The Motion of Barotropic Disturbances in the Upper Troposphere

By GEORGE W. PLATZMAN, University of Chicago

(Manuscript received 16 April 1949)

Abstract

Oscillations of a barotropic model of the upper troposphere are studied, taking into account the existence of a zone of intense westerly circulation found in this region of the atmosphere. The fundamental frequencies of these oscillations are determined and some comments are made with regard to their dispersive properties.

Introduction

Since its discovery ten years ago, the planetary type of undulation often prominent in westerly circulations of the troposphere has provided a background against which theoretical and synoptic studies of the middle latitudes have been illuminated in many interesting ways. These undulations occur on a scale which in linear dimensions approximates one-half the planetary radius, while the period of individual 'waves' has an order of magnitude n times the planetary period of rotation, n being the number of waves in a continuous train compassing the hemisphere.

The observations initially providing confirmation of ROSSBY's (1939) theory were involved mainly in 10,000-ft pressure patterns over the western hemisphere and to a certain extent, particularly in recent years, at somewhat higher elevations. Such "long waves" in the belt of westerlies are found inherently stable, in conformity with the theory, as is evidenced in part by the fact that they are traced frequently through a succession of five to ten days. The interruption of their continuity takes place through the development of smaller-scale disturbances whose operation, although not as yet accessible to satisfactory theoretical analysis, is generally understood to be governed by barocline processes.

The success of the "long-wave" theory and its subsequent evolvements is of particular importance not merely because the theory has provided a satisfactory basis for interpreting certain types of observed motions, but perhaps primarily because it has established so clearly the usefulness of treating a broad class of circulations by means of a dynamical theory in which barotropic, nondivergent, and two-dimensional fluid strata replace the complex structure of the real atmosphere. The kinematical behavior of such a fluid sheet is controlled exclusively by the fact that the vorticity of individual fluid elements cannot be altered. This permanence of vorticity has constituted a central principle which, though achieved through a rigid framework of limitations, is found to extend beyond this framework in its practical usefulness.

The original form of the theory in question is set forth with the help of two simplifications, of which the first is the familiar approximation whereby the zone between two latitude circles not too far separated is supposed mapped on a plane surface so that the language of Cartesian co-ordinates may be used without serious error. The second is the very penetrating device of assuming the latitude variation of the vertical component of the earth's rotation to be uniform throughout the zone, thereby incorporating a primary feature of the planetary character of the fluid stratum in a simple and effective manner. The equivalent problem involving motions within a truly spherical sheet in which these simplifications are not present was solved subsequently in a well-known investigation by HAURWITZ (1940), although this analysis unavoidably entails complications arising from the necessity of securing cyclic continuity of the motions and from the indeterminacy between different possible modes of oscillation.

In his analysis HAURWITZ retained the full effect of the stabilizing influence of the earth's rotation by postulating a zonal current initially in solid rotation, analogous to the laterally uniform current employed by ROSSBY. Somewhat later, the Cartesian simplifications were reintroduced in a study by GARSTENS (1941) of the coupling between the stabilizing planetary effect and the destabilizing kinematical effect of shearing motion. Recently, a more general inquiry into this problem has been undertaken in an important investigation by KUO (1949).

It has long been recognized that because of the earth's strong gravitational attraction and high rate of rotation, large-scale circulations of the atmosphere are very nearly hydrostatically and geostrophically balanced, and that in consequence, the northward decrease of tropospheric mean temperature must be associated with a predominance of westerly currents in middle latitudes, their intensity increasing until the stratosphere is reached, where the temperature gradient reverses direction. Recently, however, data analyzed by PALMÉN and collaborators have revealed in a striking manner that middle-latitude zonal circulations cannot be accounted for solely by this type of reasoning. It is now known that the intensity and lateral concentration of westerlies in the upper troposphere are in fact so great, particularly during rather regularly recurring periods, as to constitute a distinct and unique phenomenon.

An investigation published by PALMÉN and NEWTON (1948) provides satisfactory material for the purpose of illustrating certain features of the zonal-wind distribution to which attention will be drawn here. In their study, data from twelve synoptic cross sections along the meridian 80° W (all confined to the month December 1946) were averaged by a process in which, at a given pressure level, distance from the polar front was used as a coordinate to establish correspondence between points in the twelve sets of synoptic data. (As a consequence of this procedure it is difficult to assign definite latitudes to the mean cross section but the authors, though cognizant of this awkward feature of their method, have provided such a latitude scale which will be applied in the following discussion without further comment.) In fig. 1 are presented the results of computations based upon the data of PALMÉN and NEWTON[1]; curves A, B, and C show latitude distributions of the zonal wind, absolute angular momentum (per unit mass), and absolute potential vorticity, respectively. These curves were computed from values of zonal geostrophic wind in a quasihorizontal sheet limited above and below by potential temperatures 350° K and 310° K.

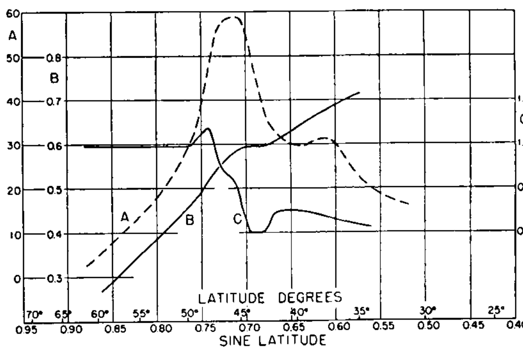

Fig. 1. Latitude distribution of relative zonal wind (curve A), absolute angular momentum (curve B), and absolute potential vorticity (curve C), computed from the semisynoptic data of PALMÉN and NEWTON (1948). The scale for curve A is in meters per second; the scale for curve B is such that the value 1.0 corresponds to the absolute momentum of a unit mass at relative rest on the equator; the scale for curve C is such that the value 1.0 corresponds to the absolute vorticity of the earth at the pole. The method of computation is described in the text.

