - \frac{h}{2\sqrt{2}}(2-d) - \frac{h^2}{2d}(1-d)^2 \quad (28)$$

and for all $h \neq 0$ and $d < 2$ this is greater in magnitude but of the same sign as Schlichting's result.

As $\gamma \to \infty$, the limit is finite only for $d = 1$, and then

$$F_{12a}'(\infty) = -1/4. \quad (29)$$

## 3. Further Generalisations

In problems of tidal oscillations in estuaries it is not permissible to ignore the effects of friction, one of which is to cause a progression in the time of high water. The motion is therefore, to the first order, a number of superimposed standing oscillations whose phases are functions of the distance, and so in equation (1) we must regard $U_0(x)$ as complex. For a uniform fluid the steady second order drift velocity is from (27)

$$\bar{u}_1 = -3U_0U_0'/4\sigma$$

where $U_0$ is real. When generalised to complex $U_0$ one readily finds

$$\bar{u}_1 = -\frac{3}{8\sigma}\{U_0' U_0^* + U_0 U_0'^*) - \\ - i(U_0' U_0^* - U_0 U_0'^*)\} \qquad (30)$$

and for a stratified fluid we may modify (25) in the same way, replacing $-3/8$ in (30) by $\frac{1}{2}F_{12a}'(\infty)$.

Equation (30) gives the steady term in the horizontal component of fluid velocity at a given position $x$. The corresponding mean velocity for a particle initially at this position was shown by Longuet-Higgins (1953) to be

$$\bar{U}_1 = \bar{u}_1 + \overline{\frac{\partial u_0}{\partial x}\int u_0\,dt + \frac{\partial u_0}{\partial y}\int v_0\,dt}$$

which gives the result

$$\overline{U}_1 = -\frac{1}{8\sigma}\{3(U_0' U_0^* + U_0 U_0'^*) - \\ - 5i(U_0' U_0^* - U_0 U_0'^*)\}. \qquad (31)$$

$\overline{U}_1$, the particle drift velocity, differs from $\bar{u}_1$ only when $U_0$ is complex, so that in the case of standing oscillations the difference is due to friction. For a stratified fluid the particle drift velocity is

$$\overline{U}_1 = \bar{u}_1 + \frac{i}{4\sigma}(U_0' U_0'^* - U_0 U_0'^*) \qquad (32)$$

where $\bar{u}_1$ is the modified form of (30) referred to above.

Another consideration in applying this theory to tidal motion is that the first order motion consists of more than one harmonic component. If we consider the oscillatory motion to be of the form

$$U(x,t) = \sum_n U_n(x)e^{i\sigma_n t} \qquad (33)$$

then it is not difficult to show that no contributions to the drift velocity arise from cross products of terms in the series (33), and that the total drift is the sum of expressions of the form (31) generalised as indicated in the case of a stratified fluid.

## 4. Application to Tidal Estuaries

The simplest representation of tidal motion is as a standing oscillation, with an antinode at the landward tidal limit, and a node or amphidromic point situated far out to sea beyond the region of present interest. The length of the estuary is therefore less than a quarter-wavelength. Although friction causes the time of high water to become progressively later as the tidal limit is approached, the phase relation between current and surface elevation remains basically that of a standing oscillation and not a progressive wave [see for example PROUDMAN (1953), § 158]. As RAYLEIGH (1883) and LONGUET-HIGGINS (1953) have pointed out, there exists beneath a standing oscillation a circulation within the boundary layer, the drift being towards the antinodes at the outer edge of the layer and towards the nodes very close to the bed. The drift towards the nodes is confined to a very narrow region and its magnitude is much less than the antinodal drift above it, which asymptotically is given by the Schlichting solution (27). It is to be expected then that the drift just outside the boundary layer, towards the antinodes, is the more efficient in transporting sediment when this is present in suspension to a depth greater than the boundary layer thickness. This is a landward drift throughout an estuary of constant or slowly varying section. The preceding results show that the existence of a more viscous layer near the bed increases the magnitude of this landward drift, so we are assured that provided the sediment concentration near the bed decreases upwards, then (27) will give an underestimate of the drift velocity.

In addition to the general landward drift, we can also deduce certain local effects. The cross-sectional area of an estuary varies along its length not monotonically but as a fluctuating function of the distance from the tidal limit. It follows that $dU/dx$ will vanish at a number of points along the estuary, and in the case of a pure standing oscillation sediment will accumulate in regions where the function $U(x)$ is a minimum. The effect of friction and fresh water river flow is to displace these regions by an amount which can be computed from the mean current data for any particular estuary.

Finally, regarding the accuracy of the analysis in section 2, it should be added that in accordance with the usual formulation of the boundary layer equations gravitational terms have been omitted. Since a small density difference is required to maintain the stratification, a density current will arise in the presence of an inclined boundary. We therefore suppose the slope to be sufficiently small for the density current to be neglected by comparison with the second order viscous drift velocity. Hence we require that $g \sin \alpha (\varrho_1 - \varrho)/\varrho \ll \nu U/h^2$.

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