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Trapped waves on the mid-latitude β-plane

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ABSTRACT

A new type of approximate solutions of the Linearized Shallow Water Equations (LSWE) on the mid-latitude β-plane, zonally propagating trapped waves with Airy-like latitude-dependent amplitude, is constructed in this work, for sufficiently small radius of deformation. In contrast to harmonic Poincare and Rossby waves, these newly found trapped waves vanish fast in the positive half-axis, and their zonal phase speed is larger than that of the corresponding harmonic waves for sufficiently large meridional domains. Our analysis implies that due to the smaller radius of deformation in the ocean compared with that in the atmosphere, the trapped waves are relevant to observations in the ocean whereas harmonic waves typify atmospheric observations. The increase in the zonal phase speed of trapped Rossby waves compared with that of harmonic ones is consistent with recent observations that showed that Sea Surface Height features propagated westwards faster than the phase speed of harmonic Rossby waves.

1. Introduction

The Linearized Shallow Water Equations (LSWE) on the β-plane describe the dynamics of fluid on a tangent plane to a sphere, when the non-linear terms in the shallow water equations on the sphere are neglected and the Coriolis parameter is linearized about some mean latitude. The resulting linear differential equations (see eq. 1) have variable coefficients due to the latitudinal variation of Coriolis frequency. In the prevailing theory, additional assumptions regarding the magnitude of the different terms in the LSWE (e.g. quasi-geostrophy, near non-divergence) are imposed, and these assumptions lead to Planetary (Rossby) and Inertia-Gravity (Poincare) harmonic waves (see e.g. Gill, 1982).

Exact and approximate analytical solutions of the LSWE that describe non-harmonic waves, have been previously shown to exist on a sphere (Longuet-Higgins, 1965) and on the equatorial β-plane (Matsuno, 1966). The latter are exact solutions in which the eigenfunctions are expressed in terms of Hermite Polynomials and the dispersion relations have simple formulae. However, no such non-harmonic wave solutions are known to exist on the unbounded mid-latitudes β-plane.

The present study addresses the LSWE on the β-plane in mid-latitudes and shows that the meridional velocity component, described exactly by the Parabolic cylinder functions, is accurately approximated by sinusoidal waves for some values of the model parameters and by Airy functions for other values of these parameter. The Airy function approximation to the original Parabolic cylinder function yields a simple analysis of trapped wave solutions and provides explicit expressions for their dispersion relation, which cannot be derived when the solutions are expressed by the Parabolic cylinder functions.

The approximation by sinusoidal waves corresponds to the classical harmonic Poincare and Rossby waves whereas the approximation by Airy function corresponds to a new class of approximate wave solutions of LSWE on β-plane that are the trapped analogues of the harmonic Poincare and Rossby waves. The analogy between these two trapped and harmonic waves is determined by the similarity of their corresponding dispersion relations. However, in contrast to the classical harmonic waves, the trapped waves vanish within several radii of deformation north of the latitude where they attain their maximal value. The existence of these trapped waves on the mid-latitude β-plane can be anticipated both from the properties of the spheroidal solutions of the LSWE on the sphere (see Longuet-Higgins, 1965) and from the numerical solutions of the LSWE in a channel on the mid-latitude β-plane (Paldor et al., 2007). However, neither analytical expressions for the eigenfunctions of the trapped waves nor their dispersion relations were calculated in these earlier studies.

In contrast, the current work develops analytical expressions for the eigenfunctions of the trapped waves, as well as for their dispersion relations and investigates, in detail, the conditions

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for their existence. The conclusion of this investigation is that for the range of LSWE parameters that typify the ocean, the trapped waves are the relevant solutions whereas the classical harmonic waves are relevant to parameter values that typify the atmosphere. The analytical approximate results are verified by exact numerical solutions of the eigenvalue problem.

Recent observations reported by Chelton and Schlax (1996) and Osychny and Cornillon (2004) show that the phase speed of long Rossby waves as inferred from the westward propagation of Sea Surface Height (SSH) features is higher than that of harmonic Rossby waves and that the difference between the phase speeds increases with latitude. The increased phase speed of trapped Rossby waves relative to that of harmonic waves is consistent with these observations. Other theories were developed to explain the reported gaps between observations and the harmonic wave theory. However, in contrast to the approach of the present study, these theories do not modify the harmonic wave theory itself and invoke, instead, additional physical processes such as meridionally varying zonal currents (Killworth et al., 1997), a combination of a free Rossby wave with a forced one (White, 1977) or a coupling between waves and wind stress curl (White et al., 1997).