If Ω_a denote absolute angular velocity of a zonal current, the vertical component of absolute vorticity at latitude Φ when computed along a surface of constant potential temperature (Θ) is

$$Z_\Theta \equiv -\frac{\delta}{\delta \xi}\left[\Omega_a\left(1-\xi^2\right)\right],$$

[1] Through the kindness of the authors the writer had access to the original data.

21533490, 1949, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/j.2153-3490.1949.tb01266.x by NICE, National Institute for Health and Care Excellence, Wiley Online Library on [05/02/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where δ signifies a variation for which Θ is constant and $\xi \equiv \sin \Phi$ is the sine of the latitude. The corresponding potential vorticity is

$$Z^\star \equiv Z_\Theta \frac{\partial p_0}{\partial p}$$

In this expression p is the pressure associated with the potential temperature Θ at latitude Φ and p_0 the pressure associated with the same potential temperature at an arbitrarily assigned but fixed latitude Φ_0, while ∂ denotes a variation for which the latitude is constant. In applying this formula to the meridional cross section of PALMÉN and NEWTON, it is convenient and natural to place the latitude at the extreme northern end of their section (58° N) since in this region nearly barotropic conditions prevail; thus, to the north of Φ_0 and also somewhat to the south $\partial p_0/\partial p \approx 1$.

Let $\Theta_1 = 350°$ K and $\Theta_2 = 310°$ K. Curve A in fig. 1 was obtained by computing first a mean value of Ω_a at each latitude from the formula

$$\bar{\Omega}_a \equiv \frac{1}{\Delta p_0} \int_1^2 \Omega_a dp_0,$$

in which Δp_0 is the baric thickness of the layer between Θ_2 and Θ_1 at latitude Φ_0; in the present instance $\Delta p_0 = 100$ mb. The expression $a\,(\bar{\Omega}_a - \omega) \cos \Phi$ then gives the *relative* linear zonal velocity represented in fig. 1, a and ω being the earth's radius and angular velocity.

Curve B in fig. 1 is obtained from the following nondimensional measure of the absolute angular momentum per unit mass:

$$\bar{M}_a \equiv (\bar{\Omega}_a \cos^2 \Phi)/\omega.$$

The value $\bar{M}_a = 1$ corresponds to the momentum of a unit mass at relative rest on the equator.

Absolute potential vorticity was computed as an average value

$$\frac{1}{\Delta p} \int_1^2 Z^\star \, dp = \frac{1}{\Delta p} \int_1^2 Z_\Theta dp_0 = \frac{\Delta p_0}{\Delta p} \bar{Z}_\Theta,$$

where Δp is the baric thickness of the layer between Θ_2 and Θ_1 at latitude Φ. In curve C the latter quantity is expressed nondimensionally through division by 2ω, the polar value of the earth's vorticity. Reference to the preceding formulae shows that

$$\frac{\Delta p_0}{\Delta p} \frac{\bar{Z}_\Theta}{2\omega} = -\frac{1}{2} \frac{\Delta p_0}{\Delta p} \frac{\delta \bar{M}_a}{\delta \xi};$$

from this last expression the distribution pictured in fig. 1 was derived. It should be noted also that the values of Ω_a used in these computations were obtained from the relative angular geostrophic velocity Ω_g by means of the gradient wind formula

$$\Omega_a = \omega\,(1 + 2\Omega_g/\omega)^{1/2},$$

which provides for the effect of curvature in zonal flow.

The curves illustrate several features of the wind distribution that have been noted in previous investigations. Roughly 250 km south of the relative wind maximum is a narrow zone of uniform absolute momentum and zero absolute vorticity, while approximately the same distance north of the relative wind maximum the potential vorticity attains a maximum value which in this case is about 20 per cent in excess of the very uniform values prevailing to the north. It is of course difficult to assign this maximum value with certainty because curve C is essentially a second derivate of the basic data, i.e., of the isobaric contour heights, but the reliability of the estimate is perhaps somewhat enhanced by the method of averaging through a relatively deep layer as described in the paragraphs preceding.

Similar computations were performed utilizing mean data published recently by HESS (1948); the results are shown in fig. 2. These data, also along the meridional plane 80° W, comprise eight months of radiosonde records, January and February 1942—45, for twelve stations located between latitudes 3° S and 74° N. The curves in fig. 2 were based on the following choice of parameters: $\Phi_0 = 70°$ N, $\Theta_1 = 375°$ K, $\Theta_2 = 325°$ K; again in this case $\Delta p_0 = 100$ mb. As would be expected, the features so apparent in the semisynoptic data assembled by PALMÉN and NEWTON are much less prominent here, particularly the rather abrupt transition from high values of absolute vorticity on the north side of the current axis to low values on the south, as shown by curve C in fig. 1.

21533490, 1949, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/j.2153-3490.1949.tb01266.x by NICE, National Institute for Health and Care Excellence, Wiley Online Library on [05/02/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

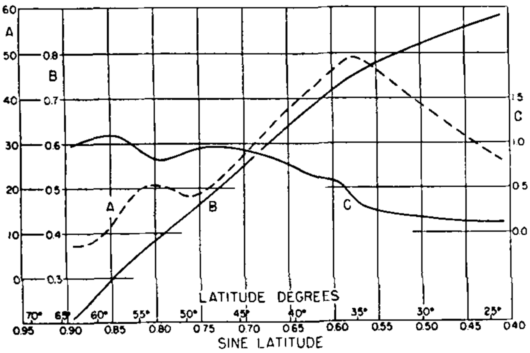

Fig. 2. Latitude distribution of relative zonal wind (curve A), absolute angular momentum (curve B), and absolute potential vorticity (curve C), computed from the mean data of HESS (1948). For further description see legend under fig. 1.

