Dynamical Aspects of the General Circulation and the Stability of Zonal Flow

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Abstract

From solutions of the linearized vorticity equation applied to systems with a variable basic zonal current, it is shown that the effect of damped disturbances is to feed energy into the zonal flow and transfer angular momentum from zones of easterlies into zones of westerlies, thereby strengthening the zonal circulation of the atmosphere. Amplifying disturbances withdraw energy from the zonal flow and transfer angular momentum from zones of westerlies into zones of easterlies, thereby weakening the strong zonal circulation and transforming it into cellular vortex motions. Thus the migratory cyclones and anticyclones are necessary elements for developing and maintaining the zonal circulation, while, on the other hand, they also act occasionally as brakes on the general circulation, keeping it from growing indefinitely.

The neutral disturbance corresponding to a typical zonal wind profile in winter is investigated by numerical integration. The stream line pattern is found to be rather similar to a typical flow pattern in the upper troposphere during a period of little change, indicating that this flow pattern is associated with the nearly neutral or slightly damped oscillations.

1. Introduction

When the motions of the atmosphere have been smoothed by taking averages along the latitude circles and over longer periods to eliminate the transient irregularities, we obtain systems of zonal currents, the so called prevailing westerlies and easterlies, which represent the more prominent overall pattern of the general circulation. This zonal wind system is interrupted from time to time by the presence of disturbances, both in the westerlies and easterlies, the number of which is much greater at lower levels than at upper levels. One of the fundamental problems that has always confronted meteorologists is the explanation of the nature of the physical processes that take place in the general circulation and also in the secondary circulations associated with it. It is evident that a complete answer to this problem can be achieved only through many different approaches which take into account physical processes in the atmosphere, among which those maintaining the global balance of energy and of momentum for the zonal wind systems are fundamental.

The concept of the balance of angular momentum in the general circulation was used first by JEFFREYS (1926), who pointed out that the maintenance of the prevailing winds in a steady state against surface friction necessitates a transfer of westerly angular momentum from the zone of surface easterlies into the zone of westerlies. However, the prevailing zonal winds are by no means steady. They go through cycles of decay and rebuilding, in addition to the mean seasonal changes in

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intensity and location. Statistical investigations have shown that the general circulation in middle latitudes tends to oscillate between extremes of strong and weak zonal motions, the time interval between consecutive maxima being of the order of magnitude of 4—6 weeks (ALLEN, 1940; WILLETT, 1947) Thus the study of the general circulation should not be restricted to the problem of maintaining the steady zonal winds against ground friction, but should also include the mechanisms for their growth and decay. However, it is most probable that the process that maintains the zonal winds is also responsible for their growth and decay and therefore the two problems can be studied simultaneously. But in view of the complexities of the natural process, we cannot expect to obtain a complete solution of the problem from any single set of equations; the physical processes themselves probably work through many different stages, so that a reasonable solution can only be achieved through many different steps. The purpose of this paper is to investigate some essential mechanisms pertaining to the overall atmospheric circulations, that are directly connected with the development of the zonal winds.

In the past, the mechanism of the general circulation which maintains the energy and momentum balances, has by and large been looked upon as consisting of a large-scale convective process, with ascending motions in lower latitudes and descending motions in higher latitudes, the development of a large zonal velocity being attributed to the action of the deflecting force of the earth's rotation. Recently the importance of the mean vertical motion for the maintenance of the zonal wind systems has been questioned by STARR (1948 a, 1949), who suggests that the horizontal energy and momentum transfer can be accounted for by the horizontal components of the general circulation and of the lesser circulations, or by the eddies with quasi-vertical axes. This idea has also been suggested by Rossby in his theory of horizontal mixing (1947, 1949). The investigations of WIDGER (1949), MINTZ (1949) and WHITE (1950) have substantially confirmed this point of view; therefore the possibility of the existence of mean meridional circulations will be disregarded in this study.

In the atmosphere the zonal wind systems are most clearly defined above about 5 kms. Above this level, especially around the level of the tropopause where the maximum of the meridional momentum transfer occurs, the transfer of heat (WHITE, 1950) is relatively of little importance and the motion of the air is controlled largely by the equation of vorticity about the vertical and the continuity equation. Below this level, the heat supply and heat transfer must play an important role and the instability of this layer, which can be inferred from the presence of the large number of disturbances there, is probably associated with these factors, although the exact nature of this instability is still not very clear to us. This type of instability has been discussed by CHARNEY (1947) EADY (1949) JAW (1946) and BERSON (1949). On the other hand, in the upper troposphere and lower stratosphere, the atmosphere generally moves with a persistent flow pattern, indicating that it is in a relatively stable state, while very unstable conditions happen only occasionally (for example, the formation of cut-off lows and highs). Since in this study we are mainly concerned with the processes which govern the maintenance and development of the zonal wind system, it is sufficient if we restrict the investigation to those processes which are prominent in the upper part of the troposphere, assuming the existence and not concerning ourselves with the origin of disturbances in this field. We shall thus divide the atmosphere roughly into two layers and study the motions within the upper layer only. However, this partition into two layers is taken only as a convenient means of analysis rather than a real barrier for transfer from one region to the other. Energy can be received from the lower layer through irregular disturbances, and can also be given by the upper to the lower layer on some occasions. Thus there is a mechanism in replenishing the kinetic energy of the upper system, even though we treat it as barotropic. In this way, we consider the lower layer merely as a source or sink of disturbances for the upper layer, and the divergence and vertical motion as a means of communication between these two layers.

In order to separate the divergence effect from the overall processes, we shall assume that it takes place intermittently within some specific time intervals, so that after some disturbances have been produced in the upper layer, the subsequent motions are horizontal nondivergent and therefore governed by the simple vorticity equation without divergence. In this way, the most important problems concerning the dynamics of the upper atmosphere can be discussed by the solutions of the simple vorticity equation. It can be shown by integrating the equations with respect to the mass distribution along the vertical that the mean motion of the whole atmosphere can also be approximately described by this two-dimensional barotropic model. Therefore the results obtained may also be considered to represent the mean motion.

Since there is no other energy source in this simplified model, the total kinetic energy must remain constant during the time under consideration. Thus any change of the kinetic energy of the mean zonal flow must come from the kinetic energy of the disturbances, and the study amounts to an investigation of the interactions between the lesser circulations and the zonal wind systems. In the following, the method of harmonic perturbations (both in x and in time t) will be adopted. The problem is therefore reduced to the discussion of the stability of the zonal wind profiles with respect to the disturbances. Here the term "Stability" is defined according to the reactions of the disturbances to the zonal wind systems in the horizontal plane. In this respect three kinds of disturbances can be distinguished: the disturbances which withdraw kinetic energy from the mean zonal flow are said to be *amplified*, those which supply kinetic energy to the mean zonal flow are said to be *damped* and those which neither give nor take energy from the mean zonal flow are *neutral*. Since this study is not aimed at explaining the origin of the disturbances (although it may help to explain the development of some large disturbances), these disturbances are taken to have been added to the upper layer from below. It then becomes physically feasible to discuss the different effects produced by damped as well as amplified disturbances. We shall demonstrate later that one of the effects is to transfer westerly angular momentum from the belts of slower zonal flow into belts of stronger zonal flow, therefore to strengthen the prevailing zonal winds and to overcome the frictional effects, although the disturbances themselves carry no resultant momentum. On the other hand, the amplified disturbances take kinetic energy from the main flow and transfer zonal momentum away from the zone of strong westerlies. Thus it appears that the zonal mean motion is not merely a statistical result of these irregular disturbances, as is frequently stated in descriptive studies of the general circulation. Rather the general circulation is built up and regulated by the disturbances (migratory cyclones and anticyclones). Its growth and decay are closely connected with the activity and development of the disturbances and these in return are connected with the stability of the zonal flow.