2. The eigenvalue problem for zonally propagating waves

The LSWE on the β-plane with the (x, y) Cartesian coordinates pointing to the east and north, respectively, are

$$\frac{\partial u}{\partial t} - f(y)v = -g'\frac{\partial \eta}{\partial x},$$
$$\frac{\partial v}{\partial t} + f(y)u = -g'\frac{\partial \eta}{\partial y},$$
$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{1}$$

where u and v are the velocity components in the eastwards and northwards directions, respectively, η is the deviation of fluid height from its mean value H, g' is the reduced gravity (or the gravitational acceleration in barotropic cases), R is Earth's radius and

$$f(y) = f_0 + \beta y = 2\Omega\left(\sin\varphi_0 + \frac{\cos\varphi_0}{R}y\right) \tag{2}$$

is the Coriolis parameter, expanded linearly about mean latitude φ_0.

In the following, we restrict our attention to the mid-latitude β-plane, where $f(y)$ satisfies $|f(y) - f_0| < |f_0|$, hence according to (2) in the Northern Hemisphere,

$$\cot\varphi_0 \frac{|y|}{R} < 1. \tag{3}$$

To reduce the number of free parameters in the problem, we scale time on f_0^{-1}, length on $R_D = \frac{\sqrt{g'H}}{f_0}$ (the radius of deformation), height (i.e. η) on H and velocity on $\sqrt{g'H}$. Using these scales in system (1) one arrives at the non-dimensional form of the LSWE on the β-plane:

$$\frac{\partial u}{\partial t} - (1 + by)v = -\frac{\partial \eta}{\partial x},$$
$$\frac{\partial v}{\partial t} + (1 + by)u = -\frac{\partial \eta}{\partial y},$$
$$\frac{\partial \eta}{\partial t} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{4}$$

where all variables are non-dimensional and where

$$b = \frac{\beta R_D}{f_0} = \cot\varphi_0 \frac{R_D}{R} = \frac{\cos\varphi_0}{\sin^2\varphi_0}\frac{\sqrt{gH}}{2\Omega R}. \tag{5}$$

Since the dimensional and non-dimensional y variables are related to one another via R_D, the non-dimensional counterpart of (3) (when eq. 5 is employed) is

$$|\, y \,|\; b = \cot\varphi_0 \frac{R_D}{R}\; |\, y \,| < 1. \tag{6}$$

At $\varphi_0 = \pi/4$, in the ocean ($R_D \sim 30$ km) $b \simeq 0.005$ whereas in the atmosphere ($R_D \sim 250$–300 km) $b \leq 0.05$.

We now let the (u, v) velocity components and η in the LSWE vary with x and t as a zonally travelling wave with y-dependent amplitude, that is, $(u, v, \eta) = (U(y), V(y), \eta(y))e^{i(kx-\omega t)}$ where k and ω are the wavenumber and frequency of the propagating wave, respectively. Substituting this form of solution in system (4) and eliminating $U(y)$ and $\eta(y)$ yields the second-order differential equation for $V(y)$ (see Longuet-Higgins, 1965; LeBlond and Mysak, 1978 for a dimensional analogue of this equation and Paldor et al., 2007 where a similar non-dimensional equation is derived with different scaling):

$$\frac{d^2V}{dy^2} + [E - (1 + by)^2]V = 0, \tag{7}$$

where

$$E = \omega^2 - k^2 - \frac{bk}{\omega}. \tag{8}$$

(Note that in both Longuet-Higgins, 1965; LeBlond and Mysak, 1978, the analysis on the β-plane is facilitated by letting the coefficient of the V-term in eq. 7 to be constant, i.e. $b = 0$). Equation (7) is a Schrödinger equation with potential $(1 + by)^2$ and eigenvalue (or energy level) E. Each eigenvalue, E, defines, according to eq. (8), three waves distinguished by the values of ω—the two large roots (one positive and one negative) correspond to the high-frequency Poincare waves and the small negative root corresponds to the low-frequency Rossby wave. Though the differential eq. (7) has known solutions for any value of E (the Parabolic cylinder functions), the unique determination of the eigensolution follows from the imposition of boundary conditions $V(y) = 0$ at two y-points, say y_0 and y_1 (where $y_1 > y_0$), both of which have to satisfy the $b|y| < 1$ constraint. It should be noted that for $V(y)$ that decays sufficiently fast at large positive y-values (e.g. exponential, see below), the exact value of y_1 does not affect the solution. The same is not true for large and negative y_0 since, when the potential $(1 + by)^2$ is small, the solution $V(y)$ is oscillatory for positive E.

