

Pressure image assimilation for atmospheric motion estimation

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ABSTRACT

The complexity of the laws of dynamics governing 3-D atmospheric flows associated with incomplete and noisy observations make the recovery of atmospheric dynamics from satellite image sequences very difficult. In this paper, we address the challenging problem of estimating physical sound and time-consistent horizontal motion fields at various atmospheric depths for a whole image sequence. Based on a vertical decomposition of the atmosphere, we propose a dynamically consistent atmospheric motion estimator relying on a multilayer dynamic model. This estimator is based on a weak constraint variational data assimilation scheme and is applied on noisy and incomplete pressure difference observations derived from satellite images. The dynamic model is a simplified vorticity-divergence form of a multilayer shallow-water model. Average horizontal motion fields are estimated for each layer. The performance of the proposed technique is assessed using synthetic examples and using real world meteorological satellite image sequences. In particular, it is shown that the estimator enables exploiting fine spatio-temporal image structures and succeeds in characterizing motion at small spatial scales.

1. Introduction

1.1. Overview

Geophysical motion characterization and image sequence analysis are crucial issues for numerous scientific domains involved in the study of climate change, weather forecasting and climate prediction or biosphere analysis. The use of surface station, balloon or in-flight aircraft measurements has improved the estimation of wind fields and has been a further step for an improved understanding of meteorological phenomena. However, the network's temporal and spatial resolutions may be insufficient for the analysis of mesoscale dynamics. Recently, in an effort to avoid these limitations, high-resolution satellites sensors have been designed to provide image sequences characterized by finer spatial and temporal resolutions. Nevertheless, the analysis of atmospheric motion from image sequences remains particularly challenging due to the complexity of atmospheric dynamics observed at such scales. Thus, advanced techniques are needed to exploit the latest generation of satellite images.

1.2. Related works

For image-based geophysical motion analysis standard techniques from computer vision, originally designed for bi-dimensional quasi-rigid motions with stable salient features, are not well adapted (Leese et al., 1971; Horn & Schunck, 1981). The design of techniques specific to fluid flows has been a step forward, towards the elaboration of reliable methods to extract characteristic features of flows (Zhou et al., 2000; Corpetti et al., 2002, 2006; Cuzol et al., 2007; Yuan et al., 2007; Cuzol & Mémin, 2008). However, for geophysical applications, existing fluid-dedicated methods are all limited to frame-to-frame estimation and do not rely on physical conservation laws.

Geophysical flows are quite well described by appropriate physical models. As a consequence, in such contexts, the inclusion of physical evolution laws should constitute a very powerful means for the motion analysis of satellite image data, when compared with standard variational or statistical generic image based motion estimation techniques.

Recently, a layered motion estimator based on the shallow-water mass conservation equation has been proposed in Héas et al. (2007) and Héas and Mémin (2008). In these studies, time consistency is reinforced by the introduction of a frame-to-frame temporal regularization based on a simplified

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vorticity-divergence form of the shallow-water momentum conservation equations. This dynamic model provides an additional constraint guiding the estimation process from the results obtained using the previous frames. Nevertheless, it is important to outline that this constraint is not imposed globally on the sequence of observations but only on a pair of images.

Variational data assimilation (Le Dimet & Talagrand, 1986; Talagrand and Courtier, 1987), derived from optimal control theory (Lions, 1971), offers a global optimal formulation allowing the combination of physical models and different kinds of observations. Since its introduction, the variational assimilation technique, commonly known as 4D-Var, has been widely used for several atmospheric applications (Bennet, 1992; Talagrand, 1997) and especially for numerical weather forecasting or climate numerical modelling (Courtier & Talagrand, 1990; Ghil & Malanotte-Rizzoli, 1991; Courtier et al., 1994; Bennet et al., 1996; Rabier et al., 1998; Le Dimet & Blum, 2002). Different variations of the original principles have been proposed over the last 15 yr, and this technique is routinely used in several operational meteorological centres. The exploitation of the spatial information of satellite data has been shown to be very promising for improving Numerical Weather Prediction (NWP; see, for instance, some applications concerning assimilation of top-clouds in Bayler et al., 2000; Kim & Benjamin, 2000; Benjamin et al., 2002 or radiances in Andersson et al., 1994).

Image observations possess very good spatial resolutions compared with spatial scales of standard data and are largely used in assimilation systems. However, only a few attempts have been made to date to consider their use for the direct estimation of dynamic parameters (like the wind fields) that are indirectly observed through the image sequences. Even if some assimilation procedures are able to capture a wind field (viewed principally as a control parameter of a dynamic system like in Andersson et al., 1994), none of them assimilate this quantity directly through an observation operator linking the motion to the image data. This issue is indeed particularly difficult due to the phenomenon responsible for the image formation and to the inherent projection on the image plane of the three-dimensional observed quantity. As a consequence, the operational estimation of wind fields is currently done using correlation techniques, such as the Atmospheric Motion Vectors (AMV) product supplied by Eumetsat (the European agency that supplies the METEOSAT satellite data). The height is then determined from the infrared temperature and converted to pressure using an ECMWF forecast profile (Menzel et al., 1983; Schmetz et al., 1993, 1995; Holmlund, 1998). Once estimated, these AMV can either be the input of data assimilation algorithms or be assimilated themselves. In this paper, we propose the suppression of the intermediary step of estimating AMV and to assimilate directly the wind field from the sequence of images. Indeed, as shown in a recent study on a turbulent 2-D flow (Papadakis et al., 2007), the use of a direct image measurement operator clearly leads to

a great improvement of the quality of the estimation compared with an assimilation process based on motion fields given by an external process (pseudo-observations of the Eulerian velocity fields).

Variational data assimilation techniques relying on image data or on features extracted from image data have recently been proposed. A technique for the assimilation of a low-order dynamic system, obtain from a POD-Galerkin projection, has been proposed in D'Adamo et al. (2007). This technique makes use of noisy flow motion measurements estimated from a set of images of particles. In (Ruhnau and Schnörr, 2007), an optimal control strategy has also been proposed for the recovery of fluid motion. This approach based on a Stokes flow model nevertheless remains an estimation technique that works only on two consecutive frames and no dynamic coherency can be guaranteed along the image sequence. Taking a different approach, two variational assimilation strategies have been proposed in the computer vision community for the tracking of curves (Papadakis and Mémin, 2008) and the estimation of fluid motion fields (Papadakis et al., 2007). Both of these methods directly assimilate image data. They do not depend on pseudo-measurements obtained by external techniques but rather propose to directly include a differential observation operator borrowed to image features estimation techniques.

The technique proposed in this paper consists of an extension of these studies for atmospheric motion estimation from satellite image sequences.

1.3. Contributions

The method we propose differs significantly from previous studies on remote sensing motion analysis. In contrast to previous image-based techniques relying on two consecutive frames, the proposed approach estimates dynamically consistent motion fields through a whole image sequence.

The method exploits a simplified version of a divergence vorticity formulation of a shallow-water dynamic system. This inexact dynamics is complemented by the use of uncertainty terms. The assimilation is embedded into an incremental optimal control scheme and relies on sparse pressure data of atmospheric layers. The choice of this simplified dynamics is guided by the nature of the observations we have at our disposal. The efficiency of this process will be demonstrated using synthetic benchmarks and analysed using real world satellite sequences.

This article is organized as follows. After a section describing the context of the study, the data and the main dynamic assumptions, the assimilation methodology is presented in Section 3. Section 4 is devoted to the exploitation of this framework to the present problem. Experimental results, on synthetic and real examples, are provided in Section 5. A summary, some perspectives as well as appendixes conclude this paper.

Before entering into the detailed description of the proposed technique, let us first situate more precisely the aims of our study,

the assumptions inherited from these objectives and the nature of the available image data.

2. Context, objectives and ensuing dynamic assumptions

In this study, we aim to recover as accurately as possible a dense description of horizontal wind fields in the 3-D physical space. Nevertheless, as satellite images usually only provide sparse data of partially visible 3-D cloud layers, from which only inaccurate pressure vertical coordinates can be indirectly inferred, we will focus on the simpler task of estimating integrated horizontal wind fields for successive atmospheric layers.

2.1. Layers decomposition of the atmosphere

The layering of atmospheric flow in the troposphere is valid as long as horizontal scales are much greater than the vertical scale-height. Thus, for layers of about 1 km thick, this hypothesis is roughly valid for horizontal scales greater than or equal to 100 km. It is thus impossible to truly characterize a layered atmosphere with a local analysis performed in the vicinity of a pixel representing a kilometre order scale. Nevertheless, one can still decompose the 3-D space into elements of variable thickness, where only sufficiently thin regions of such elements may really correspond to common layers. An analysis based on such a decomposition has the principal advantage of operating at different atmospheric pressure ranges and avoids the mixing of heterogeneous observations.

The 3-D space decomposition used here is one introduced in earlier articles (Héas et al., 2007; Héas and Mémin, 2008), where the k^{th} ($k \in [1, K]$) layer corresponds to the volume lying in between an upper surface s^{k+1} and a lower surface s^k . These surfaces s^{k+1} are defined by the height of the top of clouds belonging to the k^{th} layer. They are thus only defined in areas where clouds belonging to the k^{th} layer exist and remain undefined elsewhere. Cloud classification maps, as illustrated in Fig. 1, determine the allocation of the top of clouds to the different layers. The EUMETSAT consortium routinely provides such classifications, which are based on thresholds of top of cloud pressure.

2.2. Sparse pressure difference image observations

Top of cloud pressure images of the order of one kilometre are also routinely provided by the EUMETSAT consortium.¹ We denote the class corresponding to the k^{th} layer by C^k . It should be noted that the top of cloud pressure image is composed of

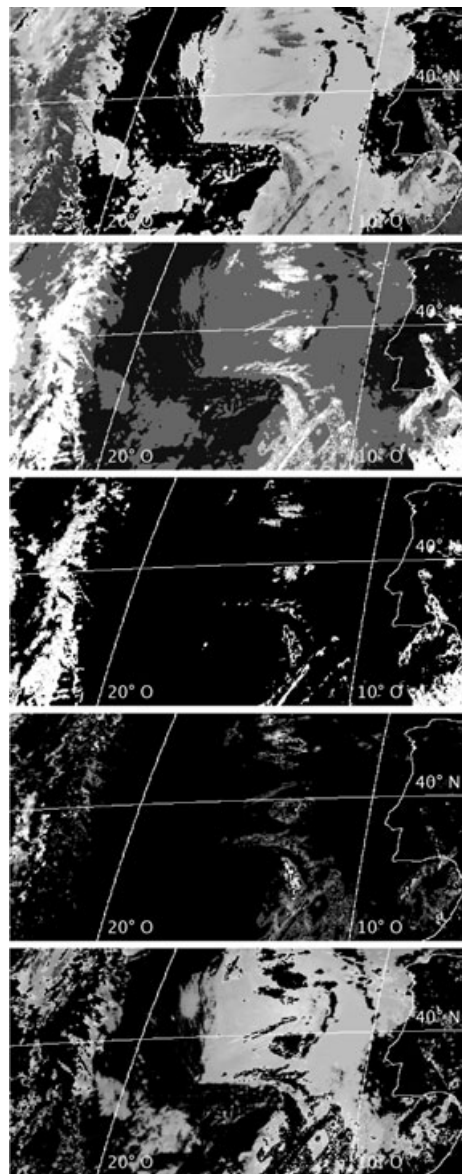


Fig 1. 'Image observations' issued from METEOSAT (Northern Atlantic Ocean on June 2004 from 13h30 until 15h45 UTC). From top to bottom: cloud top pressure image; classification into low (in dark grey), medium (in light grey) and high (in white) clouds; pressure difference of the higher layer; pressure difference of the intermediate layer; pressure difference of the lower layer. Black regions correspond to missing observations and white lines represent coastal contours, meridians and parallels (every 10°).