2. The rates of change of the mean zonal velocity and kinetic energy

The equations of motion over a spherical earth for horizontal and frictionless flow may be written

$$\frac{\partial u}{\partial t} - Zv = -\frac{1}{\varrho}\frac{\partial p}{a\cos\Phi\partial\lambda} - \frac{1}{2}\frac{\partial v_h^2}{a\cos\Phi\partial\lambda} \tag{1a}$$

$$\frac{\partial v}{\partial t} + Zu = -\frac{1}{\varrho}\frac{\partial p}{a\partial\Phi} - \frac{1}{2}\frac{\partial v_h^2}{a\partial\Phi} \tag{1b}$$

where $Z = f + \zeta = 2\,\omega\,\sin\Phi + \frac{1}{a\cos\Phi}\left[\frac{\partial v}{\partial\lambda} - \frac{\partial u\cos\Phi}{\partial\Phi}\right]$ is the vertical component of the absolute vorticity, hereafter referred to simply as the absolute vorticity, ζ the relative vorticity, u and v the zonal and meridional components of the horizontal velocity, $v_h^2 = u^2 + v^2$, ϱ the density, p the pressure, a the mean radius of the earth, and λ and Φ are the longitude and latitude respectively.

As only large-scale motions connected with the general circulation will be discussed, all quantities may be averaged along latitude circles. Thus the velocity components and vorticity can be written as $u = \bar{u} + u'$, $v = \bar{v} + v'$, and $Z = \bar{Z} + \zeta'$. Here the bars denote average values defined as $\bar{u} = (2\pi)^{-1}\int_0^{2\pi} u d\lambda$, and u', v', ζ' are the departures from the average values, which can also be taken as representing a perturbation. The motion is assumed to be purely horizontal and nondivergent within the time interval under consideration. This implies $\int_0^{2\pi} v\,d\lambda = 2\,\pi\bar{v} = 0$. In this respect the motion is similar to a geostrophic velocity field. Since the effect of the variation of the density ϱ in x-direction is small, ϱ will be taken as constant unless otherwise stated. The above condition then means that there is no net mass transport across the latitude circle at that level. If the motion at this level is representative of the mean atmospheric flow, this is equivalent to the condition

$$\int_0^\infty \int_0^{2\pi} \varrho v \cos \Phi \, d\lambda \, dz = 0.$$

When equation (1a) is integrated along the latitude circle and use is made of the nondivergent assumption, we find that the rate of change of the mean zonal velocity is given by

$$\frac{\partial u}{\partial t} = \frac{1}{2\,\pi} \int_0^{2\pi} v\zeta\, d\lambda = -\int_0^{2\pi} \frac{\partial (uv \cos^2 \Phi)}{2\,\pi a \cos^2 \Phi \partial \Phi} d\lambda = $$

$$= \frac{1}{a \cos^2 \Phi \varrho} \frac{\partial \tau}{\partial \Phi} \tag{2}$$

where $-\ 2\,\pi a^2\,\tau = \int_0^{2\pi} a^2 \cos^2 \Phi \varrho u v \, d\lambda =$ $= 2\,\pi a^2\, \overline{\varrho u v} \cos^2 \Phi$ represents the total northward transfer of angular momentum across the latitude circle. When the convergence of meridians is neglected, linear momentum can be used and $\tau = -\,\overline{\varrho u v}$. Thus the time rate of change of the mean zonal momentum in the narrow zone contained between the latitudes Φ and $\Phi + \delta\Phi$ is proportional to the net inward transport of angular momentum into this zone, namely $\partial \tau / a \cos^2 \Phi \partial \Phi$, or alternatively to the northward transport of vorticity.

In discussing the development of the general circulation it is useful to consider also the time rate of change of the kinetic energy of the mean zonal flow. This may be dealt with quite generally as follows. Multiplying the general momentum equation in the zonal direction (obtained by combining the first equation of motion and the continuity equation, both in three-dimensional forms) by the mean zonal velocity $\bar{u}$ and integrating over the whole volume Σ of air contained between the two latitude surfaces $\Phi = \Phi_1$, and $\Phi = \Phi_2$, we get

$$\frac{\partial \bar{E}_x}{\partial t} = \frac{1}{2} \int_\Sigma \frac{\partial \varrho \bar{u}^2}{\partial t}\, d\Sigma = -$$

$$-\ 2\,\pi a^2 \iint \left[\bar{\dot{\lambda}} \frac{\partial (\overline{\varrho u v} \cos^2 \Phi)}{\partial \Phi} + \right.$$

$$\left. + \bar{u} \cos \Phi \frac{\partial (\overline{\varrho u w})}{\partial z} - f \bar{u} \overline{\varrho v} \cos \Phi \right] d\Phi \, dz =$$

$$= \int_\Sigma \left[\overline{\varrho u' v'} \cos \Phi \frac{\partial \lambda}{\partial \Phi} + \overline{\varrho u' w'} \frac{\partial \bar{u}}{\partial z} \right] d\Sigma +$$

$$+ \int_\Sigma \left[\left(f + \cos \Phi \frac{\partial \bar{\dot{\lambda}}}{\partial \Phi} \right) \overline{\varrho v} \bar{u} + \right.$$

$$\left. + \overline{\varrho w} \bar{u} \frac{\partial \bar{u}}{\partial z} \right] d\Sigma + \int \overline{\varrho u v_n} \bar{u} \, dS \tag{3}$$

Here it is assumed that the variation of ϱ with longitude and with time are both small and have been neglected. Equation (3) shows that the kinetic energy of the mean zonal flow can change by a conversion from the kinetic energy of the eddy motions, an inflow of this energy through the boundaries, and possibly by the presence of a mean meridional cell. The pressure force does not occur in this equation, therefore this energy cannot be produced directly by the pressure force.

The significance of this equation for the study of the maintenance and development of the mean zonal winds has been discussed by the writer elsewhere (KUO, 1951). It has been shown that only the horizontal eddy stress $\overline{u'v'}$ and the mean meridional motion can serve as sources of kinetic energy of the mean zonal flow, and that the term $\overline{\varrho u' v'} \cos \Phi \dfrac{\partial \bar{\dot{\lambda}}}{\partial \Phi}$ in the last member of this equation is of the right order of magnitude and sign to account for the total frictional loss. We shall therefore restrict our investigations merely to this effect, which can be discussed adequately by using our simplified model.

Since the production of the total kinetic energy is also of paramount importance in understanding the physical processes, we may also write down the equation for the time rate of change of the total kinetic energy of the horizontal motion:

$$\frac{1}{2}\frac{\partial \varrho v_h^2}{\partial t} = -\nabla \cdot \left[p\vec{v}_h + \frac{1}{2}\varrho v_h^2 \vec{v}_h\right] + + p\nabla\cdot\vec{v}_h - \frac{\partial}{\partial z}\left(\frac{\varrho}{2}v_h^2 w\right) \quad (4)$$

where $\nabla\cdot$ denotes the horizontal divergence. The significances of the different terms have been discussed by Starr (1948 b). It may be noted that the pressure force is certainly a source for the total kinetic energy although not for the mean zonal kinetic energy, hence it may be concluded that its effect is to produce eddy and meridional motions.

Suppose we now integrate equation (4) over the whole region Σ bounded by two latitude surfaces along which $v = 0$, the earth's surface where $w = 0$ and the top of the atmosphere where $\varrho v_h^2 w = 0$ and the motion is assumed to be nondivergent, then all the terms on the right disappear. The equation then expresses the constancy of the total kinetic energy for the entire region. Suppose now the complete motion is divided into a mean zonal flow $\bar{u} = (2\pi)^{-1}\int_0^{2\pi} u\,d\lambda$ and a perturbation u', v' and denote the kinetic energy of the horizontal perturbation in Σ by E' then we find by making use of (2) and (4) that

$$\frac{\partial E'}{\partial t} = \frac{1}{2}\iint\frac{\partial}{\partial t}\left[\varrho(u'^2 + v'^2)\right]a^2\cos\Phi\,d\lambda d\Phi =$$

$$= -\frac{\partial \bar{E}_x}{\partial t} = -a^2\iint\varrho\bar{u}\frac{\partial\bar{u}}{\partial t}\cos\Phi\,d\lambda d\Phi =$$

$$= 2\pi a^2\int_{\Phi_1}^{\Phi_2}\tau\frac{\partial\dot{\lambda}_0}{\partial\Phi}\,d\Phi \quad (5)$$

where the integration along z has been omitted and where $\dot{\lambda}_0$ denotes the initial value of $\bar{u}/a\cos\varphi$. The same equation can be obtained from (3). It should be pointed out that equation (5) is applicable only when the entire region is considered. For any particular region or a small belt, the transport of kinetic energy through the boundaries must be taken into consideration and therefore we must use equation (4).