3. Airy-function approximation

Equation (7) can be simplified by applying the condition, $by < 1$, to eliminate the quadratic term of the potential, b^2y^2, which yields the equation

$$\frac{\mathrm{d}^2V}{\mathrm{d}y^2} + [E - (1 + 2by)]V = 0. \tag{9}$$

(Note that neglecting the $b^2\ y^2$ term in the transformation of eqs. 7–9 is consistent with neglecting of the $O(y^2)$ terms in the derivation of eq. 1 from the original LSWE on the sphere.) Substituting

$$z(y) = -(2b)^{-2/3}[E - (1 + 2by)] \tag{10}$$

for y in eq. (9) yields Airy equation (see Abramowitz and Stegun, 1972, eq. 10.4.1):

$$\frac{\mathrm{d}^2V}{\mathrm{d}z^2} - zV = 0. \tag{11}$$

This equation has a solution $Ai(z)$ that oscillates in the interval $(-\infty, 0)$ and decays to zero faster than exponential in the $(0, +\infty)$ interval (see Fig. 1). Being second order, eq. (10) has an additional solution $Bi(z)$ that increases faster than exponential in the $(0, +\infty)$ interval (not shown in Fig. 1) so that the general solution is a linear combination of these two solutions. However, in trapped waves, the contribution of $Bi(z)$ in the linear combination is zero (which is commensurate with the northern boundary being located at sufficiently large positive argument, see below).

The absolute maximal value of $Ai(z)$ occurs at $z < 0$, close to the first negative zero. Denote the zeros of $Ai(z)$ by Z_n, arranged in decreasing order and let $\zeta_n = -Z_n$, so that $\zeta_n > 0$ (since Z_n are

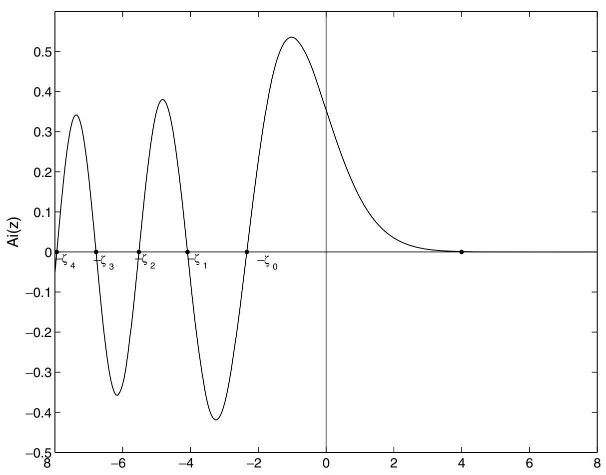

Fig 1. The Airy function Ai(z) oscillates in the negative half-axis, where all its zeros are located, and decays faster than exponential in the positive half-axis.

all negative, see Fig. 1). Evaluating the right-hand side (RHS) of eq. (10) at the southern boundary $y = y_0$ (recall, $V(y_0) = 0$) and letting $z(y) = -\zeta_n$ on its left-hand side (LHS) (so that $E_n = E$), we obtain:

$$E_n = 1 + (2b)^{2/3}\zeta_n + 2by_0. \tag{12}$$

Denote the function (10) for $E = E_n$ by $z_n(y)$, then $V_n(y) = Ai(z_n(y))$ is a solution of the differential eq. (9) that vanishes at $y = y_0$, since $Ai(z)$ vanishes at Z_n, and at $y = +\infty$ (since $Ai(z)$ vanishes at $+\infty$). The zeros of Airy functions are well documented in the literature (e.g. $\zeta_0 = 2.3381$, $\zeta_1 = 4.08795$, $\zeta_2 = 5.52050$, $\zeta_3 = 6.7867$, $\zeta_4 = 7.9441$); so, formula (12) provides the exact eigenvalues of eq. (9) for any given values of b and y_0.