The vorticity distribution pictured in fig. 1 serves as a guide in constructing an appropriate distribution of zonal velocity for the purpose of investigating two-dimensional barotropic oscillations of the upper troposphere. From an inspection of this distribution it is apparent that there exist three relatively distinct zones, each with a more or less characteristic type of vorticity variation: a northern zone of uniformly high vorticity, a transition zone in which the vorticity decreases from a high to a low value, and a southern zone of low vorticity. In order to simplify the mathematical analysis, only the northern and southern zones have been incorporated in the theoretical model, which is illustrated by the dotted line in fig. 3. It is evident that the transition zone

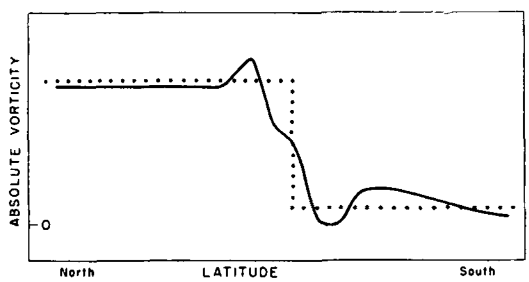

Fig. 3. Schematic hypothetical distribution of absolute vorticity (dotted line) compared with the distribution computed from the data of PALMÉN and NEWTON described in fig. 1.

is replaced by a vorticity discontinuity coinciding with the axis of the relative zonal current, while to the north and to the south the vorticity is independent of latitude.

Theoretical velocity distribution of the mean motion

Fixing attention upon a level surface in the upper troposphere, north of a given latitude the zonal flow in this level is characterized by uniform absolute vorticity, which for simplicity is assumed to prevail throughout a polar cap whose southern boundary coincides with the axis of the zonal current. The zonal velocities corresponding to this condition are easily calculated.

Denoting by Ω_a the absolute angular velocity of the zonal current in latitude Φ, the total circumpolar circulation of the absolute linear velocity will be

$$2\pi a^2 \Omega_a \cos^2 \Phi,$$

where a is sensibly the mean radius of the earth. This circulation must evidently be equal to the surface integral of the absolute vorticity, taken over the polar cap north of latitude Φ. Since the absolute vorticity is assumed uniform, the integral in question has the value

$$2\pi a^2 Z (1 - \sin \Phi),$$

where Z is the value of the absolute vorticity. On comparing the two expressions it is found that[1]

$$\Omega_a \cos^2 \Phi = Z (1 - \sin \Phi);$$

the corresponding relative angular velocity is $\Omega = \Omega_a - \omega$, where ω is the earth's angular velocity.

At the latitude Φ_0 of the current axis[2] the absolute vorticity passes abruptly from the value Z prevailing in the polar cap to the smaller value $Z - \Delta Z$, denoting by ΔZ the positive value of the discontinuous change of vorticity. At a latitude Φ, south of Φ_0, the total circumpolar circulation of the absolute linear velocity is, as before,

$$2\pi a^2 \Omega_a \cos^2 \Phi,$$

while, assuming continuity in *velocity* at

[1] This angular-velocity distribution was first given by ROSSBY (1947), who postulated the existence of a polar cap of uniform vorticity on theoretical grounds.

[2] The symbol Φ_0 as used henceforth has no connection with the same symbol involved previously in discussion of figs. 1 and 2.

21533490, 1949, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/j.2153-3490.1949.tb01266.x by NICE, National Institute for Health and Care Excellence, Wiley Online Library on [05/02/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

latitude Φ_0, the absolute circulation at Φ_0 is given by

$$2\pi a^2 Z (1 - \sin \Phi_0).$$

The difference between these two expressions must be equal to the surface integral of $Z - \Delta Z$ taken over the zone between Φ and Φ_0. Since $Z - \Delta Z$ is supposed uniform, this integral has the value

$$2\pi a^2 (Z - \Delta Z) (\sin \Phi_0 - \sin \Phi);$$

hence, to the south of Φ_0,

$$\Omega_a \cos^2 \Phi =$$
$$= Z (1 - \sin \Phi) - \Delta Z (\sin \Phi_0 - \sin \Phi).$$

If symmetry is required between the northern and southern hemispheres, the southern-hemisphere polar cap will possess the uniform vorticity $- Z$; further, the absolute vorticity $Z - \Delta Z$ in the equatorial zone between latitudes Φ_0 and $-\Phi_0$ will, on the present hypothesis, be uniformly zero, so that $\Delta Z = Z$. The preceding equation then reduces to

$$\Omega_a \cos^2 \Phi = Z (1 - \sin \Phi_0),$$

which corresponds to uniform absolute angular momentum in the equatorial zone.

Barotropic disturbances in the upper troposphere

Consider a horizontal stratum of the atmosphere in which the motion is purely zonal and possesses the angular-velocity distribution described in the preceding paragraphs. Neglecting the effects of heterogeneity, suppose a disturbance to be initiated in this layer, the velocities being purely horizontal and non-divergent. Under these conditions the absolute vorticity of each fluid element is conserved. If the disturbance is initiated without the injection of additional vorticity, it follows that the initially uniform distribution of absolute vorticity cannot be altered.[1] Therefore, a dynamically possible oscillation of a steady current possessing an initially uniform distribution of absolute vorticity may be obtained by superposition of an *arbitrary irrotational disturbance*, subject only to such restrictions as may be imposed by boundary conditions.[1]

In the absence of eddy stresses the dynamical equations for horizontal motion may be expressed in the form

$$\frac{du}{dt} - \left(2\omega \sin \Phi + \frac{u}{a} \tan \Phi\right) v = -\alpha \frac{\partial p}{\partial x}$$

$$\frac{dv}{dt} + \left(2\omega \sin \Phi + \frac{u}{a} \tan \Phi\right) u = -\alpha \frac{\partial p}{\partial y}$$

The time is denoted by t; u and v are the eastward and northward velocity components relative to the earth; Φ is the latitude and ω the earth's angular velocity; a is the mean radius of the level surface, or effectively, the earth's mean radius; α is the specific volume; p is the pressure. In this and subsequent notation the variations $\partial/\partial x$ and $\partial/\partial y$ are not intended as symbols for differential coefficients, but should be interpreted as directional derivatives; formally, $\partial x = a \cos \Phi \, \partial\lambda$ and $\partial y = a \partial \Phi$, where λ is the longitude angle. The equations are thus essentially in spherical coordinates.