3. The perturbation equations and boundary conditions

In the upper troposphere and lower stratosphere, the large-scale motions are to a large extent horizontal, nondivergent and barotropic, we therefore assume that this motion may be described by the vorticity equation and the equation of continuity. Because of the difficulty of finding general solution of non-linear partial differential equation, the vorticity equation will be linearised and it then becomes

$$\left(\frac{\partial}{\partial t} + \dot{\lambda}_0\frac{\partial}{\partial\lambda}\right)\zeta' + \frac{v'}{a}\frac{dZ_0}{d\Phi} = \nu\nabla^2\zeta' \quad (6)$$

where u_0 is the basic current depending upon Φ only, Z_0 is the absolute vorticity of the basic current, ν is the kinematic viscosity. The suffix zero occuring in $\dot{\lambda}_0$ and z_0 denotes initial values of $\dot{\lambda} = \bar{u}/a\cos\Phi$ and $\bar{Z}$. This equation will be used to investigate the changes from the initial state. The frictional term on the right is generally small and can be neglected except as discussed below. Since the motion is assumed nondivergent, the continuity equation takes the form

$$\nabla\cdot\vec{v}_h = \frac{1}{a\cos\Phi}\left[\frac{\partial u'}{\partial\lambda} + \frac{\partial(v'\cos\Phi)}{\partial\Phi}\right] = 0 \quad (7)$$

and a stream function Ψ for the perturbation velocities may be introduced, defined by

$$\frac{v'}{a}\cos\Phi = \frac{\partial\Psi}{\partial\lambda},\ \frac{u'}{a} = \dot{\lambda}\cos\Phi = -\frac{\partial\Psi}{\partial\Phi} \quad (8)$$

where the dots indicate time derivatives. In terms of this stream function, the relative vorticity ζ' of the perturbation is given by

$$\cos^2\Phi\,\zeta' = \frac{\partial^2\Psi}{\partial\lambda^2} + \frac{\partial^2\Psi}{\partial y^2} \quad \text{where } y = = \log(\tan\Phi + \sec\Phi).$$

The perturbation stream function can be represented by the sum of a number of elementary independent harmonics of the form

$$\Psi = \Psi(y)\,e^{in(\lambda-\mu t)} \quad (9)$$

where n is the wave number around the globe and μ may be complex, $\mu = \mu_r + i\mu_i$; μ_r represents the angular phase velocity and $n\mu_i$ the amplifying factor of the particular harmonic disturbance. When (9) is substituted in the linearised vorticity equation (6) we find

$$(\dot{\lambda}_0 - \mu)(\Psi'' - n^2 \Psi) + \frac{dZ_0}{dy}\Psi =$$
$$= -i\nu n^{-1}[\Psi^{iv} - 2n^2\Psi'' + n^4\Psi] \quad (10)$$

where the primes denote differentiation with respect to y. This equation must be satisfied by all the component harmonic perturbations. Since ν is extremely small in the atmosphere, the terms on the right-hand side of (10) can generally be neglected. In this case we get the nonviscous differential equation

$$(\dot{\lambda}_0 - \mu)\left(\frac{d^2\Psi}{dy^2} - n^2\Psi\right) + \frac{dZ_0}{dy}\Psi = 0 \quad (11)$$

However, it should be pointed out that if the frictional term is neglected entirely in the vorticity equation (6), then no distinction can be drawn between amplified and damped disturbances, since either the harmonic factor $e^{in(\lambda-\mu t)}$ or $e^{-in(\lambda-\mu t)}$ will give the same differential equation (11). Therefore we must take (11) as the limit of (10) with ν extremely small, with the necessary corrections for its solutions in the vicinity of the singular points where $\mu_r = \dot{\lambda}_0$.

The general solution of (11) for each harmonic contains two arbitrary constants. No specific information can be obtained without imposing some restrictions on their values. The necessary restrictions are the boundary conditions that must be satisfied by each harmonic. In the atmosphere, the only natural horizontal boundary conditions are those needed to make the perturbations finite at every point, therefore Ψ and $d\Psi/dy$ must be equal to zero at the two poles. However, since the motion over the southern hemisphere is generally unknown to us, our discussion is necessarily restricted to the northern hemisphere. It therefore appears best to introduce the equator as a line of symmetry, and disregard all the seasonal variations. Thus we may use the following different set of boundary conditions under different circumstances: (a) $\Psi = 0$ at the pole, which must be satisfied, and either $d\Psi/dy = 0$ (symmetric) or $\Psi = 0$ (antisymmetric) at the equator. Each of these conditions will give a zero momentum transfer ($\tau = 0$) at the equator. In addition to the study of the hemisphere as a whole, we may consider a perturbation confined to a smaller belt. For this case, we use (a) $\Psi = 0$ along two latitude circles. Although these boundaries are rather arbitrary, it might be of interest for example to examine the belt within the subtropics which separates the easterlies from the westerlies. It may be noted that τ, defined in (2), vanishes at the boundaries.

By applying one boundary condition to the general solution of the particular harmonic, one of the two arbitrary constants can be eliminated, while the other boundary condition enables us to determine the permissible value of μ for any given n or conversely. Therefore the total perturbation stream function is given by

$$\Phi(t, \lambda, y) = \Sigma c_j \Psi_j(y) e^{in_j(\lambda-\mu_j t)} \quad (12a)$$

where the arbitrary constants c_j are determined by the initial disturbances

$$\Phi(0, \lambda, y) = \Sigma c_j \Psi_j(y) e^{in_j\lambda} \quad (12b)$$

Thus, the harmonics that are present at any moment correspond to those present at the initial moment and the solution is unique. When every component disturbance is known, then (12a) can be used to forecast the velocity field. But at the present, we can only discuss the general properties of the individual harmonic components as related to the profile of the basic zonal currents.

4. The solutions of the vorticity equation and the nature of the disturbances

The solution of equation (11) has been discussed by the author in a previous study (KUO, 1949) for the motion over an infinite plane or a cylindrical earth, in which the coriolis parameter was taken as a linear function of Φ and the profile of the basic current symmetrical. Though the results obtained there are still generally valid for the motion over spherical earth and for asymmetrical profiles, certain modifications are unavoidable and consequently we shall now discuss the problem further.

Equation (11) shows that if the phase velocity μ lies within the range of values of $\dot{\lambda}_0$, the point $y = y_s$ where $\mu = \dot{\lambda}_0$ is a singular point of the equation.[1] The solutions show certain peculiarities around this point and require careful examination. These questions do not exist for those disturbances whose phase velocity is distinct from the velocity of the basic current. It is therefore necessary to divide the disturbances according to their velocities of propagation into three groups: (a) those propagating faster than the maximum velocity max. $\dot{\lambda}_0$ of the basic current, (b) those travelling slower than the minimum basic current min. $\dot{\lambda}_0$, and (c) those propagating with a velocity within these two limits.

The disturbances in the groups (a) and (b) will be discussed very briefly. It can be shown, by writing $\frac{dZ_0}{dy} = h - d^2\dot{\lambda}_0/dy^2$ (where $h = 2(\omega + \dot{\lambda}_0) \operatorname{sec} h^2 y + 2 \tanh y d\dot{\lambda}_0/dy$ corresponding to β for motion in a plane) and employing Sturm's oscillation theorem, that under the conditions existing in the atmosphere ($h > 0$ generally) there can be no neutral disturbance in (a). That is to say, in general, no neutral disturbance that is governed by the simple vorticity equation will travel faster than the maximum velocity of the basic flow in middle latitude westerlies and equatorial easterlies. On the other hand, there always exist neutral disturbances in group (b), since $h(\mu - \dot{\lambda}_0)^{-1}$ is then negative (see KUO, 1949).