¹ Top of cloud pressure images are derived from a radiative transfer model using ancillary data, namely temperature and humidity profiles obtained by analysis of short term forecasts. This model simulates the radiation by the top of an opaque cloud at different vertical levels, which might be observed by a satellite. The pressure level where the simulated radiation best fits with the observed radiation determines the pressure

of the cloud top for the corresponding pixel (Lutz, 1999). Multichannel techniques (using a thermal IR with a water vapour or CO₂ absorption channels) enable the determination of the temperature of the top of semi-transparent clouds (Menzel et al., 1983; Schmetz et al., 1993), and thus, their equivalent pressure level (with the help of analysed or forecast data) independently from foreground or background effects.

segments of top of cloud pressure functions $p(s^k)$ related to the different layers : $\{\bigcup_k p(x, y, s^{k+1}) | (x, y) \in C^k\}$.

Thus, pressure images of top of clouds are used to constitute sparse pressure maps of the upper boundaries of the layers $p(s^{k+1})$. As with satellite images the lower cloud boundaries are always occluded, we coarsely approximate the missing pressure observations $p(s^k)$ by an average pressure value $\bar{p}(s^k)$ observed on the top of the clouds of the layer underneath. Finally, for the layer $k \in [1, K]$, we define observations h_{obs}^k as pressure differences in hecto pascal (hPa) units:

$$h_{\text{obs}}^k = \begin{cases} \bar{p}(x, y, s^k) - p(x, y, s^{k+1}) & \text{if } (x, y) \in C^k, \\ 0 & \text{else.} \end{cases} \quad (1)$$

The resulting observations from a METEOSAT image (5 June 2004, 13 h 30 UTC—universal time, similar to GMT : Greenwich mean time—or 2004-06-05T13:30Z in ISO8601) are illustrated in Fig. 1. The scattered nature of these maps (less than 15% is exploitable) is due to the sparsity of cloud coverage but is also enforced by the higher altitude clouds occultation of the lower layers. We should also note that the use of pressure images, besides being coherent with our layers definition, will enable the creation of a precise data model relating the observed pressure data to the unknown motion field. In contrast to traditional atmospheric wind field extraction techniques, we do not rely on the assumption that cloud luminance patterns are passive tracers of the flow (assumption which is known to be violated in some situations). This is a key advantage of the proposed approach.

2.3. Ensuing dynamic assumptions

To construct the best possible dynamic model describing the evolution of the pressure layers analysed and in accordance with the objectives of this study (i.e. the recovering of the horizontal displacements of the set of layers), we rely on the shallow-water approximation to integrate vertically the horizontal primitive equations of atmospheric dynamics. Using the hypothesis of hydrostatic equilibrium, the atmospheric hydrostatic primitive equations read

$$\begin{cases} \frac{du}{dt} - \frac{uv \tan \phi}{a} + \frac{uw}{a} + \frac{p_x}{\rho} = +2\Omega v \sin \phi - 2\Omega w \cos \phi + \mathcal{F}_u, \\ \frac{dv}{dt} + \frac{u^2 \tan \phi}{a} + \frac{vw}{a} + \frac{p_y}{\rho} = -2\Omega u \sin \phi + \mathcal{F}_v, \\ \frac{dp}{dz} + \rho g = 0, \end{cases} \quad (2)$$

where (u, v) represents the horizontal velocity field, w its vertical component, p the pressure (p_x and p_y denoting the spatial derivatives w.r.t x and y), $(\mathcal{F}_u, \mathcal{F}_v)$ the horizontal components of viscous forces at subgrid scales (Frisch, 1995), ϕ the latitude, ρ the density, Ω and a the Earth angular velocity and its radius. The continuity equation complements the system

$$\frac{1}{\rho} \frac{d\rho}{dt} + \nabla \cdot \mathbf{V} = 0, \quad (3)$$

where $\mathbf{V} = (u, v, w)^T$ stands for the 3-D velocity.

A vertical integration of the primitive equations is not straightforward since, in our case, we only rely on pressure image analysis and vertical density profiles remain unknown. We circumvent this difficulty by relying on a shallow-water approximation of these primitive equations (De Saint-Venant, 1871).

The shallow-water approximation is based on two main assumptions: (1) negligibility of vertical motion with respect to (w.r.t.) horizontal motion and (2) incompressibility. For atmospheric dynamics, this is valid for the upper range of mesoscale analysis in a layered atmosphere. In the following section, we derive a dynamic model using these two hypotheses.

2.3.1. Negligible influence of the vertical motion. Considering horizontal scales of the order of 100 km combined with layer depths of the order of 1 km makes the shallow-water approximation relevant. Therefore, to obtain a valid dynamic model on a pixel grid, we filter the hydrostatic primitives equations with a Gaussian kernel function K_{δ_x} of standard deviation equal to $\delta_x = 100\delta_p^{-1}$, where δ_p denotes the image pixel resolution in kilometres.² Following the scale analysis presented in Appendix A, where the Rossby number is around 0.5, Coriolis terms have to be included in the momentum components of the filtered hydrostatic primitive equations adapted to mesoscale dynamics. These latter equations are written

$$\begin{cases} \frac{d\tilde{u}}{dt} + \frac{\tilde{p}_x}{\rho} - \tilde{v} f^\phi = \nu_T \Delta \tilde{u}, \\ \frac{d\tilde{v}}{dt} + \frac{\tilde{p}_y}{\rho} + \tilde{u} f^\phi = \nu_T \Delta \tilde{v}, \end{cases} \quad (4)$$

where we have introduced the filtered pressure $\tilde{p} = K_{\delta_x} \times p$, the filtered horizontal wind $\tilde{\mathbf{v}} = (\tilde{u}, \tilde{v})^T = K_{\delta_x} \times (u, v)^T$ and where ρ_0, f^ϕ and ν_T denote the local mean density, the Coriolis factor depending on latitude ϕ and the turbulent viscosity produced at subgrid scales, respectively. The induced turbulent dissipation can be approached by subgrid models proposed in Large Eddy Simulation literature (Meneveau and Katz, 2000). The simplest one is the well-known Smagorinsky subgrid model, which is in agreement with Kolmogorov ‘K41’ theory (Smagorinsky, 1963). For 2-D flows, it results in an isotropic diffusion with a varying turbulent viscosity coefficient equal to

$$\nu_T = (C\delta_x)^2 \sqrt{2(\tilde{u}_x^2 + \tilde{v}_y^2 + (\tilde{u}_y + \tilde{v}_x)^2)}, \quad (5)$$

where C is the Smagorinsky coefficient (which is usually fixed to 0.17).

2.3.2. Incompressibility. The shallow-water approximation also assumes incompressibility. Thus, we consider a constant density ρ_0^k within any layer k (k is the index related to the number of the layer). We should note that this incompressibility simplification which underlies the shallow-water modelling is reasonable in our case, whereas it may be erroneous for finer

² The Gaussian filtering operation is achieved by a convolution product with a Gaussian kernel and acts as a low-pass frequency filter: frequencies higher than a given value (which depends on the standard deviation) are removed. Its effect on the HPE equations is to keep only the ‘low-pass’ component, that is, the large-scale structures.

horizontal scales. Layer average densities ρ_0^k can be related to the average pressures $\bar{p}^k$ by vertical integration of the equation of state for dry air ($p = \rho RT$), combined with the hydrostatic relation ($dp = -g\rho dz$), using the assumption of constant lapse rate ($T = T_0 + \gamma z$), where g , R , T_0 and γ denote the standard physical constants. Indeed, as demonstrated in Appendix B, the mean density related to the k^{th} layer may be expressed as

$$\rho_0^k = \frac{p_0^2}{(\bar{p}^{k+1} - \bar{p}^k) \left(\frac{\gamma R}{g} + 2 \right) RT_0} \left[\left(\frac{p}{p_0} \right)^{\frac{\gamma R}{g} + 2} \right]_{\bar{p}^k}^{\bar{p}^{k+1}}. \quad (6)$$

This relation characterizes a ‘barotropic’ atmosphere (Holton, 2004). Expanding the total derivatives in the isobaric coordinate system (x, y, p) and using the fact that the flow is incompressible (zero local 3-D divergence), eq. (4) can be rewritten as

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t} + \frac{\partial \tilde{u}^2}{\partial x} + \frac{\partial \tilde{u}\tilde{v}}{\partial y} + \frac{\partial \tilde{u}\tilde{\omega}}{\partial p} + \frac{\tilde{p}_x}{\rho_0} - \tilde{v}f^\phi = \nu_T \Delta \tilde{u}, \\ \frac{\partial \tilde{v}}{\partial t} + \frac{\partial \tilde{u}\tilde{v}}{\partial x} + \frac{\partial \tilde{v}^2}{\partial y} + \frac{\partial \tilde{v}\tilde{\omega}}{\partial p} + \frac{\tilde{p}_y}{\rho_0} + \tilde{u}f^\phi = \nu_T \Delta \tilde{v}, \end{cases} \quad (7)$$

where $\tilde{\omega} = d\tilde{p}/dt$ denotes the filtered vertical wind component in pressure coordinates. In addition, the expression of the continuity eq. (3) in the isobaric coordinate system (x, y, p) yields the filtered relation

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{\omega}}{\partial p} = 0, \quad (8)$$

where $(\tilde{u}, \tilde{v}, \tilde{\omega})$ are the components of the filtered 3-D velocity.

We now derive the integrated shallow-water model dedicated to atmospheric layers.³ To perform the vertical integration of eqs. (3) and (7) in the pressure interval $[\tilde{p}(s^{k+1}), \tilde{p}(s^k)]$, we first fix the boundary conditions:

$$\begin{cases} \frac{\partial \tilde{p}(s^k)}{\partial t} + \tilde{u}(s^k) \frac{\partial \tilde{p}(s^k)}{\partial x} + \tilde{v}(s^k) \frac{\partial \tilde{p}(s^k)}{\partial y} = \tilde{\omega}(s^k), \\ \frac{\partial \tilde{p}(s^{k+1})}{\partial t} + \tilde{u}(s^{k+1}) \frac{\partial \tilde{p}(s^{k+1})}{\partial x} + \tilde{v}(s^{k+1}) \frac{\partial \tilde{p}(s^{k+1})}{\partial y} = \tilde{\omega}(s^{k+1}). \end{cases} \quad (9)$$

Such boundary conditions can be interpreted as the fact that boundary surfaces $\tilde{p}(s^k)$ and $\tilde{p}(s^{k+1})$ are deformed by the vertical wind $\tilde{\omega}(s^k)$ and $\tilde{\omega}(s^{k+1})$ in a ‘barotropic’ atmosphere. To achieve such a vertical integration in the pressure interval $[\tilde{p}(s^{k+1}), \tilde{p}(s^k)]$ varying with spatial coordinates, we employ the Leibnitz formula with the previous boundary conditions. This formula, which is valid for any integrable and derivable function $f(x, p)$ and for any interval $[a(x), b(x)]$ with boundaries depending on x , reads

$$\begin{aligned} \int_{a(x)}^{b(x)} \frac{\partial f(x, p)}{\partial x} dp \\ = \frac{\partial}{\partial x} \left(\int_{a(x)}^{b(x)} f(x, p) dp \right) \\ - f(x, b(x)) \frac{\partial b(x)}{\partial x} + f(x, a(x)) \frac{\partial a(x)}{\partial x}. \end{aligned} \quad (10)$$

³ Note that according to Fubini’s theorem, if the motion fields are integrable then vertically averaged horizontally filtered fields are equivalent to horizontally filtered vertically averaged fields.