By expanding the individual derivative the preceding equations can be arranged in the form

$$\frac{\partial u}{\partial t} - Z v = -\frac{\partial}{\partial x}\left(\alpha p + \frac{1}{2} q^2\right)$$

$$\frac{\partial v}{\partial t} + Z u = -\frac{\partial}{\partial y}\left(\alpha p + \frac{1}{2} q^2\right),$$

where $q^2 \equiv u^2 + v^2$ and

$$Z = 2\omega \sin \Phi + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + \frac{u}{a} \tan \Phi$$

is the absolute vorticity. This arrangement involves the assumption that the fluid stratum

[1] The distribution is uniform apart from the discontinuous change from the value Z to the value $Z - \Delta Z$ at the current axis. However, in studying the disturbance we neglect possible effects of lateral mixing, and assume that the discontinuity will not diffuse into a finite zone.

[1] It is illuminating to draw upon the analogy to autobarotropy in stratified fluids, which is established when the "equation of piezotropy" (corresponding here to the conservation of absolute vorticity) is equivalent to the "equation of barotropy" (corresponding here to the state of uniform vorticity).

21533490, 1949, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/j.2153-3490.1949.tb01266.x by NICE, National Institute for Health and Care Excellence, Wiley Online Library on [05/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

is homogeneous (α uniform). On linearization[1] the preceding equations reduce to

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - Zv &= -\frac{\partial}{\partial x}(\alpha p + Uu) \\ \frac{\partial v}{\partial t} + Zu &= -\frac{\partial}{\partial y}(\alpha p + Uu), \end{aligned}\right\} \qquad (1)$$

in which u and v are now understood to be the components of a small irrotational disturbance, p being the perturbation pressure, while U is the relative linear velocity of the undisturbed zonal motion with uniform absolute vorticity Z.

Since the perturbation motion is irrotational, there exist a stream function ψ and velocity potential φ such that

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial \varphi}{\partial x};\ v = +\frac{\partial \psi}{\partial x} = -\frac{\partial \varphi}{\partial y};$$

and if in addition to these relations it is noted that Z is uniform in time and, except for certain specified discontinuities, in space, the dynamical equations are found to possess the first integral

$$\alpha p + Uu = Z\psi + \partial\varphi/\partial t,$$

in which an arbitrary function of the time arising in the integration is, as customary, supposed merged in φ. Denoting by Ω the relative angular velocity of the undisturbed zonal current, so that $U = a\Omega \cos \Phi$, the preceding 'Bernoulli' equation may be expressed in the form

$$\alpha p = Z\psi + \frac{\partial \varphi}{\partial t} + \Omega \frac{\partial \varphi}{\partial \lambda},$$

where λ, as noted earlier, is the longitude angle in spherical coordinates.

The boundary conditions to be satisfied along the line of vorticity discontinuity at latitude Φ_0 may be formulated in a simple manner. The disturbance, being small, deforms the line of discontinuity only slightly from its original location along a latitude circle. Hence the kinematical boundary condition, which requires continuity of normal velocity across this line, is equivalent to the continuity of v, the northward velocity component; thus $\Delta v_0 = 0$, where Δ is used to signify a difference in value at essentially the same point but on opposite sides of the discontinuity, and the subscript is used to indicate values taken at the latitude Φ_0. Since $v = \partial\psi/\partial x$ it is clear that to the first order of small quantities the condition $\Delta v_0 = 0$ is equivalent to

$$\Delta \psi_0 = 0. \qquad (2)$$

The dynamical boundary condition, requiring continuity of pressure, may be obtained directly from the Bernoulli equation; thus, since $\Delta p_0 = 0$ and $\Delta \psi_0 = 0$,

$$\psi_0\, \Delta Z + \Delta \frac{\partial \varphi_0}{\partial t} + \Omega_0 \Delta \frac{\partial \varphi_0}{\partial \lambda} = 0,$$

provided also that the undisturbed zonal current velocity is continuous ($\Delta\Omega_0 = 0$). It will be helpful at a later stage to introduce here the *convective* coordinates

$$l \equiv \lambda - \Omega_0 t,\ \tau \equiv t,$$

by means of which an observer moving downstream at the angular rate Ω_0 would measure the disturbance. In terms of these coordinates the preceding equation takes the form

$$\nu\, \psi_0 + \frac{1}{2} \Delta \frac{\partial \varphi_0}{\partial \tau} = 0, \qquad (3)$$

where $\nu \equiv \frac{1}{2} \Delta Z$.

It remains now to establish the proper solution of $\nabla^2\psi = 0$ or $\nabla^2\varphi = 0$ compatible with the boundary conditions (2) and (3). To represent the two-dimensional solution of Laplace's equation appropriate for a spherical surface it is convenient to note that this solution may, by conformal transformation, be expressed in polar coordinates, by stereographic projection of the spherical surface onto the polar plane. The stereographic (polar) coordinates are r, λ where

$$r \equiv \sec \Phi - \tan \Phi,$$

and λ is the longitude. In polar coordinates

[1] More generally, the problem considered may be treated by methods known in connection with POISSON's equation, the distribution of vorticity being assigned over a two-dimensional surface. The following more restricted approach has been preferred as being simpler for the purpose in view.

21533490, 1949, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/j.2153-3490.1949.tb01266.x by NICE, National Institute for Health and Care Excellence, Wiley Online Library on [05/02/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

the characteristic solution of Laplace's equation is $r^n e^{in\lambda}$.

North of latitude Φ_0 and also south of this latitude, the complex potential $w = \varphi + i\psi$ can be represented generally by a sum of elementary independent harmonics w_n:

$$w = \sum_{n=1}^{\infty} w_n.$$

It is evident that each harmonic w_n must satisfy the boundary conditions (2) and (3), since these conditions are linear. The appropriate form of w_n, regarded as a function of r and of the convective coordinates l and τ, may be written as follows:

$$w_n = a_n r^n e^{inl} \qquad (r < r_0)$$

$$\overline{w}_n = \overline{b}_n r^{-n} e^{-inl}, \qquad (r > r_0)$$

where the dependence of the complex coefficients a_n and $\overline{b}_n$ on t (or τ) is to be determined in accordance with (3). In these equations w_n applies to the region north of Φ_0 and vanishes at the north pole ($r = 0$), while $\overline{w}_n$ applies to the region south of Φ_0 and vanishes at the south pole ($r = \infty$). Since $\overline{\psi}_{n0} = \psi_{n0}$ according to (2), it follows that $\overline{b}_n = - a_n^\star r_0^{2n}$ (the asterisk designating a complex conjugate), and from this, $\overline{\varphi}_{n0} = - \varphi_{n0}$. Therefore $\Delta\varphi_{n0} = 2\,\varphi_{n0}$, so that the dynamical boundary condition (3) is equivalent to