Now let us examine the possibility of having amplified or damped disturbances in these two groups. If μ is complex, Ψ must also be complex, $\Psi = \Psi_r + i\Psi_i$. Thus, when equation (11) is multiplied by the conjugate function $\Psi^\star = \Psi_r - i\Psi_i$, we get by separating the real and imaginary parts:

$$\frac{d}{dy}\left[\Psi_i \frac{d\Psi_r}{dy} - \Psi_r \frac{d\Psi_i}{dy}\right] = \frac{dW}{dy} = g_2 \frac{dZ_0}{dy} \,|\Psi|^2 \qquad (13a)$$

$$\frac{d}{dy}\left[\Psi_r \frac{d\Psi_r}{dy} + \Psi_i \frac{d\Psi_i}{dy}\right] = \left|\frac{d\Psi}{dy}\right|^2 + \left(n^2 - g_1 \frac{dZ_0}{dy}\right) |\Psi|^2 \qquad (13b)$$

where W is the Wronskian of Ψ_r and Ψ_i, $W = \Psi_i \frac{d\Psi_r}{dy} - \Psi_r \frac{d\Psi_i}{dy}$, and $g_1 = (\dot{\lambda}_0 - \mu_r) \,|\, \dot{\lambda}_0 - \mu \,|^{-2}$, $g_2 = \mu_i \,|\, \dot{\lambda}_0 - \mu \,|^{-2}$. Equation (13b), when integrated from one boundary to the other, determines the phase velocity. When (13a) is integrated from one boundary to the other we find

$$W_2 - W_1 = \int_{y_1}^{y_2} g_2 \frac{dZ_0}{dy} \,|\Psi|^2 dy \qquad (14)$$

which shows that a necessary condition for having a complex μ, or $g_2 \neq 0$, is to have $\frac{dZ_0}{dy}$ change its sign somewhere within the region, since the boundary conditions require both W_1 and W_2 to be zero. This condition holds both for the amplified and damped disturbances in the groups (a) and (b), since there is no singular point for these perturbations.

In the atmosphere, the absolute vorticity Z_0 is largely determined by the coriolis parameter and therefore increases with latitude over most of the hemisphere. From (13 b) it can be seen that under this condition there is no disturbance at all in group (a) and only neutral disturbances in group (b). Rossby's long waves are examples of these neutral disturbances.

Group (c). This group includes nearly all the wave disturbances actually observed in the atmosphere, which travel with phase velocities within the range of velocities of the basic current, min. $\dot{\lambda}_0 \leqslant \mu_r <$ max. $\dot{\lambda}_0$. For these disturbances, the points y_s where $\mu = \dot{\lambda}_0$ are singular points of (11). (For the jet stream profiles, there are generally two singular points for every value of μ, one on each side of the jet). Only one solution of the equation (11) is regular at this point, while the second involves a term with the factor log $(y - y_s)$ (KUO, 1949), and therefore is many-valued. Since the motion cannot be many-valued, the change of the value of ψ in passing from the

[1] Suffix S will always refer to the value of a function at the singular point $y = y_s$.

negative to the positive side of y_s must be determined properly, by finding the path of continuation around y_s. This is obtained by considering the solutions of (11) as the limits of the asymptotic solutions of (10) with a very small coefficient of viscosity. Is has been shown by many investigators (e.g., LIN, 1945; WASOW, 1947; KUO 1949) that this continuation must be taken along a path below the singular point $y(s_1)$ where $d\lambda_0/dy$ is positive and above the singular point $y(s_2)$ where $d\lambda_0/dy$ is negative. Therefore, the solutions should be taken as

$$\Psi_1 = \eta + a_2\eta^2 + \dots \tag{15a}$$

$$\left.\begin{aligned} \Psi_2 &= 1 + b_2\eta^2 + \dots + G_s \Psi_1 \log \eta && \text{for } \eta > 0 \\ \Psi_2 &= 1 + b_2\eta^2 + \dots + G_s \Psi_1 (\log|\eta| - \pi i) && \text{for } \eta < 0 \end{aligned}\right\} \tag{15b}$$

where

$$\eta = y - y_s \text{ and } G_s = -\left(\frac{d\lambda_0}{dy}\right)_s^{-1}\left(\frac{dZ_0}{dy}\right)_s = 2a_2.$$

However, according to LIN (1945) and WASOW (1947), the set of solutions are not valid within a small region around the singular point for the damped disturbances. This means that although the neutral and amplified disturbances can be discussed by employing the inviscid vorticity equation (11), some other effects need be considered within the small region around the singular points in discussing the damped disturbances. This point becomes clear in the case when the absolute vorticity Z_0 is a monotone increasing function of y.

With this in mind, we can use the solutions (15). According to the two expressions of Ψ_2 on the two sides of y_s, there is a change of the value of $\dfrac{d\Psi}{dy}$ from $y_s - \varepsilon$ to $y_s + \varepsilon$, given by

$$\left(\frac{d\Psi}{dy}\right)_{y=y_s+\varepsilon} - \left(\frac{d\Psi}{dy}\right)_{y=y_s-\varepsilon} = - \\ - \Psi_s \left|\frac{d\dot{\lambda}_0}{dy}\right|_s^{-1} \left(\frac{dZ_0}{dy}\right)_s \pi i \tag{16}$$

There is, therefore, a rapid change in u' across a singular point if $\left(\dfrac{dZ_0}{dy}\right)_s$ is not zero and ε very small.

Now we may proceed to discuss the stability of the zonal winds for the disturbances in group (c). Since the inviscid vorticity equation (11) is valid for all real y when $\mu_i > 0$, a necessary condition for the existence of amplifying disturbances, as can be seen from (14), is that dZ_0/dy should change its sign somewhere in the region, the same as for the disturbances in the groups (a) and (b). Unlike the disturbances in the groups (a) and (b), this condition is not necessary for damped disturbances, because equation (11) and its solution are not valid for some parts of the real y axis.

The same condition is also necessary for the existence of neutral disturbances in this group. This has been proved in the previous study by the writer (KUO, 1949) by considering the jump of the Wronskian across the singular point, given by

$$W(y_s + \varepsilon) - W(y_s - \varepsilon) = \\ = u \left(\frac{dZ_0}{dy}\right)_s \left|\frac{d\dot{\lambda}_0}{dy}\right|^{-1} |\Psi_s|^2 \tag{17}$$

Since for neutral disturbances both Ψ_r and Ψ_i must be solutions of equation (11), the variation of W is zero at all points except the singular points where there are jumps given by (17). As W is zero at the two boundaries, the sum of these jumps must be zero. For the symmetric jet profiles the values of the two jumps are the same; therefore each must be zero. Thus either $\left(\dfrac{dZ}{dy}\right)_s$ or Ψ_s must be zero. The impossibility of the latter has been proved by the writer (KUO, 1949) by the method of comparison, which can also be used for other profiles. It can also be proved by employing the expression (13b). Thus dZ_0/dy must change its sign and go to zero in the region. Also dZ_0/dy must be zero at the singular point for symmetrical jets; that is to say, this neutral disturbance travels with the velocity of the basic current $\dot{\lambda}_0(c)$ at the critical point[2] (the point where $\dfrac{dZ_0}{dy} = 0$). This is also true for the unsymmetrical jets when the value of $\dot{\lambda}_0$ is the same at the two critical points.

[2] Suffix c will always refer to the value of a function at the critical point $y = y(c)$.

For unsymmetrical jets in which the values of $\dot{\lambda}_0$ at the two critical points $y(c_1)$ and $y(c_2)$ are not the same, the condition that the total jump of W must vanish requires $\left(\frac{dZ_0}{dy}\right)_{s1}$ and $\left(\frac{dZ_0}{dy}\right)_{s2}$ to have opposite signs, which assumes the vanishing of dZ_0/dy somewhere in the region. This neutral disturbance travels with a phase velocity between the two values of $\dot{\lambda}_0$ at the two critical points, as has been proved by FOOTE and LIN (1950).

It has thus been shown that a necessary condition for the existence of amplifying disturbances is that dZ_0/dy should equal zero somewhere in the region and change its sign. The same condition is also necessary for the existence of neutral disturbances in group (c) and of damped disturbances in the two groups (a) and (b), but it is not necessary for damped disturbances in group (c).