Relying on eqs. (9) and (10), the vertical integration of eq. (7) in the pressure interval $[\tilde{p}(s^{k+1}), \tilde{p}(s^k)]$ gives rise to a momentum conservation equation for the k^{th} atmospheric layer:

$$\begin{aligned} \frac{\partial (\tilde{\mathbf{q}}^k)}{\partial t} + \text{div} \left(\frac{1}{\tilde{h}^k} \tilde{\mathbf{q}}^k \otimes \tilde{\mathbf{q}}^k \right) + \frac{1}{2\rho_0^k} \nabla_{xy} (\tilde{h}^k)^2 \\ + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f^\phi \tilde{\mathbf{q}}^k = \nu_T^k \Delta (\tilde{\mathbf{q}}^k), \end{aligned} \quad (11)$$

with

$$\tilde{h}^k = \tilde{p}(s^k) - \tilde{p}(s^{k+1}), \quad (12)$$

$$\tilde{\mathbf{v}}^k = (\tilde{u}^k, \tilde{v}^k) = \frac{1}{\tilde{h}^k} \int_{\tilde{p}(s^{k+1})}^{\tilde{p}(s^k)} \tilde{\mathbf{v}} dp, \quad (13)$$

$$\tilde{\mathbf{q}}^k = \tilde{h}^k \tilde{\mathbf{v}}^k, \quad (14)$$

$$\text{div} \left(\frac{1}{\tilde{h}^k} \tilde{\mathbf{q}}^k \otimes \tilde{\mathbf{q}}^k \right) = \begin{bmatrix} \frac{\partial (\tilde{h}^k (\tilde{u}^k)^2)}{\partial x} + \frac{\partial (\tilde{h}^k \tilde{u}^k \tilde{v}^k)}{\partial y} \\ \frac{\partial (\tilde{h}^k \tilde{u}^k \tilde{v}^k)}{\partial x} + \frac{\partial (\tilde{h}^k (\tilde{v}^k)^2)}{\partial y} \end{bmatrix}. \quad (15)$$

By vertical integration of the continuity relation (eq. 8) in the pressure interval $[\tilde{p}(s^{k+1}), \tilde{p}(s^k)]$, we supplement the momentum conservation law of eq. (11) with the mass conservation law

$$\frac{\partial \tilde{h}^k}{\partial t} + \text{div}(\tilde{\mathbf{q}}^k) = 0, \quad (16)$$

and obtain the independent Saint–Venant equation system for any atmospheric layer $k \in [1, K]$:

$$\begin{cases} \frac{\partial \tilde{h}^k}{\partial t} + \text{div}(\tilde{\mathbf{q}}^k) = 0, \\ \frac{\partial (\tilde{\mathbf{q}}^k)}{\partial t} + \text{div} \left(\frac{1}{\tilde{h}^k} \tilde{\mathbf{q}}^k \otimes \tilde{\mathbf{q}}^k \right) + \frac{1}{2\rho_0^k} \nabla_{xy} (\tilde{h}^k)^2 + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f^\phi \tilde{\mathbf{q}}^k \\ = \nu_T^k \Delta (\tilde{\mathbf{q}}^k). \end{cases} \quad (17)$$

Note that as we are using the isobaric coordinate system, partial derivatives w.r.t. x, y and t are defined at constant pressure p . However, according to Section 2.2, pressure difference observations $\tilde{h}_{\text{obs}}^k$ correspond to rough constant pressure intervals. Therefore, such data fit the shallow-water model of eq. (17) defined in the isobaric coordinate system. This model will be the basis of our assimilation process. Before describing the developed approach, the next section presents the data assimilation principle.

3. Data assimilation

In this section, we present the main principles of variational data assimilation for an imperfect dynamic model. We refer the reader to Bennet (1992), Le Dimet & Talagrand (1986), Lions (1971), Talagrand and Courtier (1987) and Vidard et al. (2000) for complete methodological aspects of data assimilation and applications concerning geophysical flows.

Our problem consists of recovering, from an initial condition, a system's state X partially observed and driven by approximately known dynamics. This can be formalized as finding $X(\mathbf{s}, t)$, for any location $\mathbf{s}$ at time $t \in [t_0, t_f]$, that satisfies the system:

$$\frac{\partial X}{\partial t}(\mathbf{s}, t) + \mathbb{M}(X(\mathbf{s}, t)) = v_m(\mathbf{s}), \quad (18)$$

$$X(\mathbf{s}, t_0) = X_0(\mathbf{s}) + v_n(\mathbf{s}), \quad (19)$$

$$\mathcal{Y}(\mathbf{s}, t) = \mathbb{H}(X(\mathbf{s}, t)) + v_o(\mathbf{s}, t), \quad (20)$$

where $\mathbb{M}$ is the non-linear operator relative to the dynamics, X_0 is the initial vector at time t_0 and (v_n, v_m) are (unknown) additive control variables relative to noise on the dynamics and the initial condition respectively. In addition, noisy measurements $\mathcal{Y}$ of the unknown state are available through the non-linear operator $\mathbb{H}$ up to v_o . To estimate the system's state, a common methodology is the minimization of the cost function $\mathcal{J}$:

$$\begin{aligned} \mathcal{J}(X) = & \frac{1}{2} \int_{t_0}^{t_f} \|\mathcal{Y} - \mathbb{H}(X(v_m, v_n))\|_{R^{-1}}^2 dt \\ & + \frac{1}{2} \|X(\mathbf{s}, t_0) - X_0(\mathbf{s})\|_{B^{-1}}^2 \\ & + \frac{1}{2} \int_{t_0}^{t_f} \left\| \frac{\partial X}{\partial t}(\mathbf{s}, t) + \mathbb{M}(X(\mathbf{s}, t)) \right\|_{Q^{-1}}^2 dt, \end{aligned} \quad (21)$$

where we have introduced the information matrices R, B, Q relative to the covariance of the errors (v_m, v_n, v_o) . The Mahalanobis distance that has been used reads $\|X\|_{A^{-1}} = X^T A^{-1} X$. The evaluation of X can be done by cancelling the gradient $\delta \mathcal{J}_X(\theta) = \lim_{\beta \rightarrow 0} \frac{J(X+\beta\theta) - J(X)}{\beta}$ of (21). Unfortunately, the estimation of such gradient is in practice unfeasible for a large system's state since it would be necessary to compute perturbations along all the components of X . One way to cope with this difficulty is to write 'an adjoint formulation' of the problem. To that end, the 'adjoint variables' λ that express the errors of the dynamic model are introduced as

$$\lambda = Q^{-1} \left(\frac{\partial X}{\partial t} + \mathbb{M}(X + \beta\theta) \right). \quad (22)$$

Denoting

- (1) $(\frac{\partial \mathbb{M}}{\partial X})$ and $(\frac{\partial \mathbb{H}}{\partial X})$ the 'linear tangent operators' of $\mathbb{M}$ and $\mathbb{H}$ respectively,⁴
- (2) $(\partial_X \mathbb{M})^*$ and $(\partial_X \mathbb{H})^*$ their 'adjoint operators',⁵

it can be shown that cancelling the gradient $\delta \mathcal{J}_X(\theta)$ w.r.t. the adjoint variables λ leads to a retrograde integration of an adjoint evolution model that takes into account the observations. Once the adjoint variables λ are estimated, one can recover the system

state X using relation (22). Finally, recovering X leads to the following incremental algorithm (Bennett & Thorburn, 1992):

- (1) Starting from $\tilde{X}(\mathbf{s}, t_0) = X_0(\mathbf{s})$, perform a 'forward' integration: $\frac{\partial \tilde{X}}{\partial t} + \mathbb{M}(\tilde{X}) = 0$;
- (2) $\tilde{X}(\mathbf{s}, t)$ being available, 'compute the adjoint variables' $\lambda(\mathbf{s}, t)$ with the 'backward' equation
$$\lambda(t_f) = 0; \\ - \frac{\partial \lambda}{\partial t}(t) + (\partial_X \mathbb{M})^* \lambda(t) = (\partial_X \mathbb{H})^* R^{-1}(\mathcal{Y} - \mathbb{H}(\tilde{X}))(t); \quad (23)$$
- (3) 'Update the initial condition': $dX(t_0) = B\lambda(t_0)$;
- (4) λ being available, 'compute the state space' $dX(t)$ from $dX(t_0)$ with the 'forward' integration
$$\frac{\partial dX}{\partial t}(t) + \left(\frac{\partial \mathbb{M}}{\partial \tilde{X}} \right) dX(t) = Q\lambda(t); \quad (24)$$
- (4) 'Update': $\tilde{X} = \tilde{X} + dX$;
- (5) 'Loop' to step (2) until convergence.

Intuitively, the adjoint variables λ contain information about the discrepancy between the observations and the dynamic model. They are computed from a current solution $\tilde{X}$ with the backward integration (23) that encompasses both the observations and the dynamic operators. This deviation indicator between the observations and the model is then used to refine the initial condition (step 3) and to recover the system state through an imperfect dynamic model where errors are $Q\lambda$ (step 4). It should be noted that if the dynamic is perfect, the associated error covariance Q is zero and the algorithm only refines the initial condition. However, from an image analysis point of view, a perfect modelling is difficult to obtain since the different models on which one can rely are usually inaccurate due, for instance, to 3-D to 2-D projections, varying lighting conditions, completely unknown boundary conditions at the image borders, etc.

4. Data assimilation for multilayer shallow-water model

In this section, we propose to develop an image assimilation technique based on (17) to estimate the mesoscale flows. To that end, we further discuss in Section 4.1 the shallow-water model and its limitations based on the available observations. The Sections 4.2, 4.3, 4.4 and 4.5 are, respectively, devoted to the detailed description of the system state, the selected dynamic model, the observation system and the definition of the different error covariance matrices.

4.1. Dynamic model

4.1.1. Limitation of Saint Venant's shallow-water model. The dynamic model of eq. (17) describes accurately the mesoscale atmospheric flows. Therefore, one can immediately

⁴ The linear tangent of an operator $\mathbb{A}$ is the directional derivative of the operator (the Gâteaux derivative): $\lim_{\beta \rightarrow 0} \frac{\mathbb{A}(X+\beta\theta) - \mathbb{A}(X)}{\beta}$.

⁵ The adjoint $\mathbb{A}^*$ of a linear operator $\mathbb{A}$ on a space $\mathcal{D}$ is such as $\forall x_1, x_2 \in \mathcal{D}, \langle \mathbb{A}x_1, x_2 \rangle = \langle x_1, \mathbb{A}^*x_2 \rangle$.

define an assimilation scheme relying on the system variables.

$$X = \begin{bmatrix} \tilde{h}^k \\ \tilde{\mathbf{q}}^k \end{bmatrix}, \quad (25)$$

where the perfect dynamic model is given by relation (17). Remembering that images of pressure differences, h^k , are observable, it is possible to define the measurement system $\mathcal{Y} = \mathbb{H}(X)$ as

$$\begin{cases} \mathcal{Y} = \tilde{h}_{\text{obs}}^k = K_{\delta_x} * h_{\text{obs}}^k, \\ \mathbb{H} = [I_d, 0]. \end{cases} \quad (26)$$

This system has the advantage of assimilating both the sparse pressure differences $\tilde{h}_{\text{obs}}^k$ and the associated motion field $\tilde{\mathbf{q}}^k$ without observing this latter quantity. However, it is important to note that the proposed shallow-water system describes the dynamics of physical quantities under a layering assumption. In practice, our data come from the Meteosat Second Generation (MSG) sensor, where a pixel corresponds to a square of $3000 \times 3000 \text{ m}^2$. Even considering a spatial convolution by a Gaussian kernel K_{δ_x} , a scale analysis reported in Appendix C shows that the term $\frac{1}{2\rho_0^k} \nabla_{xy}(\tilde{h}^k)^2$ in (17) locally dominates. This is all the more true in areas where the layering assumption breaks down at the considered observation scale. In particular, this term dominates on the borders of segments in sparse pressure images, where a pixel may contain heterogeneous contributions from different pressure levels. This perfect model is hence too restrictive and is likely to generate some numerical instabilities. We consequently prefer to develop, from this shallow-water model (17), an imperfect dynamic system that describes the evolution of the flow while authorizing the layering assumption to be violated at some locations. This is the scope of the next section.