$$\partial\varphi_{n0}/\partial\tau + \nu\psi_{n0} = 0.$$

Thus the real part of the complex function $\partial w_n/\partial\tau - i\nu w_n$ vanishes on the circle $r = r_0$ within which the function is analytic. This function must then vanish everywhere:

$$\partial w_n/\partial\tau - i\nu w_n = 0.$$

Assembling these results, the solution compatible with (2) and (3) finally takes the form

$$w_n = c_n r^n e^{i(nl+\nu\tau)} \qquad (r < r_0)$$

$$\overline{w}_n = - (r/r_0)^{-2n} w_n^\star, \qquad (r > r_0)$$

where c_n is an absolute (complex) constant, the value of which is determined by the initial form of the disturbance, as yet unspecified. The general solution, obtained by superposition of all elementary harmonics w_n, is

$$w = e^{i\nu\tau} \sum_{n=1}^{\infty} c_n r^n e^{inl}, \qquad (r < r_0)$$

together with a similar expression for the region $r > r_0$.

It is evident by inspection of the form of this solution that with respect to the 'convective' observer ($l = \lambda - \Omega_0 t$ = constant) the disturbance is *periodic*, each configuration of the motion being repeated after a time interval $2\pi/\nu = 4\pi/\Delta Z$, where ΔZ is the discontinuity of vorticity in the axis of the current. (If ΔZ is expressed as $\Delta Z = \varkappa\omega$, this period has the value $2 \div \varkappa$ in days.) No systematic dispersion can occur in such disturbances. This is also evident by inspection of the form of the elementary harmonics w_n; for, writing

$$\mu_n = \Omega_0 - \nu/n, \qquad (4)$$

it is found that in terms of λ and t,

$$w_n = c_n r^n \exp\,[in\,(\lambda - \mu_n t)].$$

The periodic wave train represented by the nth harmonic is therefore propagated eastward, without change of structure, at the angular rate μ_n given by (4). With respect to the convective observer this wave train is propagated westward at the angular rate ν/n and hence in the time interval $t = 0$ to $t = 2\pi/\nu$ (independent of n) has traveled an angular distance $2\pi/n$ equal to one wave cycle. Superposition of all harmonics must then produce the same disturbance as existed for $t = 0$.

Equation (4) is the fundamental frequency equation for disturbances in a zonal current for which the absolute vorticity is uniform to the north and to the south of a latitude where the relative velocity of the current reaches a maximum value. The vorticity to the north exceeds that to the south by an amount ΔZ. In equation (4), μ_n is the angular velocity of an elementary wave train (positive when the waves move eastward) and Ω_0 is the angular velocity in the axis of the zonal current, while $\nu \equiv \frac{1}{2}\,\Delta Z$ and n is the total number of in-

21533490, 1949, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/j.2153-3490.1949.tb01266.x by NICE, National Institute for Health and Care Excellence, Wiley Online Library on [05/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

dividual waves encompassing the hemisphere.[2]

ROSSBY (1947) has pointed out that a regime of uniform absolute vorticity within an extensive polar cap would tend to be established through the operation of large-scale lateral mixing, particularly during periods of fully developed meridional exchange by cyclonic and anticyclonic eddies. Further, he has suggested that a limiting value of the vorticity of such a polar cap would be given by 2ω, the polar vorticity of the earth. If $Z = 2\omega$ and if, for example, the absolute vorticity $Z - \Delta Z = 0$ south of the current axis, then $\Delta Z = Z = 2\omega$ so that $\nu \equiv \frac{1}{2}\,\Delta Z = \omega$. With these values of Z and ν it is possible to obtain a simple expression for the number of waves (hemispheric wave number) in a stationary simple harmonic wave train. Setting $\mu_n = 0$ and $\nu = \omega$ in (4), the stationary wave number is given by

$$n_s = \omega/\Omega_0.$$

If the value of Ω_0 is calculated from the angular-velocity distribution described earlier, then, provided $Z = 2\omega$, it is easily found that

$$n_s = \frac{1 - \sin \Phi_0}{1 + \sin \Phi_0}.$$

For $n_s = 3$ this gives $\Phi_0 = 30°$; for $n_s = 4$, $\Phi_0 \approx 37°$; for $n_s = 5$, $\Phi_0 \approx 42°$. The latter values appear to be in fair conformity to conditions frequently prevailing in the upper troposphere, as observed on isobaric contour charts.

Dr GEORGE CRESSMAN has drawn the writer's attention to his studies of long-wave patterns at the 500-mb level, which indicate that in most cases southward displacement of the latitude of maximum wind is accompanied by an *increase* in hemispheric wave number. However, the purpose of the above formula relating n_s and Φ_0 is merely to indicate that the present theory provides reasonable correspondence between plausible values of hemispheric wave number and the latitude of maximum wind. Without making any special assumption as to the value of ν in (4), the stationary wave number would be

$$n_s = \nu/\Omega_0.$$

It is the writer's impression that with southward displacement, there is a concurrent intensification of the maximum wind and of the difference in shear north and south — that is, southward displacement is accompanied by an increase in both ν and Ω_0. CRESSMAN's results would then be consistent with the relation $n_s = \nu/\Omega_0$, provided ν increases more rapidly than Ω_0.

The analysis leading to equation (4) is defective in at least one obvious respect. It was tacitly assumed that south of latitude Φ_0 the solution $\overline{w}_n$ extends to the south pole, whereas if this is the case, the main zonal motion, corresponding to $Z - \Delta Z =$ constant, would also extend to the south pole, giving infinite velocities there. If symmetry is required between the two hemispheres, the frequency equation (4) is altered somewhat. To demonstrate this, assume that south of latitude $-\Phi_0$ in the southern hemisphere, the uniform vorticity $-Z$ prevails, while in the intermediate (equatorial) zone the absolute vorticity is zero. The discontinuity in vorticity is then $\Delta Z = Z$; this applies to both discontinuities, at Φ_0 in the northern hemisphere and at $-\Phi_0$ in the southern hemisphere.