5. Equations of angular momentum transfer, energy exchange and the stability of zonal flow

The relation between the meridional transport of angular momentum and the stability of the zonal flow has been discussed by the writer in a previous investigation (KUO, 1949). While the results obtained there are correct for the particular profile studied, they need some modifications and elaborations. The same problem has also been discussed by MACHTA (1949) and HASSANEIN (1949). These authors show that even with a constant basic current, some perturbations are able to produce a constant meridional transfer of momentum. However, these perturbations do not satisfy the boundary conditions of this paper, and furthermore, a constant momentum transfer without accumulation does not help the momentum budget, as can be seen from equation (2). Because of the importance of this momentum transfer and the energy exchange between the basic zonal current and the perturbations, they will be discussed in some detail in the following two sections.

The perturbation velocity components u' and v' can be expressed in terms of the stream function,

$$u' = -\left[\frac{d\Psi_r}{dy}\cos n\xi - \frac{d\Psi_i}{dy}\sin n\xi\right]e^{n\mu_i t}$$

and $v = -n[\Psi_r \sin n\xi + \Psi_i \cos n\xi]e^{n\mu_i t}$, where $\xi = \lambda - \mu_r t$ is the phase angle. From these expressions and by making use of equation (13a), the average northward angular momentum transfer across unit length of the latitude circle is found to be given by

$$\varrho \overline{u'v'}\cos^2\Phi = -\tau =$$

$$= \frac{1}{2}\varrho n\mu_i e^{2n\mu_i t}\int_{y_1}^{y}\frac{|\Psi|^2}{|\dot{\lambda}_0 - \mu|^2}\frac{dZ_0}{dy}dy \quad (18)$$

This transfer can also be expressed in terms of the tilting of the troughs and ridges, defined as the curves along which the meridional velocity is zero. The location of the trough and ridge curves is determined by $\tan n\xi = -\Psi_i/\Psi_r$ and therefore the angle α which they make with the northerly direction is given by

$$|\Psi|^2\tan\alpha = \Psi_r^2\sec^2 n\xi\,\frac{d(n\xi)}{ady} = W =$$

$$= \mu_i\int_{y_1}^{y}\frac{|\Psi|^2}{|\dot{\lambda}_0 - \mu|^2}\frac{dZ_0}{dy}dy \quad (19)$$

Since $|\Psi|^2$ is proportional to $\overline{v^2}$, we see that the angular momentum transfer given by equation (18) is proportional to the product of the mean kinetic energy of the perturbation and the tilting of the troughs and ridges.

The vorticity transport can either be obtained in the same way, or simply by differentiating (18) with respect to y. Thus we have

$$\overline{v'\zeta'} = \frac{1}{2\pi}\int_0^{2\pi} v'\zeta' d\lambda = \frac{dW}{dy}e^{2n\mu_i t} = -$$

$$-\frac{\mu_i}{|\dot{\lambda}_0 - \mu|^2}\frac{dZ_0}{dy}|\Psi|^2 \quad (20)$$

This equation shows that the vorticity transfer produced by the damped disturbances is in the direction of increasing Z_0 while that produced by the amplifying disturbances is in the direction of decreasing Z_0, therefore it is not always in the direction of the absolute vorticity gradient. When the zonal flow is stable, it actually is transported in the direction opposite to the gradient of vorticity, therefore has the effect of increasing this gradient; only a very unstable flow will result in a thorough mixing and a reduction of the existing vorticity gradient.

Substituting equation (18) into (2), we get the time rate of change of the mean zonal velocity:

$$\frac{\partial \bar{u}}{\partial t} = -\frac{n\mu_i}{2} e^{2n\mu_i t} \frac{dZ_0}{dy} \frac{|\Psi|^2}{|\dot{\lambda}_0 - \mu|^2} \quad (21)$$

It also follows that the effect of the damped disturbances is to produce an increase of $\bar{u}$ in the region where dZ_0/dy is positive and a decrease where dZ_0/dy is negative, thus their effect is to intensify the zonal flow. The effect of the amplifying disturbances is in the opposite direction, that is to flatten the jet-stream of westerlies and easterlies.

When the expression (18) for τ is substituted nto the energy equation (5), it becomes

$$\frac{\partial E'}{\partial t} = -a \int_{\Phi_1}^{\Phi_2} \dot{\lambda}_0 \frac{\partial \tau}{\partial \Phi} \partial \Phi =$$

$$= \frac{1}{2} \varrho n \mu_i e^{2n\mu_i t} \int_{\Phi_1}^{\Phi_2} \dot{\lambda}_0 \frac{|\Psi|^2}{|\dot{\lambda}_0 - \mu|^2} \frac{dZ_0}{d\Phi} d\Phi \quad (22)$$

From (22) it may be seen that if the perturbation kinetic energy is to increase ($\mu_i > 0$), then, besides the condititon that dZ_0/dy should change its sign somewhere in the region, we must also have positive dZ_0/dy associated with higher values of $\dot{\lambda}_0$ and negative dZ_0/dy with lower values of $\dot{\lambda}_0$. This may be considered as a second necessary condition for the occurrence of amplified disturbances and is equivalent to the condition given by FJÖRTOFT (1950). For the profile of a jet-stream of westerlies, this condition is automatically satisfied when dZ_0/dy changes its sign within the region. It is not necessarily satisfied however, for example, an easterly current around the equator can satisfy these requirements simultaneously only if the current is sufficiently strong.

It can now be shown that for the jet stream profiles, these two conditions are sufficient for the occurrence of amplified disturbances. That is to say, when they are fulfilled, equation (11) and the given boundary conditions actually permit solutions with a positive μ_j.

The existence of amplified disturbances for a symmetric jet stream velocity profile has been proved in a previous study (KUO, 1949) by applying the Sturm oscillation theorem. This is done by first establishing the existence of a neutral disturbance with $\mu_r = \dot{\lambda}_0(c)$. For the jet-type velocity profile in which the current velocities at the two critical points c_1 and c_2 are the same, and dZ_0/dy is positive in the central part and negative in the outer parts of the region, the ratio $[\dot{\lambda} - \dot{\lambda}(c)]^{-1} dZ_0/dy$ is always positive and finite. It follows that a neutral disturbance with $\mu = \dot{\lambda}_0(c)$ exists, since the solutions of the equation

$$\frac{d^2\Psi_1}{dy^2} = \left[n^2 - \frac{\frac{dZ_0}{dy}}{\dot{\lambda}_0 - \dot{\lambda}_0(c)} \right] \Psi_1, \ \Psi_1(0) = 0 \quad (23)$$

become oscillatory for sufficiently small values of n^2. When we investigate the eigen value $\mu = \mu_r + i\mu_i$ in the neighborhood of this eigen-value $\mu = \dot{\lambda}_0(c)$ we find that (KUO, 1949) if $\mu_r > \dot{\lambda}_0(c)$ then $\mu_i < 0$, and if $\mu_r < \dot{\lambda}_0(c)$, then $\mu_i > 0$. Hence the disturbances which are travelling with a phase velocity greater than the critical values $\dot{\lambda}_0(c)$ will be damped and those which are travelling with a phase velocity less than the critical value $\dot{\lambda}_0(c)$ will be amplified.

The existence of neutral and amplified disturbances for the profiles with $\dot{\lambda}_0(c_1)$ and $\dot{\lambda}_0(c_2)$ differing slightly from each other can be discussed by the argument used by FOOTE and LIN (1950). From the above discussions one can summarize the necessary and sufficient conditions for the existence of amplified and neutral disturbances (with $\mu_r = \dot{\lambda}_0$) as follows:

If dZ_0/dy equals zero and changes its sign somewhere in the region and if $\frac{dZ_0}{dy} > 0$ where $\dot{\lambda}_0$ is large and < 0 where $\dot{\lambda}_0$ is small, then amplified and neutral disturbances may exist.

A necessary and sufficient condition for the existence of damped disturbances is that $\frac{dZ_0}{dy}$ *is not identically zero in the entire region.*

In applying the equations (18) — (22) to the damped disturbances propagating with the current velocity of some point, it must be remembered however that the solution does not hold in the small region around the singular point. Since this interval is generally small, we can neglect the large fluctuations within it and just compute the change of τ across this interval from (17). Then equation (22) can also be used to discuss the damping effect. The dissipation of kinetic energy through friction within the narrow viscous strip can only increase this damping effect, and not reverse it.