4.1.2. Divergence-vorticity shallow-water dynamic model.

To overcome the limitations mentioned above, we describe the evolution of the filtered state variables from a simplified version of the previously introduced shallow-water equations and perform assimilation using an imperfect modelling scheme. To that end, we assume filtered horizontal motion components to be homogeneous within the layer. In other words, we neglect their vertical derivatives and consider that filtered horizontal winds $\tilde{\mathbf{v}}^k$, which have been vertically averaged, are equal to filtered horizontal winds on layer upper surfaces $\mathbf{s}^{k+1}$. Using such an assumption yields

$$\tilde{\mathbf{v}}_t^k + \nabla(\tilde{\mathbf{v}}^k) \tilde{\mathbf{v}}^k - \rho_0^{-1} \nabla \tilde{p}(s^{k+1}) + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f^\phi \tilde{\mathbf{v}}^k = \nu_T \Delta(\tilde{\mathbf{v}}^k), \quad (27)$$

with the notations $\nabla(\tilde{\mathbf{v}}^k) = (\nabla \tilde{u}^k, \nabla \tilde{v}^k)^\top$ and $\Delta(\tilde{\mathbf{v}}^k) = (\Delta \tilde{u}^k, \Delta \tilde{v}^k)^\top$. Let us denote the vorticity by $\tilde{\zeta}^k = \text{curl}(\tilde{\mathbf{v}}^k)$ and the divergence by $\tilde{D}^k = \text{div}(\tilde{\mathbf{v}}^k)$. The previous system may be expressed in its vorticity–divergence form

$$\begin{cases} \tilde{\zeta}_t^k + \tilde{\mathbf{v}}^k \cdot \nabla \tilde{\zeta}^k + (\tilde{\zeta}^k + f^\phi) \tilde{D}^k = \nu_T \Delta(\tilde{\zeta}^k), \\ \tilde{D}_t^k + \tilde{\mathbf{v}}^k \cdot \nabla \tilde{D}^k + (\tilde{D}^k)^2 - 2|J| \\ \quad - \rho_0^{-1} \Delta \tilde{p}(s^{k+1}) + f^\phi \tilde{\zeta}^k = \nu_T \Delta(\tilde{D}^k), \end{cases} \quad (28)$$

where $|J|$ is the determinant of the Jacobian matrix of variables $(\tilde{u}^k, \tilde{v}^k)$. For vorticity based large eddy simulation formulations, we may rely on enstrophy-based subgrid models (Mansour et al., 1978) instead of using the Smagorinsky model (eq. 5). This subgrid dissipation model is based on Taylor’s vorticity transfer and dissipation by small scales theory (Taylor, 1932) and reads

$$\nu_T = (C\delta_x)^2 |\tilde{\zeta}^k|. \quad (29)$$

In the momentum conservation formulations of eqs (27) and (28), dynamic models predict the evolution of velocity components $(\tilde{u}^k, \tilde{v}^k)$ and of divergence and vorticity $(\tilde{\zeta}^k, \tilde{D}^k)$. In both models, one of the major difficulties is created by the dependence of the pressure variable $\tilde{p}(s^{k+1})$ which is an unknown variable of the k^{th} layer state. Adding pressure as a new state variable driven by the mass conservation law (16) constitutes a solution for achieving the model integrations. However, in Section 4.4, we will need this mass conservation law to define a proper observation operator. Thus, we search instead to derive a dynamic model that is independent of the pressure unknown. In contrast to the classical formulation, the vorticity–divergence equations have the advantage of providing such a model for the vorticity evolution. As regards the divergence, since at large-scales, it can be considered weak almost everywhere, we will rely on an approximate evolution law. We assume here that the divergence is advected by the flow and a noise variable that encodes the uncertainty on the model. More precisely, we will assume that the divergence map is a function of a stochastic process representing a particle position and is driven by the following stochastic differential equation:

$$d\mathbf{x}(t) = \tilde{\mathbf{v}}^k(\mathbf{x}(t))dt + \sqrt{2\nu_T} dB(t). \quad (30)$$

This equation states that the particle position is known only up to an uncertainty that grows linearly with time. Here B_t denotes a standard Brownian motion of $\mathbb{R}^2$. The process $\mathbf{x}_t$ starts at point, $\mathbf{x}_0$. It can be shown through the Ito formula and Kolmogorov’s forward equation, that the expectation at time t of such a function, $\xi(t, \mathbf{x}) = \mathbb{E}[\text{div} \tilde{\mathbf{v}}^k(\mathbf{x}(t))]$ follows an advection diffusion equation (Oksendal, 1998)

$$\begin{aligned} \xi_t + \tilde{\mathbf{v}}^k \cdot \nabla \xi + \xi \text{div} \tilde{\mathbf{v}}^k - \nu_T \Delta \xi &= 0, \\ \xi(0, \mathbf{x}_0) &= \text{div} \tilde{\mathbf{v}}^k(\mathbf{x}_0). \end{aligned} \quad (31)$$

Assuming that the divergence of the flow is given by its expectation ($\tilde{D}^k \approx \xi$) allows us to write the simplified filtered vorticity–divergence model for the layer k as

$$\begin{cases} \tilde{\zeta}_t^k + \tilde{\mathbf{v}}^k \cdot \nabla \tilde{\zeta}^k + (\tilde{\zeta}^k + f^\phi) \tilde{D}^k = \nu_T \Delta \tilde{\zeta}^k, \\ \tilde{D}_t^k + \tilde{\mathbf{v}}^k \cdot \nabla \tilde{D}^k + (\tilde{D}^k)^2 = \nu_T \Delta \tilde{D}^k. \end{cases} \quad (32)$$

In this model, we assume that the divergence of the flow is weak and is similar to the divergence expectation. The divergence equation no longer describes the evolution of the flow divergence but the evolution of its expectation. The expectation of the divergence value is advected by the flow and dissipates

due to a subgrid isotropic uncertainty. This hypothesis is quite natural in large-scale modelling.

4.2. State variables

We choose to represent the system state X (i.e. the velocity field) through the curl and divergence components:

$$X = \begin{bmatrix} \tilde{\zeta}^k \\ \tilde{D}^k \end{bmatrix}, \quad (33)$$

as their evolution can be described by eq. (32) and since they completely determine the underlying 2-D velocity up to a harmonic transportation component. Denoting the orthogonal gradient by $\nabla^\perp = (-\partial/\partial y, \partial/\partial x)^\top$ and the 2-D Green kernel G associated with the Laplacian operator ($G(\cdot) = \frac{1}{2\pi} \log(\cdot)$), the Helmholtz decomposition of the field can be expressed as

$$\tilde{\mathbf{v}}^k = \underbrace{\nabla^\perp(G * \tilde{\zeta}^k)}_{\tilde{\mathbf{v}}_{\text{sol}}^k} + \underbrace{\nabla(G \times \tilde{D}^k)}_{\tilde{\mathbf{v}}_{\text{irr}}^k} + \tilde{\mathbf{v}}_{\text{har}}^k, \quad (34)$$

$$\begin{cases} \lambda_\zeta^k(t_f) = 0, \\ \lambda_D^k(t_f) = 0, \\ -\partial_t \begin{bmatrix} \lambda_\zeta^k \\ \lambda_D^k \end{bmatrix} + \underbrace{\begin{bmatrix} -\tilde{\mathbf{v}}^* \cdot \nabla + \tilde{D}^* - \nu_T \Delta & \tilde{\zeta}^* + f^\phi \\ 0 & -\tilde{\mathbf{v}}^* \cdot \nabla + \tilde{D}^* - \nu_T \Delta \end{bmatrix}}_{(\partial_{X^*} \mathbb{M})^*} \begin{bmatrix} \lambda_\zeta^k \\ \lambda_D^k \end{bmatrix} \\ = \partial_{X^k} \mathbb{H}^* R^{-1}(Y - \mathbb{H}(X)). \end{cases} \quad (37)$$

where $\tilde{\mathbf{v}}_{\text{har}}^k$ is a harmonic transportation part ($\text{div} \tilde{\mathbf{v}}_{\text{har}}^k = \text{curl} \tilde{\mathbf{v}}_{\text{har}}^k = 0$). In our applications, we will assume that this component of the velocity is stationary within the temporal interval of the analysed image sequence. This component may be recovered by subtracting its solenoidal and irrotationnal parts from the initial field $\tilde{\mathbf{v}}_0^k$. As a consequence, the field can be represented by its div-curl components as

$$\tilde{\mathbf{v}}^k = \nabla^\perp(G \times \tilde{\zeta}^k) + \nabla(G \times \tilde{D}^k) = \underbrace{[\nabla^\perp G \times, \nabla G \times]}_{\mathbb{H}_G} \underbrace{\begin{bmatrix} \tilde{\zeta}^k \\ \tilde{D}^k \end{bmatrix}}_X, \quad (35)$$

where the operator $\mathbb{H}_G$ can efficiently be computed in the Fourier domain.

4.3. Dynamic operator and its adjoint

The dynamic model required for the assimilation in (18) is defined by relation (32). This model is associated with an imperfect dynamic modelling where the model errors are related to the Coriolis effect, to the pressure difference dissipation of the

divergence component and to the subgrid stress tensor error of the vorticity and divergence component.

The associated tangent linear system (eq. 24) around the current solution $X^* = [\tilde{\zeta}^*, \tilde{D}^*]^T$ is

$$\begin{aligned} \partial_t \begin{bmatrix} \tilde{\zeta}^k \\ \tilde{D}^k \end{bmatrix} + \underbrace{\begin{bmatrix} \tilde{\mathbf{v}}^* \cdot \nabla + \tilde{D}^* - \nu_T \Delta & \tilde{\zeta}^* + f^\phi \\ 0 & \tilde{\mathbf{v}}^* \cdot \nabla + \tilde{D}^* - \nu_T \Delta \end{bmatrix}}_{\partial_{X^*} \mathbb{M}} \begin{bmatrix} \tilde{\zeta}^k \\ \tilde{D}^k \end{bmatrix} \\ = Q^k \lambda, \end{aligned} \quad (36)$$

where $\nu_T = (C\delta_x)^2 |\tilde{\zeta}^*|$. This equation includes an error covariance matrix Q^k that will be further defined in Section 4.5. At each time increment $d t$, the current velocity solution $\tilde{\mathbf{v}}^*$ is updated according to the vorticity $\tilde{\zeta}^k$, the divergence $\tilde{D}^k$ and the harmonic transportation $\tilde{\mathbf{v}}_{\text{har}}^k$ using the operator defined in eq. (35). The adjoint variables $[\lambda_\zeta^k, \lambda_D^k]^T$ are obtained through the following adjoint system:

We should note that the turbulent viscosity ν_T and the velocity $\tilde{\mathbf{v}}^*$ are computed during the direct integration and maintained in the backward integration. The discretization of the bi-dimensional advection system must be done carefully to prevent numerical explosions. In this application, it has been achieved using the non-oscillatory schemes proposed in (Xu and Shu, 2006). The temporal integration has been implemented through a third-order Runge–Kutta scheme and the time step dt is defined to respect the total variation diminishing property (Kurganov & Tadmor, 2000). Once the direct dynamic model has been discretized, the matrix $(\partial_{X^*} \mathbb{M})$ is available and the adjoint system involved in (37) corresponds to the transpose of this matrix. More details about the construction of adjoint models can be found in (Talagrand and Courtier, 1987).

The observation components $\mathcal{Y} = \mathbb{H}(X)$ and the error covariance matrices are defined in the next sections.