If transverse velocities do not appear across the equator ($\overline{\psi} = 0$ for $r = 1$), it suffices to restrict consideration to the form of the disturbance in the northern hemisphere, since the southern-hemisphere disturbance will follow from symmetry. The appropriate form of w_n, regarded as a function of r and of the convective coordinates l and τ, may be expressed as:

$$w_n = a_n\, r^n\, e^{inl}$$
$$\overline{w}_n = \overline{a}_n\, r^n\, e^{inl} + \overline{b}_n\, r^{-n}\, e^{-inl},$$

the former applying to the polar cap north of Φ_0 ($0 < r < r_0$) and the latter to the symmetrical equatorial zone between Φ_0 and $-\Phi_0$ ($r_0 < r < r_0^{-1}$). The dependence of the complex coefficients a_n, $\overline{a}_n$, and $\overline{b}_n$ on τ is to be determined as before, by means of (3). From the symmetry condition $\overline{\psi}_n = 0$ for

[2] In his early studies of the stability of sensitive jets, RAYLEIGH (*Theory of sound*, chapter 21) obtained frequency equations similar in form to (4); indeed, the physical model of a jet employed by RAYLEIGH is, apart from obvious adaptations to the geophysical setting of the present discussion, essentially equivalent to the model from which (4) has resulted.

21533490, 1949, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/j.2153-3490.1949.tb01266.x by NICE, National Institute for Health and Care Excellence, Wiley Online Library on [05/02/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$r = 1$, and from the condition (2) that $\psi_n = \bar{\psi}_n$ for $r = r_0$, it is found that

$$\bar{a}_n = -\bar{a}_n\, r_0^{2n}/\sigma_n\,;\ \bar{b}_n = -a_n^{\star} r_0^{2n}/\sigma_n,$$

where $\sigma_n \equiv 1 - r_0^{2n}$. With these values for the coefficients of $\bar{w}_n$, it follows that

$$\Delta\varphi_{n0} \equiv \varphi_{\ 0} - \bar{\varphi}_{n0} = 2\,\varphi_{n0}/\sigma_n.$$

The dynamical boundary condition (3) is then equivalent to

$$\partial\varphi_{n0}/\partial\tau + \nu_n\,\psi_{n0} = 0,$$

in which $\nu_n \equiv \sigma_n\,\nu = (1 - r_0^{2n})\,\nu$. Thus the real part of the complex function $\partial w_n/\partial\tau - i\nu_n w_n$ vanishes on the circle $r = r_0$ within which the function is analytic. This function must then vanish everywhere:

$$\partial w_n/\partial\tau - i\nu_n w_n = 0.$$

It follows that the solution providing symmetry between the hemispheres finally takes the form

$$w_n = c_n\, r^n\, exp\,[i\,(nl + \nu_n\tau)]$$
$$\bar{w}_n = -\,(r/r_0)^{-2n}\,(w_n^{\star} + r^{2n} w_n)/\sigma_n,$$

in which c_n, as before, is an absolute constant, the value of which is determined by the initial form of the disturbance.

The frequency equation in this case is evidently

$$\mu_n = \Omega_0 - \nu_n/n. \qquad (5)$$

This differs from (4) in that ν is replaced by $\nu_n = (1 - r_0^{2n})\nu$, but since $r_0^2 < 1/3$ for $\Phi_0 > 30°$, equation (5) will give values of μ_n differing only slightly from (4) except for the very lowest harmonics, $n < 4$.

It is possible that equation (4) may prove useful prognostically as a means for estimating the displacement of long-wave patterns in the upper troposphere, during periods when such patterns are well developed. In applying this formula, a profile of the zonal wind must first be constructed and plotted, for example, in meters per second versus degrees latitude. The displacement may then be computed by measuring the following quantities:

1. The latitude Φ_0 of the maximum zonal wind,
2. The maximum zonal wind speed U_0,
3. The difference ΔZ between the mean absolute vorticity to the north and to the south of the maximum wind,
4. The wave length or hemispheric wave number n estimated from analysis of 500- or 300-mb contours.

From these four quantities the rate of displacement of the wave pattern in degrees longitude per day may be determined from (4) or appropriate modifications thereof. Preliminary tests of this formula in a few cases indicate that it is capable of providing correct orders of magnitude of the wave displacements.

Energy propagation

Considerable attention in recent years has been directed toward the dispersive properties of "planetary" disturbances in connection with the growth and decay of the tropospheric westerlies and in relation to the coupling between widely separated zones of activity. This interest has been aroused mainly by the provocative investigation published about four years ago by Rossby (1945). As a consequence of this and later studies by the same author, it has become evident that the purely barotropic component of all large-scale circulations may perform an important function by operating as channels through which local concentrations of atmospheric energy are, in part, permitted to travel outwards in advance of the parent disturbance and activate latent supplies of energy successively in different longitudes, thus serving, in this sense, as perpetual catalytic agents. Many aspects of the dispersion problem have been examined critically in a study published recently by Yeh (1949).

Rossby's analysis has involved in large measure the well-known correspondence between the so-called kinematical and dynamical group velocities. If μ_n is the elementary wave speed (in angular measure) and n the corresponding wave number, then from a *purely formal* point of view the kinematical group velocity would be expressed by $d(n\mu_n)/dn$. But it must be observed that this expression cannot be interpreted strictly in accordance with the ordinary interference theory of wave groups, since the elementary components into which an arbitrary phase distribution on the sphere can ultimately be decomposed are necessarily of integral order, and thus constitute a discrete rather than a continuous spectrum of wave numbers. Further, since the present dis-

21533490, 1949, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/j.2153-3490.1949.tb01266.x by NICE, National Institute for Health and Care Excellence, Wiley Online Library on [05/02/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

cussion concerns disturbances in which harmonics of low order are predominant, this difficulty is apparently not easily disposed. In dealing with wave-like disturbances it is, of course, possible to define a phase angle π so that an expression such as $\cos \pi$, combined with an appropriate amplitude factor, adequately represents the disturbance. The phase velocity would then be given by

$$-\partial\pi/\partial t \div \partial\pi/\partial\lambda,$$

but there is no reason to suppose that the two quantities forming this ratio represent, respectively, the local frequency and local wave number, assuming the possibility of defining these local wave properties kinematically. This identification between phase gradients and kinematically relevant wave properties is limited evidently to those portions of the disturbance where these gradients are sufficiently small, measured in terms of time and space units comparable to the local period and wave length.