6. Momentum and vorticity transfer and energy exchange for different velocity profiles

We shall next consider the meridional momentum and vorticity transfer and energy exchange between the perturbations and the mean flow for some typical profiles.

(a) *Jet-stream profile in which the values of $\dot{\lambda}_0$ at the two critical points are the same.*

Assume that there are two critical points $\left(\text{points where } \frac{dZ_0}{dy} = 0\right)$, $y(c_1)$ and $y(c_2)$, one

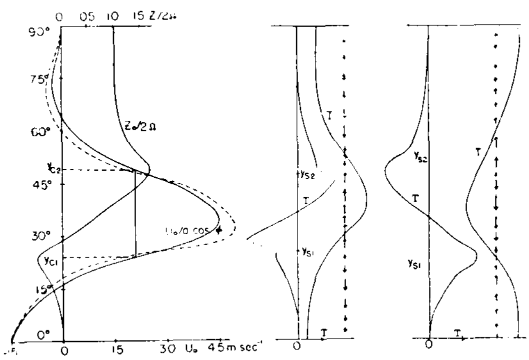

Fig. 1. (a) The mean zonal current and the mean absolute vorticity distribuions. The dashed curve represents the mean zonal winds produced by damped disturbances at some later time. (b) Schematic orientation of the troughs and ridges (T) and the meridional distribution of the stress $\tau = -\overline{\varrho u'v'}\cos^2\Phi$ for damped disturbances; the arrows indicate direction of vorticity transport. (c) Same as (b) for amplified disturbances.

to each side of the jet stream of westerlies. The profile must be such that Z_0 decreases with latitude in the outer parts of the region and increases in the central part, as is shown in fig. 1. For this profile, a neutral disturbance exists which propagates with the velocity $\dot{\lambda}_0(c)$, which has its troughs directed from north to south, which gives zero momentum transfer at all points, and which causes no energy exchange between the basic flow and the perturbation.

Now we discuss the damped disturbance which moves with a velocity only slightly greater than the critical velocity $\dot{\lambda}_0(c)$, and having singular points. The eddy stress τ (southward angular momentum transport) produced by this damped disturbance ($\mu_i < 0$) decreases slowly from zero at $y = y_1$ to a negative value up to the point $y(s_1) - \varepsilon$, according to equation (18); then it decreases rapidly across the singular point, corresponding to the change of W given by (17), up to $y(s_1) + \varepsilon$, since dZ_0/dy is positive at $y(s_1)$. Although the inviscid theory fails to be valid within the small belt $2\,\varepsilon$ around $y(s_1)$ we may approximate to the distribution of τ within this small interval by the dotted curve, as is indicated in fig. 1b. As the change of τ across this interval is known, its values at all other points where the inviscid theory holds can be computed from (18). Thus it increases from a negative value at $y(s_1) + \varepsilon$ to a positive value at $y(s_2) - \varepsilon$, where it has another rapid decrease up to $y(s_2) + \varepsilon$, and then decreases to zero as the other boundary is approached. The vorticity transfer is northward in the central region between $y(c_1)$ and $y(c_2)$ and southward in the outer regions, as denoted by the small arrows. The direction of the trough and ridge lines for these damped disturbances is represented by the curve T. It is seen that the troughs and ridges are oriented with their convex-side toward the east in the vicinity of the jet stream, and momentum is transferred from the outer parts into the central part, (from regions where the zonal current is small into the regions where it is great), therefore having the effect of intensifying the zonal currents. It is also seen that τ and the wind shear $\frac{d\dot{\lambda}_0}{dy}$ have opposite signs in most parts of the region;

therefore the damped disturbances supply their energy into the main zonal flow. The dashed curve in fig. 1a represents the zonal wind distribution produced by the damped disturbances at some later time.

For the amplifying disturbances, τ is given by (18) at every point. It increases from zero at $y = y_1$ to a positive value, has a higher rate of increase near the singular point, and attains its maximum value at $y(c_1)$, then decreases to a negative minimum value at $y(c_2)$, and increases to zero as the other boundary is approached, as is shown in fig. 1c. Thus positive τ is associated mainly with positive $d\dot{\lambda}_0/dy$ and negative τ with negative $d\dot{\lambda}_0/dy$, showing that the kinetic energy of the perturbation will increase with time. The vorticity transfer is in the opposite direction as produced by the damped disturbances. The trough lines and ridge lines are oriented with their convex side toward the west. There is a flow of angular momentum from the central part into the outer parts, with the effect of flattening the profile of the zonal motion.

For the disturbances without singular points ($\mu_r \neq \dot{\lambda}_0$), the general features of the distribution of τ and the orientation of the trough and ridge-lines are the same as have been discussed above.

The results obtained above agree with those obtained in a previous study (KUO, 1949) for symmetric profiles and over a plane or a cylindrical earth. When the asymmetry is not great, the general conclusions obtained also hold for the profile with different values of $\dot{\lambda}_0$ at the two critical points. For this kind of profile, a constant momentum transport can be produced between the two critical points by a neutral disturbance as has been discussed by FOOTE and LIN (1950). However, no other effect on the mean zonal flow will be produced by this neutral disturbance except a gradual reduction of this difference of $\dot{\lambda}_0$ between the two critical points.

(b) *Jet is limited to middle latitudes while the vorticity distribution at higher and lower latitudes is dominated by the coriolis parameter.*

In the atmosphere, even when the westerlies are strong, and the absolute vorticity has a maximum to the north and a minimum to the south of the jet, the negative gradient of vorticity generally does not extend to the equator, but Z_0 attains a second weaker maximum at a lower latitude, showing the predominance of the effect of earth's rotation. This type of vorticity distribution, such as is represented in fig. 2, occurs much oftener than the case (a) and therefore deserves separate discussion.

In this case, there and four critical points in the zonal wind profile where dZ_0/dy vanishes, the two inner ones having higher values of $\dot{\lambda}_0$. If $\dot{\lambda}_0$ is the same at these two inner critical points, $y(c_2)$ and $y(c_3)$, a neutral disturbance will travel with this velocity. Another neutral disturbance corresponding to the two outer critical points may also exist.

For damped disturbances, the distribution of τ and the direction of the troughs and ridges are represented by the curves in fig. 2b. It starts from zero and increases to a maximum at $y(c_1)$, then decreases from $y(c_1)$ to $y(c_2)$. Assuming that this damped disturbance is travelling with a velocity greater than $\dot{\lambda}_0(c_2)$, so that $\left(\frac{dZ_0}{dy}\right)_{s1}$ is positive, then τ will have a sudden decrease across this singular point, and reach a large negative value at $y(s_1) + \varepsilon$. It increases from there to the point $y(s_2) - \varepsilon$, where it reaches a positive value, and then decreases rapidly across the singular point $y(s_2)$. It attains a small negative value as $y(c_4)$ is approached and increases to zero near the pole. It is seen that $\tau \frac{dZ_0}{dy}$ is mainly negative,

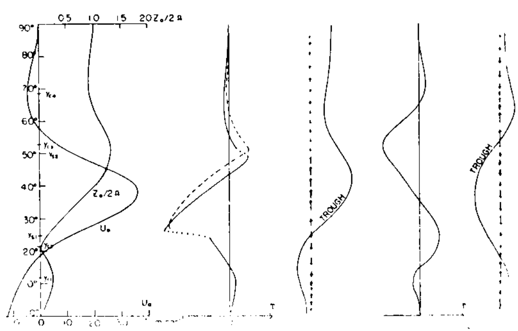

Fig. 2. Same as Fig. 1 for a weaker zonal wind profile.

therefore the total kinetic energy of the perturbation decreases with time, being fed into the zonal flow, at least in part. Also, angular momentum is transported into the central part of the zone of westerlies, strengthening the westerlies and also developing an easterly current north of the equator, thus gradually increasing the degree of instability. For this case, the troughs and ridges are tilted with their convex side toward the east in the zone of westerlies, but the direction is reversed in the equatorial belt and also near the pole.