The inequality (6) provides the range of y where solutions of eq. (9) are physically meaningful. Thus, although all functions $V_n(y)$ vanish at $+\infty$, only those that vanish inside the interval defined in eq. (6) with physically acceptable b-values, may be regarded as correct solutions of our problem. Until we establish the corresponding conditions for b, even the existence of such waves is not guaranteed.

To find this condition, we note that $Ai(z)$ becomes negligibly small as z becomes sufficiently large. We select $z = 4$ as such a 'sufficiently large' argument since at $z = 4$, $Ai(z)$ has dropped to 10^{-3} of its maximal value. Similarly, the solution $V_n(y)$ is negligibly small at the y-point that corresponds to $z = 4$ via eq. (10), when E is defined by eq. (12). Thus, denoting the value of y corresponding to $z = 4$ by Y_n when $E = E_n$ in eq. (10) (i.e. $z_n(Y_n) = 4$), we obtain

$$Y_n = y_0 + (2b)^{-1/3}(4 + \zeta_n). \tag{13}$$

To guarantee that $V_n(y)$ is physically meaningful i.e., that it vanishes inside the interval defined in eq. (6), we must ensure that inequality (6) is satisfied at $y = Y_n$. Substituting Y_n in inequality (6), we obtain that b and y_0 are related by

$$(2b)^{2/3}(4 + \zeta_n) + 2by_0 < 2, \tag{14}$$

which is guaranteed to be satisfied for sufficiently small $b = \cot\varphi_0 \frac{R_D}{R}$. In particular, for $\zeta_5 = 7.9$ and $y_0 = 0$ this condition is satisfied provided $b < 0.035$.

A comparison between the exact eigenvalues E_n for $n = 0, 1, 2$ calculated from the numerical solutions of eq. (7) in the interval $(y_0, y_1), y_0 = -10, y_1 = Y_n$ (see eq. 13), and their analytic approximations given by eq. (12) (that provides the eigenvalues of the approximate eq. 9) is shown in Fig. 2, which confirms the accuracy of formulae (12) whenever inequality (14) is satisfied. For values of b near 0.005 (typical for the ocean), the approximate solution nearly coincides with the exact one.

4. Trapped waves versus harmonic waves in a channel

Equation (8) shows that the value of E determines the dispersion relation $\omega(k)$. Therefore, to establish the accuracy of our

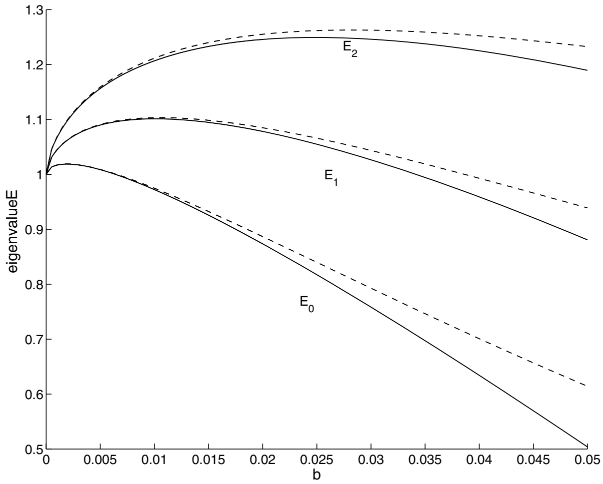

Fig 2. The variation with b of the first three eigenvalues, E_0, E_1 and E_2, of eq. (7) (solid line) and those of eq. (9) given explicitly by eq. (12) (dashed line), for $y_0 = -10$ and y_1 given by eq. (13) for the corresponding n. The value of $b = 0.005$ corresponds to the oceanic case where $R_D = 30$ km while the value of $b = 0.05$ corresponds to the atmospheric case where $R_D = 300$ km. The graphs clearly verify that eq. (12) provides an accurate approximation to the eigenvalues of eq. (7), especially in the oceanographic range of R_D (and b) values.

approach, we have to compare the eigenvalues of the original problem, eq. (7), with those of the Airy-type approximation (given by eq. 12) and of the classical harmonic wave approximation (obtained by setting $b = 0$). This comparison requires that the physical setup, including f_0, β and boundary conditions, be identical in all three problems. To complete the setup of the boundary value problem on the β-plane, we apply the conventional assumption that $v(y)$ vanishes at the boundaries of a zonal channel $-\delta < y < +\delta$. Setting $\delta = -y_0$ ensures that the solution $V_n(y)$ of eq. (9) vanishes at $-\delta$ and if (14) is satisfied, then it vanishes at $+\delta$ also. Substituting the expression (12) for $E_n(k)$ into eq. (8) one obtains

$$E_n = \omega^2 - \left(k^2 + \frac{bk}{\omega}\right) = 1 + (2b)^{2/3}\zeta_n - 2b\delta. \tag{15}$$