It has been customary in hydrodynamical studies, following particularly the work of RAYLEIGH, to consider the dynamical group velocity as a rate of propagation of energy. Since the analytical properties are known for the particular type of disturbance investigated in the preceding section of this paper, it is possible to compute directly the rate of propagation of wave energy for each elementary component of the disturbance, although such a computation does not in itself lead necessarily to information regarding the presence or absence of dispersion, since the energy distribution in a simple harmonic wave train is always uniform. The procedure in question is based upon the following line of reasoning.

Consider a fixed meridian F and a moving meridian M (fig. 4) with angular rate of

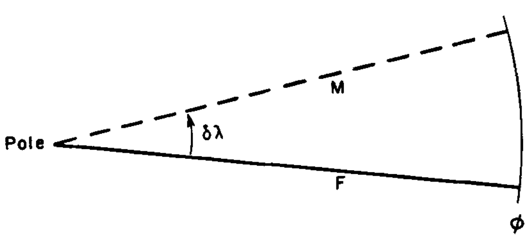

Fig. 4. Schematic diagram illustrating fixed meridan (F) and moving meridian (M) used in computation of energy velocity.

displacement γ. Let $E_t(\lambda, t)$ represent the rate of transmission of wave energy through the fixed meridian between the pole and an arbitrary latitude Φ and let $E_\lambda(\lambda, t)$ represent the "density" of wave energy in this range of latitude, i.e., $E_\lambda\, \delta\lambda$ is the amount of wave energy in the space between the pole, latitude Φ and two meridians separated by an angle $\delta\lambda$. At the instant when M coincides with F the rate of transmission of energy through M is

$$E_t(\lambda, t) - \gamma E_\lambda(\lambda, t).$$

If the total flux of energy through the moving meridian is made to vanish after a time interval T during which this meridian has been displaced from λ_0 to $\lambda_0 + \gamma T$, then

$$\int_0^T E_t(\lambda_0 + \gamma t, t)\, dt = \gamma \int_0^T E_\lambda(\lambda_0 + \gamma t, t) dt.$$

For periodic waves propagated without change of form at the angular rate μ, every wave property is such that, for example, $E_t(\lambda, t) = E_t(\lambda - \mu t)$; hence

$$\int_0^T E_t(\lambda_0 - \gamma' t)\, dt = \gamma \int_0^T E_\lambda(\lambda_0 - \gamma' t)\, dt,$$

where $\gamma' \equiv \mu - \gamma$. Writing $\xi \equiv \lambda_0 - \gamma' t$,

$$\int_{\lambda_0}^{\lambda_0 - \gamma' T} E_t(\xi)\, d\xi = \gamma \int_{\lambda_0}^{\lambda_0 - \gamma' T} E_\lambda(\xi)\, d\xi.$$

The moving meridian will traverse one complete wave if $\gamma' T$ is equal to the length of the individual waves, in which case the choice of λ_0 is immaterial in view of the periodicity of the wave train. It follows that

$$\gamma = \overline{E_t}/\overline{E_\lambda},$$

the bar signifying an average value taken over one complete wave cycle. This formula is equivalent to that employed by RAYLEIGH (1877). On the average, there is no net flux of energy through a plane moving uniformly with speed γ, which in this sense therefore represents the rate of propagation of wave energy.

To determine the appropriate expressions for E_t and E_λ it is convenient to return to the perturbation dynamical equations (1). On multiplying the first of these by u, the second by v, and adding, it is found that

$$\frac{\partial}{\partial t}\left(\frac{1}{2} q^2\right) + \operatorname{div}(\chi u, \chi v) = 0,$$

21533490, 1949, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/j.2153-3490.1949.tb01266.x by NICE, National Institute for Health and Care Excellence, Wiley Online Library on [05/02/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where $\chi \equiv \alpha p + Uu$ and $\frac{1}{2} q^2 \equiv \frac{1}{2} (u^2 + v^2)$ is the perturbation kinetic energy (per unit mass). This is one form of the continuity equation for wave energy. Upon integration over a fixed area and transformation of the divergence integral into a line integral, the equation expresses that the rate at which perturbation energy within the area is increasing, must equal the rate at which energy is transmitted across the boundaries of the area. The perturbation energy is transmitted in part by the perturbation pressure p and in part advectively by the basic current U. In spherical coordinates,

$$\frac{\partial}{\partial t}\left(\frac{1}{2} q^2\right) + \frac{1}{a \cos \Phi}\left(\frac{\partial \chi u}{\partial \lambda} + \frac{\partial \chi v \cos \Phi}{\partial \Phi}\right) = 0.$$

The wave energy in an elementary region $\delta\lambda$, $\delta\Phi$ is $\frac{1}{2} q^2 a^2 \cos \Phi \; \delta\Phi\delta\lambda$ (apart from certain constant factors), so the energy "density" defined previously is

$$E_\lambda = \int_{\Phi_0}^{\pi/2} \frac{1}{2} q^2 a^2 \cos \Phi \; \delta\Phi.$$

Further, it is evident from the form of the energy continuity equation that $\chi u a \delta\Phi$ expresses the rate at which wave energy is transmitted in the direction of wave propagation, normal to the segment $\delta\Phi$, so that

$$E_t = \int_{\Phi_0}^{\pi/2} \chi u a \delta\Phi.$$

The latitude Φ_0 in these expressions for E_t and E_λ will be taken as that of the maximum wind, where the regime of uniform vorticity Z is interrupted. On multiplication of the preceding energy continuity equation by $a^2 \cos \Phi \; \delta\Phi$ and integration from Φ_0 to $\pi/2$,

$$\frac{\partial E_\lambda}{\partial t} + \frac{\partial E_t}{\partial \lambda} = \chi_0 v_0 a \cos \Phi_0.$$

The right-hand member of this form of the continuity equation determines the rate at which energy is accumulated or depleted due to lateral transfer across the latitude Φ_0. In the case of a periodic wave train this must on the average be equal to zero since the wave does not propagate laterally, as is easily verified by noting that χ and v are 90 degrees out of phase.