4.4. Image observation operator

4.4.1. Extracting motion fields from pressure images. To define an observation operator that links the unknown motion field $\tilde{\mathbf{v}}^k$ to images of pressure differences $h_{\text{obs}}^k(\mathbf{s}, t)$, we use the

mass conservation law of eq. (16):

$$\frac{\partial h_{\text{obs}}^k(\mathbf{s}, t)}{\partial t} + \nabla h_{\text{obs}}^k(\mathbf{s}, t) \cdot \tilde{\mathbf{v}}^k(\mathbf{s}, t) + h_{\text{obs}}^k(\mathbf{s}, t) \text{div} \tilde{\mathbf{v}}^k(\mathbf{s}, t) \approx 0. \quad (38)$$

However, this formulation alone cannot be used to estimate $\tilde{\mathbf{v}}^k$ as it provides only one equation for two unknowns at each spatio-temporal location $(\mathbf{s}, t)$. To remove such ambiguities, a common approach is to assume a spatial coherence of wind field estimates in a given neighbourhood (similar to the well-known approach of (Lucas and Kanade, 1981) used in computer vision). In the present case, this assumption is coherent since unknowns are velocity vectors $\tilde{\mathbf{v}}^k(\mathbf{s}, t)$ that are spatially filtered within a neighbourhood of size δ_x . The measured motion field should thus obey the following constraint:

$$K_{\delta_x} \times \left(\frac{\partial h_{\text{obs}}^k}{\partial t} + \nabla h_{\text{obs}}^k \cdot \tilde{\mathbf{v}} + h_{\text{obs}}^k \text{div} \tilde{\mathbf{v}}^k \right) \approx 0, \quad (39)$$

where K_{δ_x} is the Gaussian kernel of standard deviation δ_x . We should note that in practice, to prevent the influence of areas without any data in the border of the regions of interest C^k , all the points without pressure information have been removed from the Gaussian smoothing. From the relation (39) and recalling that $\tilde{\mathbf{v}}^k = \mathbb{H}_G X$ (with $X = [\tilde{\zeta}^k, \tilde{D}^k]^T$), one can easily define our observation system $\mathcal{Y} = \mathbb{H}(X)$ as

$$\begin{cases} \mathcal{Y} = K_{\delta_x} * \frac{\partial h_{\text{obs}}^k(\mathbf{s}, t)}{\partial t}, \\ \mathbb{H}(X) = - (K_{\delta_x} * \nabla h_{\text{obs}}^k)^T \mathbb{H}_G(X), \\ \quad - (K_{\delta_x} * h_{\text{obs}}^k) \nabla^T \mathbb{H}_G(X). \end{cases} \quad (40)$$

The observation operator $\mathbb{H}$ is linear w.r.t. the state variable $X = [\tilde{\zeta}^k, \tilde{D}^k]^T$ and is then identical to its linear tangent operator. Noting that the adjoint of $\mathbb{H}_G$ is $-\mathbb{H}_G$ (as shown in Papadakis and Mémin, 2007), the adjoint $\partial_{X^*} \mathbb{H}^*(Y)$ of $\mathbb{H}(X)$ therefore reads

$$\partial_{X^*} \mathbb{H}^*(Y) = \mathbb{H}_G ((K_{\delta_x} * \nabla h_{\text{obs}}^k) Y) - \mathbb{H}_G ((\nabla((K_{\delta_x} * h_{\text{obs}}^k) Y)) \quad (41)$$

4.4.2. Observation retrieval and representativeness. Most of the motion estimation techniques, which aim at extracting wind fields from satellite images (like AMV, Atmospheric Motion Vectors from Eumetsat), are based on cloud luminance pattern matching over consecutive frames. Such techniques assume that clouds behave like tracers of the underlying motion field and that the cloud brightness function remains constant over time. It is, unfortunately, well known that these hypotheses are not always valid for satellite images (Schmetz et al., 1993; Corpetti et al., 2002). A worthwhile benefit of the present study is to dispense with these assumptions since the observed pressure differences are directly linked to the local Eulerian wind fields through a relation derived from the continuity equation.

However, these pressure difference maps have been obtained through a sequence of manipulations (cloud classification, conversion from pixel to temperature, from temperature to pressure)

and simplifications (the pressure value of the unobserved ‘lower’ level of the layer is arbitrarily assigned to a mean value). These manipulations are likely to generate some errors in the input data. These errors are quantifiable from Eumetsat products through three possible values of a quality parameter η of the cloud height assignment. These values are set to $\eta(\mathbf{s}) = 0$ (no pressure in $\mathbf{s}$), 1 (pressure of good quality) or 2 (poor quality). This information will be taken into account for the definition of the error covariance matrices in the next section.

4.5. Error covariance matrices and initialization issues

4.5.1. Observations. The inverse error covariance matrix related to the observations $(R^k)^{-1}$ is diagonal and defined using the mask of observation C^k :

$$(R^k(\mathbf{s}, \mathbf{s}'))^{-1} = \begin{cases} (Z^k)^{-1} (P^k)^{-1} & \text{if } \mathbf{s} = \mathbf{s}' \text{ and } \mathbf{s} \in C^k \\ 0 & \text{else.} \end{cases} \quad (42)$$

The inverse error covariance matrices $(Z^k)^{-1}$ and $(P^k)^{-1}$ are, respectively, related to a confidence measure related to the observation operator and on the pressure height assignment errors. We chose to define $(Z^k)^{-1}$ as

$$(Z^k(\mathbf{s}, \mathbf{s}'))^{-1} = \alpha_{\text{obs}} \begin{cases} \exp \frac{[\mathcal{Y} - \mathbb{H}(X)]^2}{\sigma^2} & \text{if } \mathbf{s} = \mathbf{s}' \text{ and } \mathbf{s} \in C^k \\ 0 & \text{else.} \end{cases} \quad (43)$$

where σ and α_{obs} are parameters to be fixed. As shown in Corpetti et al. (2008), this penalization amounts to considering a robust norm on the first term of the cost-function in (21). Such a robust function error $[\mathcal{Y} - \mathbb{H}(X)]$ (called ‘outliers’ in the Robust Statistics literature (Huber, 1981)). Compared with a quadratic cost function (i.e. $R = I_d$), the use of such robust semi-norms permits limiting the influence of those erroneous points. The robust norm, $\Phi(\cdot)$, is expressed here in an equivalent (w.r.t. its minimization) weighted quadratic expression, in which the weight function involved, $z(\cdot) = \Phi'(\sqrt{\cdot}) = (Z^k)^{-1}(x, y)$, is iteratively updated at every grid point. Different types of cost functions (convex or non-convex) that obey this semi-quadratic reformulation can be chosen under mere assumptions (mainly strict concavity of $\Phi'(\sqrt{\cdot})$). Interested readers may consult a number of courses to obtain deeper insights into the use of robust functions and on the convergence of the associated semi-quadratic minimization process (Huber, 1981; Geman & Reynolds, 1992; Delanay & Bresler, 1998). Such robust estimators have demonstrated significant improvements for computer vision applications (Geman & Reynolds, 1992; Mémin and Pérez, 1998). In our application, it enables us to properly deal with corrupted areas that do not fit our data model exactly.

As for matrix $(P^k)^{-1}$, it is directly defined from the quality index η :

$$(P^k)^{-1}(\mathbf{s}, \mathbf{s}) = \begin{cases} 0 & \text{if } \eta(\mathbf{s}) = 0, \quad \mathbf{s} = \mathbf{s}' \text{ and } \mathbf{s} \in C^k, \\ 1 & \text{if } \eta(\mathbf{s}) = 1, \quad \mathbf{s} = \mathbf{s}' \text{ and } \mathbf{s} \in C^k, \\ 0.5 & \text{if } \eta(\mathbf{s}) = 2, \quad \mathbf{s} = \mathbf{s}' \text{ and } \mathbf{s} \in C^k, \\ 0 & \text{else.} \end{cases} \quad (44)$$

In this way, the impact of the measurements in areas associated with a poor quality of the pressure level is lowered.

4.5.2. Initial conditions. As no guarantee of convergence towards a global minimum can be ensured for such a non-convex functional, the quality of results depends on the initial conditions given to the system. We chose to initialize the velocities $\tilde{\mathbf{v}}_0^k$ and the associated divergence and vorticity $[D_0^k, \xi_0^k]$ by filtering wind fields provided by the optic-flow algorithm dedicated to atmospheric layers proposed in Héas et al. (2007) with the Gaussian Kernel K_{δ_x} . Two inverse diagonal error covariance matrices $(B_D^k)^{-1}$ and $(B_\xi^k)^{-1}$ are defined as

$$\begin{cases} (B_D^k)^{-1}(\mathbf{s}, \mathbf{s}) = \alpha_D, \\ (B_\xi^k)^{-1}(\mathbf{s}, \mathbf{s}) = \alpha_\xi, \end{cases} \quad (45)$$

where α_D and α_ξ are fixed parameters defining the inverse of the initial variable error covariances.

4.5.3. Dynamics. A nine-diagonal inverse error covariance matrix $(Q^k)^{-1}$ fixing the model inaccuracy with a given parameter α_Q has been defined for the dynamic model in a similar way as for the initial conditions.

All the components needed for the definition of our assimilation process have now been successively defined in Sections 4.1, 4.2, 4.3, 4.4 and 4.5. The next section presents some experiments using synthetic and real data.

5. Experiments

For a quantitative evaluation of the proposed image assimilation scheme, we have first performed experiments on synthetic image observations generated by short time numerical simulation of atmospheric layer motion. This evaluation is described in Section 5.1. As the dynamic model of our assimilation process in (32) is a simplified version of the primitive shallow-water model presented in (17), two experiments related to different synthetic observations have been carried out. The first synthetic experiment (Section 5.1.1) uses images generated with the simplified model whereas the images of the second one (Section 5.1.2) have been produced following the shallow-water model. Section 5.2 is devoted to the application of the assimilation scheme on real Meteosat data. The core of Section 5.3 is a discussion of practical aspects.

5.1. Synthetic image sequences

For the sake of clarity, we denote the proposed simplified dynamic model in (32) as ‘model-a’ and the primitive shallow-water model (17) as ‘model-b’. For both cases, we selected realistic initial conditions (for images and motion fields) to integrate the models in an equivalent time period of 2 h 30 min and to create two different sequence benchmarks composed of 10 frames each, which are denoted respectively as ‘seq-a’ (for model-a) and ‘seq-b’ (for model-b). Each image is composed of 128×128 pixels and represents a spatial area of approximately 400 km^2 . These benchmarks have then been deteriorated by different noises and by masking operations corresponding to real situations (the masking area represents approximately 80% of the global area). We formed four synthetic sequences a_1, a_2, a_3, a_4 related to seq-a and four other synthetic sequences b_1, b_2, b_3, b_4 related to seq-b. Sequences a_1, b_1 and a_2, b_2 were composed of dense observations of h_{obs}^k in hPa, corrupted by Gaussian noises with standard deviation respectively equal to 20% and 40% of the pressure amplitude. A realistic cloud classification sequence was employed to extract regions from a_1, a_2, b_1 and b_2 to create the noisy and incomplete synthetic sequences a_3, a_4, b_3 and b_4 . To initialize the assimilation systems, we have used the synthetic ground truth values of $\tilde{\mathbf{v}}^k(t_0)$, deteriorated by Gaussian noises with standard deviation equal to 20% and 40% of the variable amplitudes, respectively, for experiments using synthetic sequences a_1, a_3, b_1, b_3 and a_2, a_4, b_2, b_4 .

Table 1 summarizes the characteristics of the two sequences used. This table also presents the initial value of root mean square error (RMSE) of state variables. These values correspond to the discrepancy between the ground truth and the state variables obtained after a first direct integration from the noisy initial conditions. Figure 2 displays examples of noisy and sparse pressure difference observations and initial values of the state variables used in the experiments.

5.1.1. Synthetic sequence generated from the proposed simplified model. In this experiment, the dynamic model involved in the generation of the image sequence exactly corresponds to the dynamics used in the assimilation process (model-a, eq. 32). The goal of this experiment is to analyse the robustness of the assimilation w.r.t. noisy data and initial settings.

Comparing Tables 1 and 2, one can note a significant decrease induced by the assimilation system on the RMSE between the actual and the estimated velocities; the final RMSE is reduced roughly by a factor of $\approx 85\%$ by the assimilation process. Moreover, the stable behaviour of the RMSE (in the four cases considered) demonstrates the robustness of the approach to cope with incomplete and noisy observations. A motion field and the associated vorticity and divergence maps obtained for experiment a_4 are plotted in the top part of Fig. 3 and compared with the actual solution. A comparison between those images and the images of their initial values, displayed in Fig. 2, clearly illustrates the estimation efficiency.