In the high-frequency limit, $\omega \gg 1$, we neglect the $\frac{bk}{\omega}$ term and obtain

$$\omega^2(k, n) = k^2 + E_n = 1 + k^2 + (2b)^{2/3}\zeta_n - 2b\delta, \tag{16}$$

which is the dispersion relations for trapped Poincare waves. In the low-frequency limit, $\omega \ll 1$, we neglect the ω^2 term in eq. (15) and obtain

$$\omega(k, n) = \frac{-bk}{k^2 + E_n} = \frac{-bk}{1 + k^2 + (2b)^{2/3}\zeta_n - 2b\delta}, \tag{17}$$

which is the dispersion relations for trapped Rossby waves.

We will consider also the classical harmonic solution obtained by setting $b = 0$ in eq. (9):

$$\frac{\mathrm{d}^2 V}{\mathrm{d}y^2} + (E - 1)V = 0. \tag{18}$$

This constant coefficient equation is a limiting case of eq. (9) for $\delta \ll 1$, implying that $y \ll 1$ and hence $by \ll 1$, everywhere. Thus, instead of an Airy eigenfunction, $Ai(y)$, one obtains the harmonic eigenfunctions

$$V_n = \sin\left[\frac{\pi(n+1)}{2\delta}(y + \delta)\right], \tag{19}$$

with the corresponding eigenvalues

$$E_n = 1 + \left[\frac{\pi(n+1)}{2\delta}\right]^2. \tag{20}$$

An important question that arises from these calculations is: which of the two expression for the eigenvalues—(15) or (20)—is more accurate? The general answer to this question is provided by the inequality (14), which implies that the domain of the Airy-type solution (i.e. the distance between its southern and northern boundaries, y_0 and Y_n) is given by

$$Y_n - y_0 = (2b)^{-1/3)}(4 + \zeta_n).$$

This domain of the solution should be smaller than 2δ for the Airy-type solution to be valid so that for $n = 0$, the condition

$$(2b)^{-1/3}(4 + \zeta_0) < 2\delta \tag{21}$$

guarantees that a solution of eq. (7) in $-\delta < y < +\delta$ is an Airy-type function.

Figure 3 shows the dependence of the first eigenvalues $E_0(\delta)$ obtained by the exact numerical solutions of eq. (7) and by its approximations (9) and (18), subject to the same channel boundary conditions at $y = -\delta$ and $y = +\delta$. These eigenvalues were calculated for $k = 0$ and for $b = 0.005$ (Fig. 3a) and $b = 0.05$ (Fig. 3b). Figure 3a clearly illustrates that the range of δ is divided into two parts: for $\delta < 8$, the eigenvalue of the harmonic solution, eq. (20), provides an accurate estimate of the eigenvalue of (7) whereas for $\delta > 8$, the solution of (7) is a trapped wave, which is accurately approximated by an Airy-type function. The transition at $\delta \approx 8$ implies that the analytical estimate in eq. (21) is quite conservative since for $b = 0.005$ it predicts that Airy-type solutions prevail for $\delta > 15$ only. One can easily get $\delta > 8$ for $b = 0.005$ in eq. (21) by placing the vanishing positive z-point of the Airy function corresponding to $y = +\delta$ at $z = 1$, instead of $z = 4$ (i.e. moving the $z = 4$ point in Fig. 1 to $z = 1$).