For the amplifying disturbances, the distribution of τ is represented by the curve in fig. 2c. It is seen that $\tau\, d\dot{\lambda}_0/dy$ is mainly positive and has large values in the central part of the profile, so that the perturbation is withdrawing kinetic energy from the main flow. But in the outer parts, $\tau\, d\dot{\lambda}_0/dy$ is negative, so that the perturbation is also supplying kinetic energy into the main flow, strengthening the zonal flow in those parts, as can be seen from equation (22). Thus this case is relatively stable as compared with the case (a).

It should be pointed out that these discussions of the distribution of τ with respect to the zonal wind profile apply only to the individual situations. Because the jet-stream of westerlies fluctuate both in their latitudinal position and strength, and because different effects are produced by different kinds of disturbances, the distribution of the average values of τ for a longer period may differ considerably from that given by the corresponding mean profile.

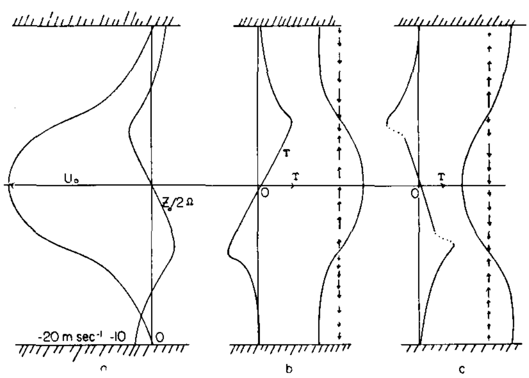

Fig. 3. Distribution of τ and vorticity transfer in the equatorial easterlies. (b) for amplified disturbances. (c) for damped disturbances.

(c) *Waves in the high level easterlies in equatorial belt*

Although there is no natural boundary between the high level westerlies in middle and higher latitudes and the high level easterlies in equatorial latitudes, different kinds of perturbations exist in these two regions. The disturbances in the westerlies move rather rapidly towards the east and are numerous, while in the easterlies there are relatively few large disturbances, and these generally move slowly towards the west or remain stationary. Therefore we may take the equatorial easterlies as a separate region.

The case when the easterly current is more concentrated in the equatorial belt and flat in the outer regions, with dZ_0/dy negative around the equator and positive in the outer parts will be discussed. According to the discussion in section 5, both damped and amplified disturbances can exist in this case. An examination of equation (18) shows that for amplified disturbances, τ is positive north of the equator and negative to the south, with the same sign as the shear of the basic current, as indicated in fig. 3b. This gives an increase of the perturbation kinetic energy. The trough and ridge lines for these amplifying disturbances are oriented with their convex side toward the east, contrary to that in the zone of westerlies. This tilting gives an equatorward flow of westerly momentum and will tend to flatten and possibly destroy the zonal flow pattern.

For damped disturbances with phase velocity within the range of $\dot{\lambda}_0$ in the belt, the distribution of τ is represented by the curve in fig. 3c, with the proper rapid change across the singular point. It is seen that τ is negative to the north and positive to the south of the equator, so that the perturbation kinetic energy will decrease. The troughs and ridges are tilted with their convex side toward the west, and momentum is transferred away from the equator, with the result that the easterly current is strengthened near the equator, increasing the degree of instability.

Besides these disturbances with μ_r within the range of $\dot{\lambda}_0$ in the belt, we might also have disturbances moving toward the west at a speed faster than the maximum easterly current.

The general features of the distribution of τ are the same as those described above.

(d) *When dZ_0/dy is positive in the entire region*

When the prevailing zonal winds are weak, the value of Z_0 is determined primarily by the coriolis parameter, therefore may increase with latitude over the entire hemisphere. According to the preceding discussions, this case is stable and cannot be disturbed spontaneously. The same conclusion may also be reached by applying the non-linearized vorticity equation $dZ_0/dt = 0$ (FJÖRTOFT, 1950; STARR, 1950; KUO, 1950). However, the linear theory also tells us that all possible neutral disturbances travel with phase velocity $\mu \leq \min. \dot{\lambda}_0$. Thus if any disturbance is added to the system and is moving with a velocity within the range of $\dot{\lambda}_0$, which is generally the case because added vorticity moves with the current, the motion cannot follow the law of conservation of absolute vorticity at every point, for some other effects necessarily enter into the problem (either viscosity or divergence), at least in some small regions. If we allow viscosity into the small region around the singular points, then damping effect will take place.

Although viscosity is important for these damped disturbances, its effect will be small outside the small strips around the singular points. Therefore the distribution of the stress τ is still given by equation (18) in these regions, and increases with latitude. Across the small strips around the singular points, τ has rapid decreases, attains a negative value at $y(s_2)+\varepsilon$, and goes to zero as the boundary $y=y_2$ is approached. Corresponding to these negative jumps of τ, the mean zonal current decreases within these strips. The mean vorticity decreases around $y(s_1)$ and increases around $y(s_2)$, indicating a tendency of creating a minimum and a maximum of $\overline{Z}$ along these latitudes respectively.

One physical explanation of this process is that high concentrations of vorticity are always present in the small-scale disturbances, which under the influence of the northward increase of Z, will result in a transport of cyclonic vorticity toward higher latitudes (KUO, 1950). The accumulative action of eddy viscosity near the singular points in spreading out the existing vorticity concentrations helps to build up extreme values of vorticity along these latitudes.

7. The neutral disturbances in the upper troposphere in winter

In order to compare the general results obtained above with actual flow patterns in the atmosphere, a solution of the perturbation equation for some normal zonal wind profile is desirable. Here the study will be restricted to the neutral disturbances which probably represent the flow pattern observed on the upper air weather charts during a period of relatively little change.

The average basic current studied in this section is given in fig. 4. It is taken to represent a mean profile of the layer from 900 mb to 100 mb, and resembles a typical profile at the 500 mb level across a section of strong westerlies. In order to simplify the integration, the profile has been greatly smoothed and adjusted to make $\dot{\lambda}_0$ have the same value at the two inner critical points.

For this neutral disturbance, the phase velocity is given by the current velocity at the inner critical points. The corresponding wave number and the eigen function can be obtained by Taylor's expansion. Thus, if Ψ and its derivatives at the starting point $y=y_1$ are known, the value of Ψ and its first derivative at the nearby point $y = y_1 + \eta$ are given by

$$\Psi(y_1+\eta) = \Psi(y_1) + \eta\Psi'(y_1) + \\ + \frac{\eta^2}{2!}\Psi''(y_1) + \frac{\eta^3}{3!}\Psi'''(y_1) + \frac{\eta^4}{4!}\Psi^{iv}(y_1) + \ldots$$

$$\Psi'(y_1+\eta) = \Psi'(y_1) + \eta\Psi''(y_1) + \\ + \frac{\eta^2}{2!}\Psi'''(y_1) + \frac{\eta^3}{3!}\Psi^{iv}(y_1) + \frac{\eta^4}{4!}\,\delta\Psi^{iv}(y_1)$$

where the primes denote differentiation with respect to y and

$$\delta\Psi^{iv}(y_1) = \Psi^{iv}(y_1) - \Psi^{iv}(y_1-\eta)$$

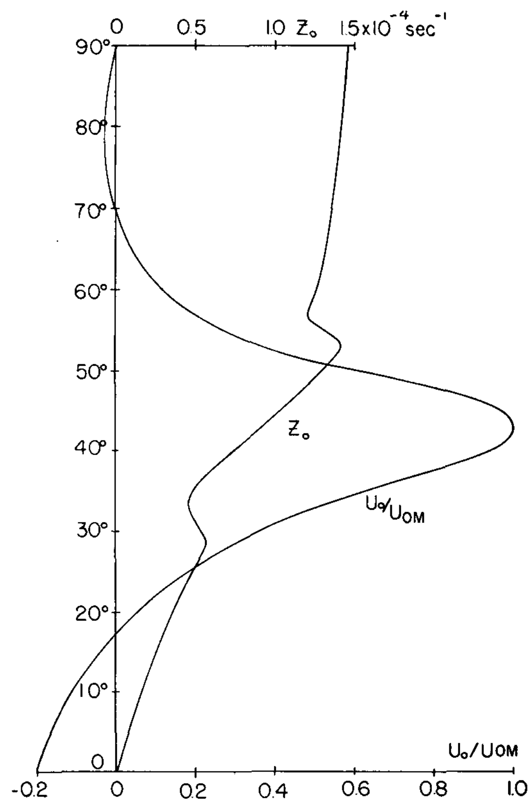

Fig. 4. The mean zonal wind profile and absolute vorticity distribution in winter, with $u_{0M} = 40$ m. sec^{-1}.