Table 1. Initial setup used for assimilation with images issued from ‘model-a’ (top) and ‘model-b’ (bottom). Root mean square error (RMSE) on the vorticity $\tilde{\zeta}^k$, divergence $\tilde{D}^k$ and wind norm $\tilde{v}^k$ obtained at all times t of the image synthetic sequences

	Mask	Noise %	RMSE on all $\tilde{\zeta}^k(t)$ (1/frame)	RMSE on all $\tilde{D}^k(t)$ (1/frame)	RMSE on all $ \tilde{v}^k(t) $ (pixel/frame)
a_1	no	20	0.01119	0.01343	0.08798
a_2	no	40	0.01481	0.01518	0.09512
a_3	yes	20	0.01119	0.01343	0.08798
a_4	yes	40	0.01481	0.01518	0.09512
b_1	no	20	0.01297	0.00932	0.09352
b_2	no	40	0.01670	0.01814	0.1009
b_3	yes	20	0.01297	0.00932	0.09352
b_4	yes	40	0.01670	0.01814	0.1009

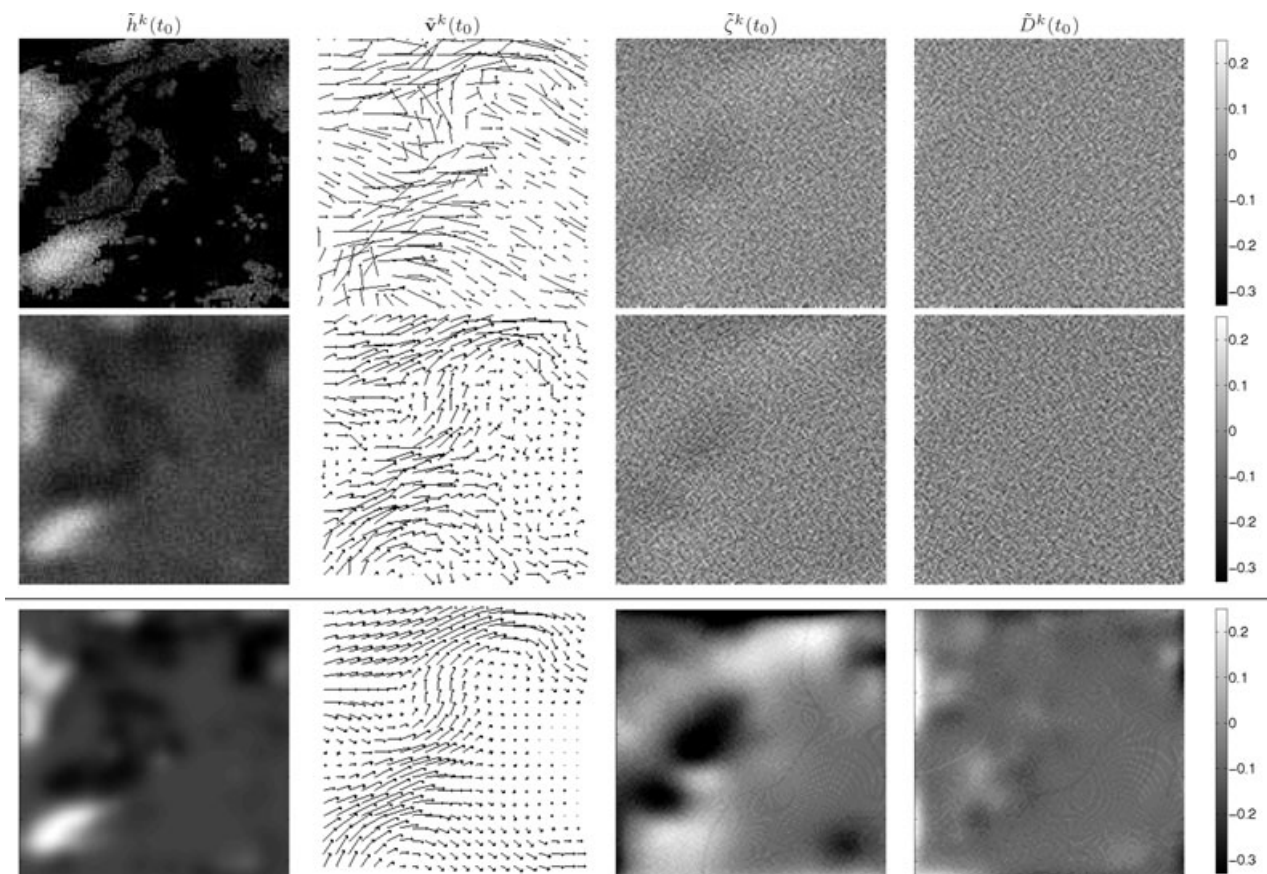


Fig 2. Examples of variables at initial time before assimilation with images issued from ‘model-a’ (top), ‘model-b’ (middle) and the corresponding ground truth (bottom). For each experiment, the columns exhibit blurred initial difference pressure image $\tilde{h}^k$, wind $\tilde{v}^k$, vorticity $\tilde{\zeta}^k$ and divergence $\tilde{D}^k$, respectively. The top part corresponds to the experiment a_4 whereas the middle part represents data of experiment b_2 .

5.1.2. Synthetic sequence generated from the primitive shallow-water model. In this application, the dynamic model used for the construction of images corresponds to the primitive shallow-water model (eq. 17). Therefore, it differs from the dynamic model considered in the assimilation process. The goal of this experiment is then twofold: (1) to analyse the ability of the

method to deal with an imperfect dynamic modelling involving additional uncertainty variables and (2) to evaluate the accuracy of the simplification considered in Section 4.1.2

Comparing Tables 1 and 2, a significant decrease of the RMSE after assimilation can still be observed. The final RMSE of this experiment is reduced by approximately 75%. The residual

Table 2. Final estimation errors obtained by our assimilation approach with images issued from ‘model-a’ (top) and ‘model-b’ (bottom). Root Mean Square Error (RMSE) and residual error (RE) on estimates $\tilde{\zeta}^k$, $\tilde{D}^k$ and $\tilde{v}^k$ obtained at all times t of the image synthetic sequences

	Errors on $\tilde{\zeta}^k(t)$		Errors on $\tilde{D}^k(t)$		Errors on $ \tilde{v}^k(t) $	
	RMSE (1/frame)	RE %	RMSE (1/frame)	RE %	RMSE (pixel/frame)	RE %
a_1	0.00379	8.71	0.00182	9.99	0.01787	9.59
a_2	0.00405	8.91	0.00214	10.87	0.01783	11.12
a_3	0.00378	9.20	0.00182	11.61	0.01782	9.58
a_4	0.00405	9.51	0.00214	11.74	0.01784	11.11
b_1	0.00514	8.94	0.00444	13.21	0.01883	9.77
b_2	0.00536	9.37	0.00450	13.62	0.02139	11.27
b_3	0.00514	9.43	0.00441	13.41	0.02000	10.61
b_4	0.00560	10.69	0.00481	14.74	0.02429	13.45

errors of the divergence and vorticity recovered are slightly above the errors obtained in the previous experiment but remains very weak. For both experiments, the estimated velocity fields are of the same order of accuracy. This result demonstrates the capacity of the simplified dynamics together with a weak constraint assimilation formulation to fit an exact shallow-water dynamic. Here again, the stable behaviour of the RMSE on vorticity, divergence and motion fields for the four experimental cases is notable. Vorticity, divergence and motion field estimates obtained from experiment b_2 are presented at the bottom of Fig. 3. The efficiency of the process can be observed by comparing those images with the images displayed in Fig. 2 and corresponding to the initial values of the state variables for the same benchmark.

The performance of the simplified dynamic model has been assessed from two different synthetic cases. These experiments have shown the capacity of the proposed assimilation scheme to estimate accurate motion fields from synthetic pressure sequences. Let us now study the behaviour of the approach on a real case.

5.2. METEOSAT Satellite image sequence

We turn now to a qualitative evaluation of our method on MSG meteorological image sequences acquired over the Northern Atlantic Ocean on 5 June 2004 from 13 h 30 until 15 h 45 UTC at the rate of one image per 15 min. This benchmark data, which has been provided by the EUMETSAT consortium, is composed of 10 frames sequence of top of cloud pressure and cloud-classification images. The image spatial resolution is $3 \times 3 \text{ km}^2$ at the centre of the whole Earth image disk. The cloud-classifications were used to segment images into $K = 3$ broad

layers, at low, intermediate and high altitude.⁶ Applying the methodology described in Section 4.4, pressure difference images for the three layers were derived from pressure images.

In Fig. 4, we present a sample of the motion fields estimated for two different layers together with the original observations. The motion fields estimated for the different layers are consistent with a visual inspection of the sequence. In particular, several motion differences between layers are very relevant. For instance, near the bottom left-hand corner of the images, the lower layer possesses a southward motion whereas the intermediate layer moves northward.

In addition, let us outline that the motion of some small cloud structures have been well characterized by this assimilation scheme. This is due to the fact that the shallow-water assumption is not assumed to be strictly respected everywhere. Thus, small-scale information from the image observations may locally generate a motion field which significantly departs from the shallow-water solution. This constitutes a benefit of our approach since our preliminary experiments in (Corpetti et al., 2008) have shown that the motions of small structures cannot be extracted from an assimilation scheme combining a primitive shallow-water model (17) and pressure image measurements. Of course, at this stage, the question of the validation is crucial and at the moment, no ground truth is available (especially concerning the small scales structures). One possible and interesting idea would be to compare our estimation with AMV products. However, this would be difficult as it would require the development of adapted procedures to deal with the resulting sparsity of AMV data when one focuses on a particular layer. The motion vectors from AMV are indeed assigned to a precise height and the way to compare this kind of data with our motion fields is not trivial. This issue will be the object of a future study. Nevertheless, we have compared our estimations with the velocity fields obtained by the layer-dedicated optic-flow method of Héas et al. (2007), whose results have been validated using several Meteosat data. The maps of divergence and vorticity corresponding to this ‘reference’ estimator are displayed in Fig. 5. It can be observed, from a visual comparison between the lower layer vorticity–divergence components estimated by (Héas et al., 2007) and by the proposed assimilation system that our estimations are in accordance with these motion fields. Indeed, time-persistent large structures of the flow are accurately estimated by both methods whereas noise and time-inconsistent structures have only been removed by the assimilation approach.

5.3. Discussion on experiments

5.3.1. Comments. This experimental part has been divided into two subsections using synthetic and real data. The two

⁶ The number of $K = 3$ layers is due to the nature of the input data—based on routines from Eumetsat—but any number of levels is supported by the approach.