The upper panel of Fig. 3 also shows that at the values of δ close to 40, the eigenvalue E of the trapped waves is about 30% lower than that of the harmonic waves. From the relationship between E and ω (eq. 8), one can immediately conclude that this decrease in E implies that the phase speed of low-frequency trapped Rossby waves is significantly

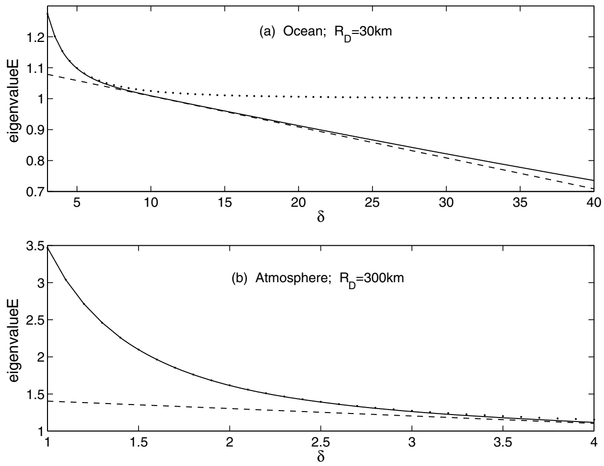

Fig 3. The dependence of the first eigenvalue E_0 on the non-dimensional half-width of the *y*-range, δ, (i.e. the dimensional meridional range divided by R_D) for, the exact solution (solid line), Airy-type approximation (dashed line) and classical harmonic approximation (dotted line). The eigenvalue of the Airy-type wave is given by eq. (12) whereas that of the harmonic wave by eq. (23). In the upper panel (ocean), $R_D = 30$ km whereas in the lower panel (atmosphere), $R_D = 300$ km, so that the dimensional meridional range corresponding to the largest δ-value in both panels, $R_D \times \delta$, is 1200 km. The upper panel shows that in the ocean at channel half-width exceeding 240 km, the Airy-type approximation is more accurate than the harmonic one whereas the lower panel shows that in the atmosphere, the classical approximation provides accurate results for half-width of up to 1200 km. At values of δ larger than those given in the two panels (40 in panel a and 4 in panel b) the meridional range is over 2400 km; so, the β-plane approximation ceases to be valid since Earth's sphericity has to be included.

larger than that of harmonic Rossby waves. It should be noted that Fig. 3a implies that in the ocean where $R_D =$ 30 km, the dimensional scale above which Airy-type solutions prevail is ($\delta \times R_D$=) 240 km only. Thus, Airy-type solutions are relevant to theories on the β-plane in the ocean where the variation of *f*(*y*) is not neglected.

In contrast, Fig. 3b shows that in the atmosphere, where R_D is about 300 km (and $b = 0.05$ at 45°), the harmonic wave theory provides an accurate approximation for the eigenvalue at $\delta \leq$ 4, which corresponds to dimensional domain of $300 \times 4 =$ 1200 km. Thus, the harmonic theory provides an accurate approximation for the eigenvalues on scales where the β-plane model applies, that is, scales smaller than Earth's radius but large enough for the variation of *f*(*y*) to be included. In this case, too, the range of δ where Airy-type solutions are accurate is $\delta > 4$ whereas eq. (8) yields $\delta > 7$ for $b = 0.05$. As in the case of the ocean, replacing $z = 4$ by $z = 1$ as the positive vanishing *z*-point of the Airy function yields $\delta > 4$ in eq. (8), in accordance with the result of Fig. 3b.

5. Discussion and summary

The boundary conditions required for closing the mathematical problem originate from the physical setting of the problem, which can be a wall (representing a coastline or a bottom mountain) or regularity/vanishing of a certain field (e.g. *V*(*y*)) at infinity. If no such boundary exists but *V*(*y*) happens to vanish at some point *y*, then this point can be considered as the required boundary even when the cause of the vanishing of *V*(*y*) there is unknown. Thus, trapped waves may exist even when no boundaries such as those assumed above exist. The same argument holds for harmonic waves in the limiting case when $b = 0$ (so, there is no explicit dependence of the equation on *y*).

In this section, we define identical boundary conditions for Airy-type and harmonic solutions that include only southern boundary (and not two boundaries as in the previous section) and compare the phase speeds of harmonic and trapped Rossby waves in this setting.

The oscillatory form of solutions to eqs. (9) and (18) assumes that these solutions vanish at some y_0, and so, we let the left-hand (southern) boundary conditions to be $V(y_0) = 0$. Since $v(t, x, y) = e^{i(kx-\omega t)}\, V(y)$, the vanishing of $V(y)$ at y_0 at a certain time implies the vanishing of $v(x, y, t)$ at y_0 at all times—that is, $v(x, y, t)$ is a standing wave. As for the right-hand (northern) boundary, we impose the weaker condition that the solution must be bounded at $+\infty$. The general solution of eq. (11) is a linear combination $Ai(y)$ and $Bi(y)$, where $Bi(y)$ is the Airy function of the second kind that grows faster than exponential when $y \to \infty$. Hence the eigenfunctions of the boundary value problem defined by eq. (9) and the above boundary conditions are the functions $V_n(y)$ of the previous sections and the corresponding eigenvalues are given by eq. (12).