Since Ψ has to satisfy the vorticity equation at every point, the second and higher derivatives have to be computed from (11) by successive differentiation and by use of the values of the lower derivatives already obtained. Thus

$$\Psi''(y_1) = \left[n^2 - (\dot{\lambda}_0 - \mu)^{-1} \frac{dZ_0}{dy}\right]_{y_1} \Psi(y_1) \equiv$$

$$\equiv (n^2 - F)_{y_1} \Psi(y)$$

$$\Psi''' = (n^2 - F)\,\Psi' - F'\Psi$$

$$\Psi^{iv} = (n^2 - F)\,\Psi'' - 2\,F'\Psi' - F''\Psi$$

and so on, and Ψ can be integrated in a step-by-step manner.

To start the integration, the solutions corresponding to the distribution of F near the equator are obtained. In our case, F is equal to -23 near the equator. Thus, for $y = 0$, (11) becomes $\Psi'' = m^2\Psi$, where $m^2 = n^2 + 23$, and the solution that satisfies the boundary condition $\Psi(0) = 0$ is

$$\Psi = A \sinh my \tag{24}$$

The first and higher derivatives for the starting point can be computed from this solution and the values of Ψ and its derivatives at $y = 0 + \eta$ computed from the Taylor expansions. The value of $\delta\Psi^{iv}$ at the starting point can either be neglected or computed from the solution given above. If the condition $u' = 0 = -\partial\Psi/\partial y$ is used for $y = 0$, we get another solution. Since the difference is small, it will not be discussed.

Since the other boundary condition applies at $y = \infty$ (the pole) in our coordinate system, it is necessary to replace it by the condition

$$\Psi' + n\Psi = 0 \tag{25}$$

for a very large value of y, i.e., Ψ should decrease exponentially as the pole is approached. For the profile we are studying, the function F can be represented by $F = -2\,(y - 0.8)^{-2} = -2x^{-2}$ for $y > 1.3$ (north of 65° N), to a high degree of approximation. Thus, for $y > 1.3$ equation (11) becomes

$$\frac{d^2\Psi}{dx^2} - \left(n^2 + \frac{2}{x^2}\right)\Psi = 0 \tag{26}$$

The solution of (26) which satisfies the condition (25) is given by

$$\Psi = C\left(n + \frac{1}{x}\right) e^{-nx} \tag{27}$$

and is valid for $y > 1.3$.

When the function Ψ with its derivatives obtained from (24) has been integrated to $y = 1.3$, the ratio Ψ'/Ψ should agree with that given by (27). The value of n that fulfils this requirement is the eigenvalue of the problem, and the corresponding Ψ the eigenfunction. It is found that the neutral oscillation corresponding to the two inner critical points of the profile in fig. 4 has the eigenvalue $n = 9.5$. The phase velocity of this neutral disturbance is $\mu_r = 0.05\,\omega$, about 12 deg. lat per day. The 500 mb and 300 mb maps during the winter months January and February show that the wave number is generally from 8 to 10, and the phase velocity is about 10—15 degrees of longitude per day. The eigenfunction Ψ and its first derivative are given in fig. 5. It is seen that the perturbations are much stronger on the north side of the jet than on the south side; this agrees with the statistical result obtained by RIEHL, YEH and LA SEUR (1950) that the maximum value of $\overline{v}^2$ occurs generally to the north of the jet.

In the perturbation stream function, the arbitrary constant has been chosen to give a maximum perturbation velocity $u' = 12$ m sec^{-1} while the maximum velocity of the basic current is 40 m sec^{-1}. The perturbation kinetic energy is less than 10 per cent of that of the basic flow.

Since only an integral wave number is permissible, we shall take $n = 9$ instead of 9.5. The total stream function is plotted in fig. 6. It is seen that closed cyclonic and anticyclonic centers are produced to the north and south of the jet stream of westerlies as a result of the

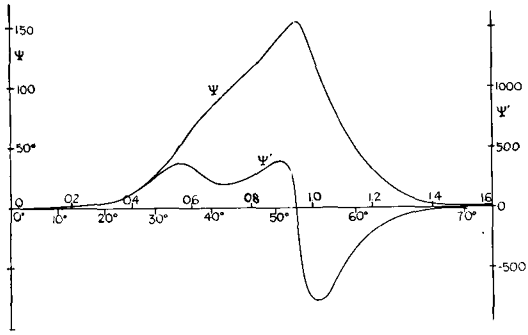

Fig. 5. The eigen function Ψ and its derivative for the neutral disturbance.

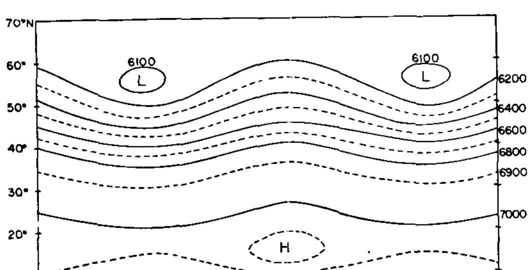

Fig. 6. Composite contour lines in meters.

perturbation. However, the anticyclonic centers are very weak, much weaker than the cyclones to the north. In order to produce the large semi-stationary anticyclonic centers in the subtropics, it is necessary to superpose another slowly moving disturbance, largely determined by the two outer critical points in the profile. Since this generally does not effect the flow pattern in middle and higher latitudes, it will not be discussed.

8. Summary

From the discussions above, it is seen that different disturbances have quite different effects on the zonal flow, according to whether the basic flow is stable or unstable for the particular disturbance. In the upper atmosphere under normal conditions, the absolute vorticity is largely determined by the coriolis parameter, increasing toward the north in most regions. This situation is stable for most disturbances. Thus, when disturbances are added to the upper layer, most of them will undergo a damping process, transport zonal momentum from regions of weaker to regions of stronger zonal motion, and supply kinetic energy to the zonal flow. Parts of these zonal momentum and energy being transported downwards to overcome ground friction. The remainder will have the effect of strengthening and sharpening the zonal westerlies and easterlies, and thereby also increase the degree of instability.

When the process of building up of the zonal wind systems through the damping disturbances has gone on for a sufficiently long period of time, the currents become strong and sharp, and maxima and minima of absolute vorticity are produced, the zonal wind distribution becomes unstable for the more effective disturbances, with longer wave lengths and smaller phase velocities. (From the vorticity distribution in fig. 5 and the solution obtained in the previous section, it may be inferred that the wave number of maximum instability is about 4 or 5 during the periods when the jet stream is strong in the winter months.) The amplitude of these disturbances will then increase to form closed cold lows to the south and warm highs to the north of the jet. As these amplified disturbances have the effect of transferring angular momentum from regions of higher to regions of lower zonal velocity, and withdraw kinetic energy from the zonal flow, the intensity of the mean zonal flow is greatly reduced by these disturbances, and the zonal wind profile is flattened.

After this breakdown, the absolute vorticity distribution is again dominated by the coriolis parameter, so that the situation is stable for other disturbances, and the damping effect operates once again, to rebuild the prevailing zonal winds. Thus according to this study, the general circulation will go through cycles of decay and rebuilding, due to the effect of these disturbances. It may be noted that their kinetic energy, at least in part, must be destroyed through the convergence effect, or transferred to the lower layer, so that the energy of the upper system will not continually increase.

From the foregoing discussions, it may be concluded that *the damped disturbances are really necessary parts in building up the prevailing zonal winds and overcoming the loss of kinetic energy and angular momentum through friction in the lower atmosphere. On the other hand, although large amplified disturbances occur only occasionally, they act in such a way as to prevent the prevailing winds from growing indefinitely, and therefore act as brakes for the general circulation.*

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