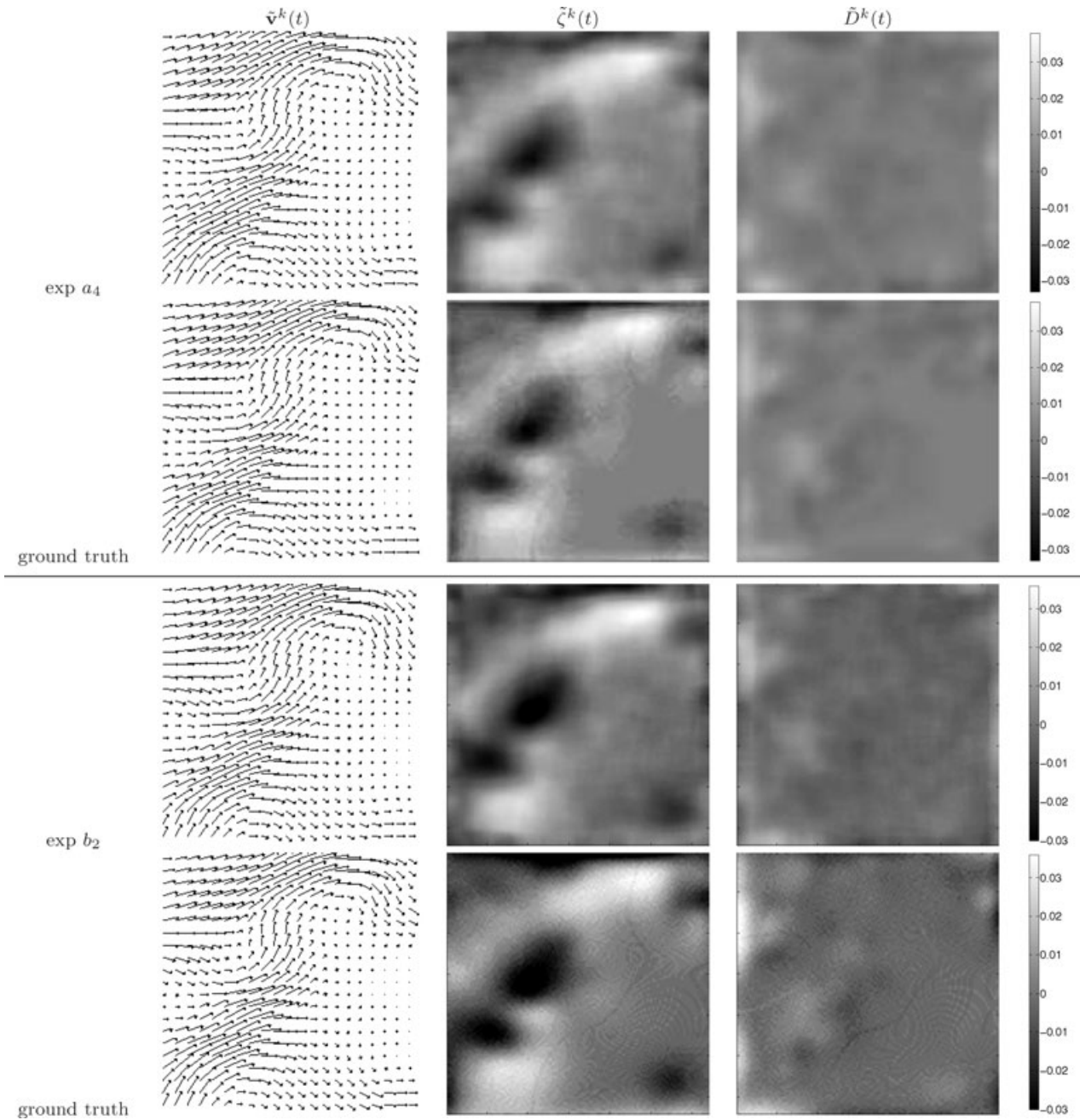


Fig 3. Variables after assimilation with images issued from ‘model-a’ (top) and ‘model-b’ (bottom). The corresponding experiments are a_4 (top) and b_2 (bottom). The first column corresponds to the wind $\tilde{\mathbf{v}}^k$, the second to the vorticity $\tilde{\zeta}^k$ and the third to the divergence $\tilde{D}^k$ fields at $t = 1h15$ min. For each experiment, the first line corresponds to the estimated variables whereas the second line corresponds to the ground truth.

experiments with synthetic data have illustrated the accuracy and the stability of our assimilation system in presence of noisy and sparse observations. The second set of experiments allowed us to assess the validity of the simplified divergence vorticity form of the shallow-water model considered in this study.

For the experiments on real situations, we have checked that our scheme supplied motion fields that are visually plausible and in accordance with a dedicated atmospheric layer motion estimator. It is obvious that the validation has to be more complete. A complete evaluation was beyond the scope of this study. In the near future, we plan to compare thoroughly our results

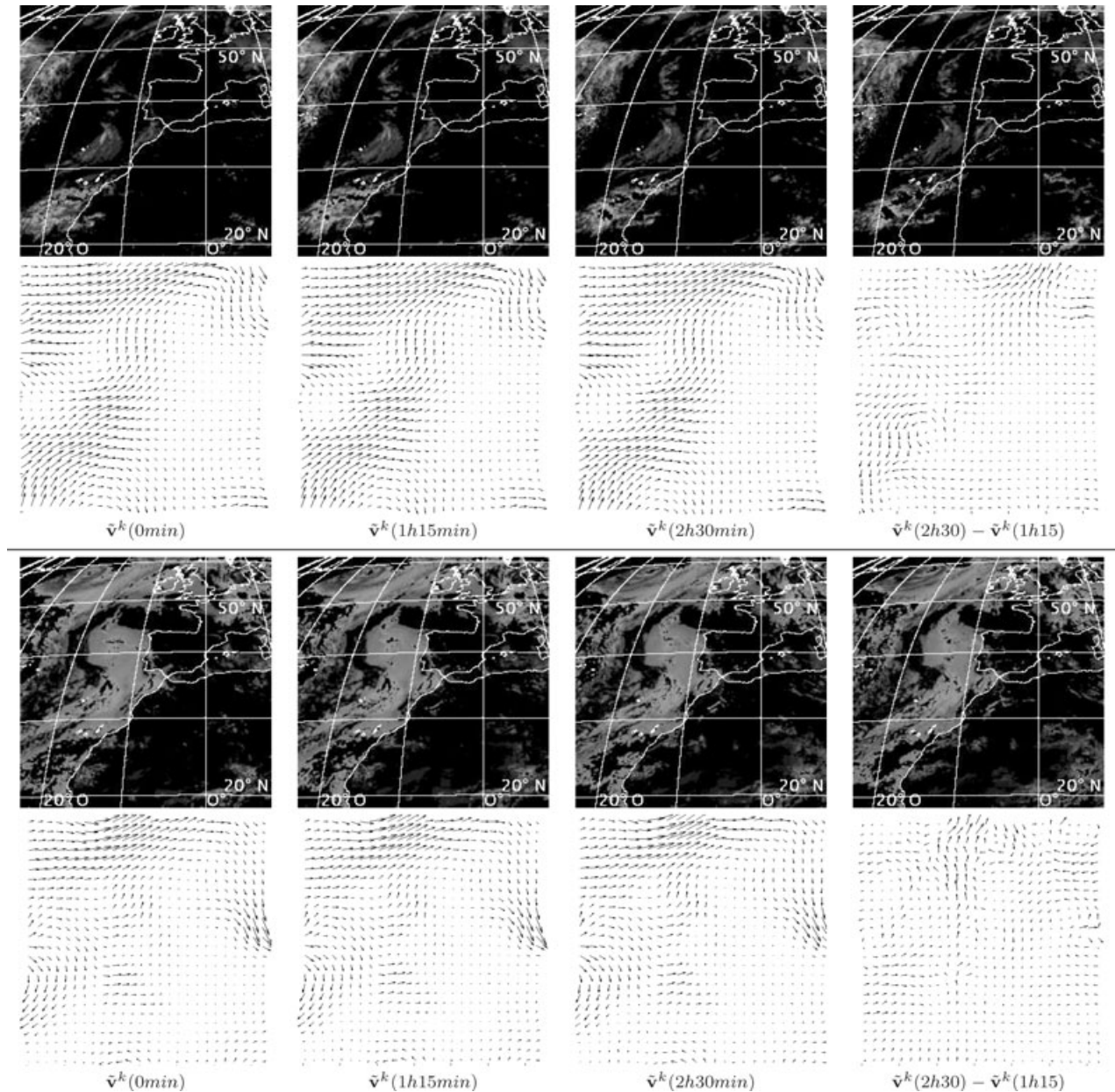


Fig 4. 'Horizontal wind fields estimated by assimilation with an imperfect modelling' for the intermediate layer (top part) and the lower layer (bottom part). For each layer, the first line represents *four* images of the sequences. The three first motion fields of the second line correspond to estimation at $t = 0$ min, $t = 1$ h 15 min and $t = 2$ h 30 min, respectively. The last motion field is the difference $\tilde{v}^k(2h30) - \tilde{v}^k(1h15)$ and illustrates the temporal changes.

with AMV data and in situ data provided by dedicated probes (radiosondes for instance) to obtain reference velocities at some points and to allow a precise statistical assessment in the vicinity of these points. Another key point of error analysis relies on the study of the adjoint variables λ defined in relation (22) when the process has converged. These components are indeed related to

the errors between the dynamic model and the observations and are likely to exhibit some interesting properties and information. This point will be also the object of future studies.

5.3.2. Computation time. The execution time for one forward-backward computation depends on (1) the size of images (2) the number of images and (3) the magnitude of the

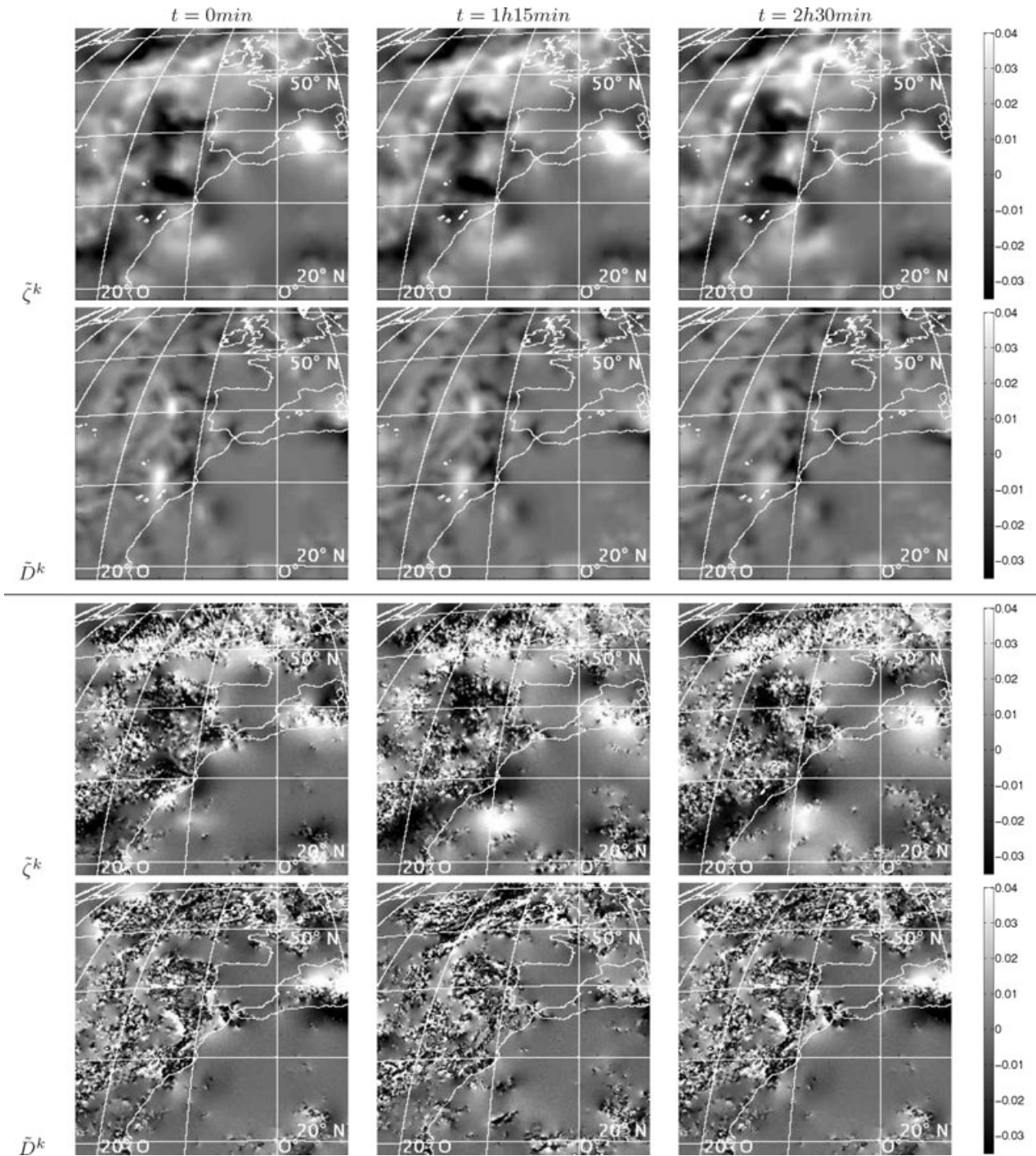


Fig 5. ‘Comparison of results from assimilation (top) and optical-flow (bottom)’ for the lower layer at $t = 0 \text{ min}$, $t = 1 \text{ h } 15 \text{ min}$ and $t = 2 \text{ h } 30 \text{ min}$, respectively. The first line represents the vorticity and the second line the divergence. One can conclude that both estimations are in accordance but the one from the assimilation has a better temporal consistency.

vorticity and divergence components; as mentioned in Section 4.3, an explicit Runge-Kutta scheme has indeed been used for the temporal (forward and backward) integration (Kurganov & Tadmor, 2000). To ensure stability of this scheme, the time-step

discretization is inversely proportional to the magnitude of the velocity. The initialization plays a key role in the computational cost also since the iterative process converges faster if one correctly initializes the system state.