From the Bernoulli equation obtained previously,

$$\chi \equiv \alpha p + Uu = Z\psi + \partial\varphi/\partial t,$$

where φ and ψ are the perturbation velocity potential and stream function. It may be shown without difficulty that for the periodic wave train progressing at the rate μ_n without change of form, in which $\psi \sim r^n$,

$$Z\psi + \partial\varphi/\partial t = (\mu_n + Z/n)\, au \cos \Phi.$$

It follows that

$$\overline{E}_t = \left(\mu_n + \frac{Z}{n}\right) \int_{\Phi_0}^{\pi/2} \overline{u^2} a^2 \cos \Phi \delta\Phi =$$

$$= \left(\mu_n + \frac{Z}{n}\right) \overline{E}_\lambda,$$

since $\overline{u^2} = \overline{v^2}$ in this case. Hence the rate of propagation of energy through a simple harmonic wave train in the region of uniform vorticity Z is given by

$$\gamma_n = \mu_n + Z/n \tag{6}$$

In this expression, either of the formulas (4) or (5) for the wave speed μ_n may be substituted, depending upon whether a symmetrical disturbance is required in the southern hemisphere. But whatever the form of the disturbance south of the vorticity discontinuity, the dynamical group velocity γ_n computed here exceeds the wave speed by an amount Z/n, which increases with increasing wave length.

It cannot be concluded, however, that an arbitrary disturbance, composed of an infinite number of such wave trains, is necessarily dispersive. In fact, it has already been pointed out that disturbances for which equation (4) gives the harmonic frequencies are nondispersive, because of the linearity of this equation. Thus, the energy distribution associated with disturbances of this kind, must be propagated without change. To demonstrate this, it is convenient to refer to the relation

$$\partial w_n/\partial \tau = i\nu w_n$$

obtained in the preceding section. Here w_n is the complex potential for the nth harmonic and $\nu \equiv \frac{1}{2} \Delta Z$. Since ν in this case is inde-

21533490, 1949, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/j.2153-3490.1949.tb01266.x by NICE, National Institute for Health and Care Excellence, Wiley Online Library on [05/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

pendent of n, the total potential $w = \Sigma w_n$ must satisfy the same condition:

$$\partial w/\partial \tau = i\nu w.$$

Differentiate this equation with respect to λ and separate real and imaginary parts:

$$\partial u/\partial \tau = +\nu v; \ \partial v/\partial \tau = -\nu u.$$

From these it is evident that

$$\frac{\partial}{\partial \tau}\left(\frac{1}{2}q^2\right) = \left(\frac{\partial}{\partial t} + \Omega_0 \frac{\partial}{\partial \lambda}\right)\left(\frac{1}{2}q^2\right) = 0,$$

the first equality resulting merely from the meaning of $\partial/\partial\tau$. The energy distribution of an arbitrary disturbance is therefore propagated convectively without change. It is interesting to note, however, that at every latitude $\Phi > \Phi_0$, where the convective velocity Ω is smaller than Ω_0, the disturbance energy is propagated at a rate $\Omega_0 > \Omega$.

In the more realistic model for which the harmonic frequencies are given by (5), the situation is quite different; in this case ν is replaced by $\nu_n = (1 - r_0^{2n})\,\nu$, and an arbitrary disturbance must, in general, be dispersive. The spreading of an initial concentration of energy may, of course, be computed from the appropriate FOURIER series, but unfortunately a simple means of describing the character of the dispersion has not been found.

Acknowledgments. — I wish to express my appreciation for the benefit of many fruitful discussions with Prof. V. P. STARR, and Dr. T. C. YEH.

REFERENCES

GARSTENS, M. A., 1941: Some effects of shearing motion on the propagation of waves in the prevailing westerlies. *Papers Phy. Ocean. Meteor., Mass. Inst. Tech. & Woods Hole Ocean. Instn*, **8**, no. 4, 32 pp.

HAURWITZ, B., 1940: The motion of atmospheric disturbances on the spherical earth. *Journal of Marine Research*, **3**, 254—267.

HESS, SEYMOUR L., 1948: Some new mean meridional cross sections through the atmosphere. *Journ. of Meteor.*, **5**, 293—300.

KUO, HSIAO-LAN, 1949: Dynamic instability of two-dimensional nondivergent flow in a barotropic atmosphere. *Journ. of Meteor.*, **6**., 105—122.

PALMÉN, E., and NEWTON, C. W., 1948: A study of the mean wind and temperature distribution in the vicinity of the polar front in winter. *Journ. of Meteor.*, **5**, 220—226.

RAYLEIGH, LORD, 1877: On progressive waves. *Proc. London Math. Soc.* (*1*), **9**, 21—26. (Reprinted in *Theory of Sound*, Appendix to vol. 1.)

ROSSBY, C.-G., 1939: Relation between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semi-permanent centers of action. *Journal of Marine Research*, **2**, 38—55.

— 1945: On the propagation of frequencies and energy in certain types of oceanic and atmospheric waves. *Journ. of Meteor.*, **2**, 187—204.

— 1947: On the distribution of angular velocity in gaseous envelopes under the influence of large-scale horizontal mixing processes. *Bull. Amer. Meteor. Soc.*, **28**, 53—68.

YEH, TU-CHENG, 1949: On energy dispersion in the atmosphere. *Journ. of Meteor.*, **6**, 1—16.

21533490, 1949, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/j.2153-3490.1949.tb01266.x by NICE, National Institute for Health and Care Excellence, Wiley Online Library on [05/02/2025]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License