Table 3. Default parameters: values of the covariance parameters used in our applications

Variable	Value	Description
α_{obs}	2	Error cov. observation (eq. 43)
σ	5	Outlier parameter (eq. 43)
α_D	10^{-3}	Error cov. on initial D (eq. 45)
α_ζ	10^{-3}	Error cov. on initial ζ (eq. 45)
α_Q	10^{-3}	Error cov. on dynamics (Section 4.5.3)

Table 4. Scale analysis of the horizontal primitive hydrostatic eqs (A1)

	A	B	C	D	E	F	G
x-eq.	$\frac{du}{dt}$	$\frac{uv \tan \phi}{a}$	$\frac{uw}{a}$	$\frac{1}{\rho} \frac{\partial p}{\partial x}$	$2 \Omega v \sin \phi$	$2 \Omega w \cos \phi$	$\mathcal{F}_u$
y-eq.	$\frac{dv}{dt}$	$\frac{u^2 \tan \phi}{a}$	$\frac{vw}{a}$	$\frac{1}{\rho} \frac{\partial p}{\partial y}$	$2 \Omega u \sin \phi$	–	$\mathcal{F}_v$
scales	$\frac{U}{\Delta t}$	$\frac{U^2}{a}$	$\frac{UW}{a}$	$\frac{\delta p}{\rho L}$	$\Omega \sin \phi U$	$\Omega \cos \phi w$	$\frac{vU}{H^2}$
(ms ⁻²)	10^{-2}	10^{-4}	10^{-5}	10^{-2}	10^{-3}	10^{-4}	10^{-10}

As an example, in our application, the estimation of the complete system state takes approximately 1 min for the synthetic sequences (128×128 pixels) whereas the estimation on a real layer (1024×1024 pixels) takes around 30 min for a MATLAB code, on a PC 2.33 Hz with 3GB of RAM. An efficient ‘C’ implementation would obviously significantly reduce the computational time.

5.3.3. *Parameter values.* The numerical values of the different parameters are set according to the range of variation of the quantity of interest. The values used in all our applications are given in Table 3.

6. Conclusion

In this paper, we have proposed an assimilation approach enabling, for the first time, a dynamically consistent estimation of a sequence of dense and layered atmospheric wind fields from entire satellite image sequences.

The motion estimator is applied to sparse pressure difference images corresponding to a stack of layers in a stratified atmosphere. A method was proposed to obtain such images from top

of cloud pressure images and classification, which are routinely supplied by the EUMETSAT consortium.

The dynamic constraint of our system suggested relying on the shallow-water assumption. We have shown that a perfect shallow-water modelling approach is likely to be unstable for the Meteosat data considered. We have then derived a simplified dynamic modelling with uncertainty terms. This model corresponds to a simplification of the vorticity–divergence form of a shallow-water dynamics for which the momentum equations are independent of the layer thickness. The mass conservation law complements these dynamics and provides the basis of an original image-adapted observation operator.

An evaluation performed on synthetic image sequences and on real METEOSAT top of cloud pressure image sequences has shown that the technique proposed enables characterizing time-consistent wind fields at large and small scales.

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8. Appendix A: Scale analysis of filtered hydrostatic primitive equations

For mesoscale atmospheric motions at horizontal scale of 100 km, the following scale characteristics hold:

U	$\sim$	10 m s^{-1}	Horizontal velocity
w	$\sim$	1 m s^{-1}	Vertical velocity
L	$\sim$	10^5 m	Length
H	$\sim$	10^3 m	Depth
δp	$\sim$	10^3 Pa	Horizontal pressure variation
ρ	$\sim$	1 kg m^{-3}	Density
Δt	$\sim$	10^3 s	Temporal sampling period
ν	$\sim$	$10^{-5} \text{ m}^2 \text{ s}^{-1}$	Friction coefficient
a	$\sim$	10^6 m	Earth radius
$\Omega \sin \phi$	$\sim$	10^{-4} s^{-1}	Coriolis coefficient
$\frac{U}{2\Omega \sin \phi L}$	$\sim$	0.5	Rossby number

The horizontal primitive hydrostatic equations of relation (2) are

Table 5. ‘Scale analysis’ of the Saint Venant’s model (C1).

	A	B	C	D	E
x-eq.	$\frac{\partial \tilde{h}^k \tilde{u}^k}{\partial t}$	$\frac{\partial (\tilde{h}^k (\tilde{u}^k)^2)}{\partial x} + \frac{\partial (\tilde{h}^k \tilde{u}^k \tilde{v}^k)}{\partial y}$	$\frac{\partial (\tilde{h}^k)^2}{2\rho_0^k \partial x}$	$-f^\phi \tilde{h}^k v$	$v_T^k ((\tilde{h}^k u)_{xx} + (\tilde{h}^k u)_{yy})$
y-eq.	$\frac{\partial \tilde{h}^k \tilde{v}^k}{\partial t}$	$\frac{\partial (\tilde{h}^k \tilde{u}^k \tilde{v}^k)}{\partial x} + \frac{\partial (\tilde{h}^k (\tilde{v}^k)^2)}{\partial y}$	$\frac{\partial (\tilde{h}^k)^2}{2\rho_0^k \partial y}$	$f^\phi \tilde{h}^k u$	$v_T^k ((\tilde{h}^k v)_{xx} + (\tilde{h}^k v)_{yy})$
scales	$\mathbf{q}/\Delta t$	$\tilde{h}^k \mathbf{v}^2/\Delta_{xy}$	$(\tilde{h}^k)^2/\rho \Delta_{xy}$	$f^\phi \tilde{h}^k \mathbf{v}$	$v_T \tilde{h}^k \mathbf{v}/\Delta_{xy}^2$
(kg.s ⁻⁴)	10^2	10^1	10^3	10^2	10^2

$$\left\{ \begin{array}{l} \underbrace{\frac{du}{dt}}_A - \underbrace{\frac{uv \tan \phi}{a}}_B + \underbrace{\frac{uw}{a}}_C + \underbrace{\frac{p_x}{\rho}}_D = \underbrace{+2\Omega v \sin \phi - 2\Omega w \cos \phi}_F + \underbrace{\mathcal{F}_u}_G \\ \underbrace{\frac{dv}{dt}}_A + \underbrace{\frac{u^2 \tan \phi}{a}}_B + \underbrace{\frac{vw}{a}}_C + \underbrace{\frac{p_y}{\rho}}_D = \underbrace{-2\Omega u \sin \phi}_E + \underbrace{\mathcal{F}_v}_G \end{array} \right. \quad (\text{A1})$$

An analysis of these terms using the characteristic scales is given in Table 4. We conclude that terms B, C, F and G can be ignored in comparison to the other terms. Moreover it is well known that for high Reynolds numbers characteristic of atmospheric flows, due to the energy dissipation at unobservable scales, turbulent viscosity dissipation $\mathcal{T}$ cannot be ignored in Navier–Stokes numerical simulations (Smagorinsky, 1963; Sagaut, 1998; Meneveau and Katz, 2000). We thus obtain the filtered momentum equations written in a vectorial form:

$$\rho \frac{d\tilde{\mathbf{v}}}{dt} = -\nabla \tilde{p} - 2\Omega \sin \phi \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tilde{\mathbf{v}} + \mathcal{T}. \quad (\text{A2})$$

9. Appendix B: Calculation of layer average densities

The integration of the hydrostatic relation between altitudes z_0 and z (or pressure p_0 and p), with a constant lapse rate may be written as

$$\int_{p_0}^p \frac{dp'}{p'} = - \int_{z_0}^z \frac{g}{R(T_0 + \gamma z')} dz'. \quad (\text{B1})$$

This expression gives rise, after some calculation, to the expression of density as a function of pressure (Holton, 2004):

$$\rho(p) = \frac{p_0}{RT_0} \left(\frac{p}{p_0} \right)^{\frac{\gamma R}{g} + 1}. \quad (\text{B2})$$

Computing the vertical density average, the mean density related to the k^{th} layer reads

$$\begin{aligned} \rho_0^k &= \frac{1}{\bar{p}^{k+1} - \bar{p}^k} \int_{\bar{p}^k}^{\bar{p}^{k+1}} \rho(p) dp \\ &= \frac{p_0^2}{(\bar{p}^{k+1} - \bar{p}^k) \left(\frac{\gamma R}{g} + 2 \right) RT_0} \left[\left(\frac{p}{p_0} \right)^{\frac{\gamma R}{g} + 2} \right]_{\bar{p}^k}^{\bar{p}^{k+1}}. \end{aligned} \quad (\text{B3})$$

Note that a constant lapse rate γ , that is to say, a linear variation of temperature with altitude, is a rough approximation in the troposphere. However, as we are averaging the obtained density law vertically and horizontally over the whole layer spatial support, such an approximation should have a minor impact on the accuracy of the modelling.

10. Appendix C: Scale analysis of Saint Venant's model with pressure image observations

For mesoscale atmospheric motions observed through a METEOSAT images sequence, which has been low-pass filtered to

obtain a characteristic horizontal scale of 100 km, the following characteristic scales hold (with $h^k = p^k - p^{k+1}$)

$\mathbf{v}$	$\sim 10 \text{ m s}^{-1}$	Horizontal velocity
h^k	$\sim 10^4 \text{ Pa}$	Pressure differences
ρ	$\sim 1 \text{ kg m}^{-3}$	Density
Δ_t	$\sim 10^3 \text{ s}$	Temporal sampling period
Δ_{xy}	$\sim 10^5 \text{ m}$	Spatial scale
f^ϕ	$\sim 10^{-3} \text{ s}^{-1}$	Coriolis coefficient
ν_T	$\sim 10^4 \text{ m}^2 \text{ s}^{-1}$	Turbulent diffusion (cf. 5)

The system to study reads:

$$\begin{aligned} \underbrace{\frac{\partial(\tilde{\mathbf{q}}^k)}{\partial t}}_A + \underbrace{\text{div} \left(\frac{1}{\tilde{h}^k} \tilde{\mathbf{q}}^k \otimes \tilde{\mathbf{q}}^k \right)}_B + \underbrace{\frac{1}{2\rho_0^k} \nabla_{xy} (\tilde{h}^k)^2}_C + \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} f^\phi \tilde{\mathbf{q}}^k}_D \\ = \underbrace{\nu_T^k \Delta(\tilde{\mathbf{q}}^k)}_E, \end{aligned} \quad (\text{C1})$$

where $(\nu_T = (C\delta_x)^2 \sqrt{2(\tilde{u}_x^2 + \tilde{v}_y^2 + (\tilde{u}_y + \tilde{v}_x)^2)})$ (cf. relation 5). Noting that the Pascal unit is $\text{hPa} = \text{kg s}^{-2} \text{m}^{-1}$ and recalling that $\mathbf{q} = h^k \mathbf{v}$, a scale analysis of this system is given in Table 5.

This corresponds to the order of magnitudes of the quantities of interest under the shallow-water approximation. From Table 5, we observe that term C of eq. (C1) is slightly higher than all the others. In practice, as soon as the shallow-water assumption breaks, the pressure difference of a layer $\tilde{h}^k$ grows, and we observed that the term $\nabla_{xy} (\tilde{h}^k)^2$ can reach the magnitude of 10^7 kg s^{-4} . The sensitivity of this term w.r.t. the assumption generates strong numerical instabilities. This primitive relation is, therefore, not adapted to the assimilation of sparse images of pressure differences when one wants to deal with corrupted input data.

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