In the classical theory, the solution of eq. (18) bounded at infinity is

$$V(y) = \sin l(y - y_0), \tag{22}$$

where $l^2 = E - 1$, and therefore,

$$E(l) = 1 + l^2. \tag{23}$$

Note that unlike trapped waves, the eigenvalue of this waves is continuous and not discrete. The dispersion relation for harmonic Poincare waves that correspond to the dispersion relation of trapped Poincare waves in eq. (16) is obtained by setting $b =$ 0 and $E = E(l)$ in (8):

$$\omega^2(k, l) = k^2 + E(l) = 1 + k^2 + l^2. \tag{24}$$

The harmonic counterpart of the dispersion relation for trapped Rossby waves (eq. 17) is obtained by setting $\omega^2 = 0$ and $E =$ $E(l)$ in (8), which yields

$$\omega(k, l) = \frac{-bk}{k^2 + E(l)} = \frac{-bk}{1 + k^2 + l^2}. \tag{25}$$

In both dispersion relations (16) and (17) for trapped Poincare and Rossby waves, respectively, the $(2b)^{2/3}\, \zeta_n + 2by_0$ term replaces the contribution of the meridional wavenumber l^2 in the corresponding harmonic dispersion relations whereas the '1' in the denominator on the RHS of eq. (25) plays the role of the radius of deformation contribution to the frequency $f_0/\sqrt{gH}$. Similarly, the '1' on the RHS of eq. (24) stands for f_0^2 in the corresponding dimensional expression.

In the previous section, the dependence of E_0 on the location of the boundary $y_0 = -\delta$ was investigated by assuming that φ_0 (and hence f_0) is fixed while $y_0(= -\delta)$ is varied. However, if we wish to compare the theoretical results with observations at different latitudes, we should let φ_0 (customarily identified with the point of observation) vary while keeping the latitude of the southern boundary (which is independent of the latitude of observation), φ_s, fixed. This determination of φ_0 from observations only (i.e. regarding it as a free parameter of the theory) is the practice used by observationalists, who applied the classical harmonic solutions to their data. The relationship between φ_s and φ_0 involves y_0 via

$$(\varphi_s - \varphi_0)R = y_0 R_D, \tag{26}$$

where both sides express the line distance between latitudes φ_s and φ_0 along a meridian. Recent observations conducted in open ocean (Chelton and Schlax, 1996; Fu, 2004; Osychny and Cornillon, 2004) show that the observed values of zonal phase speed of Rossby waves are larger than those predicted by the classical harmonic theory and that this difference increases with the latitude of observation. In terms of the waves frequency, the phase speed is given by $C = \omega/k$; so using eqs. (25) and (5), one gets the harmonic estimate for the phase speed in dimensional units:

$$C = \frac{-\beta R_D^2}{1 + k^2 R_D^2 + l^2 R_D^2}. \tag{27}$$

The corresponding dimensional expression for trapped waves is obtained from (17) using (26):

$$C = \frac{-\beta R_D^2}{1 + k^2 R_D^2 + (2b)^{2/3}\zeta_n - 2(\varphi_0 - \varphi_s)\cot\varphi_0}. \tag{28}$$

Comparing the dispersion relations for harmonic and trapped waves (eqs. 27 and 28), one can immediately notice that the trapped waves' phase speed differs from the harmonic phase speed even in the non-dispersive, long-wave limit and that it depends explicitly on the location of the southern boundary φ_s. In trapped waves, the simple contribution of the meridional wavenumber, $l^2 R_D^2$, is replaced by two terms, one of which includes the contribution of the radius of deformation (i.e. $(2b)^{2/3}$ ζ_n) and the other, negative, term proportional to $\varphi_0 - \varphi_s$. This second term decreases the value of the denominator, which increases the absolute value of the phase speed. This increase in the phase speed becomes significant for sufficiently large $\varphi_0 - \varphi_s$. We found no physical explanation to this simple mathematical result of the increase in the phase speed of trapped waves for large $\varphi_0 - \varphi_s$.

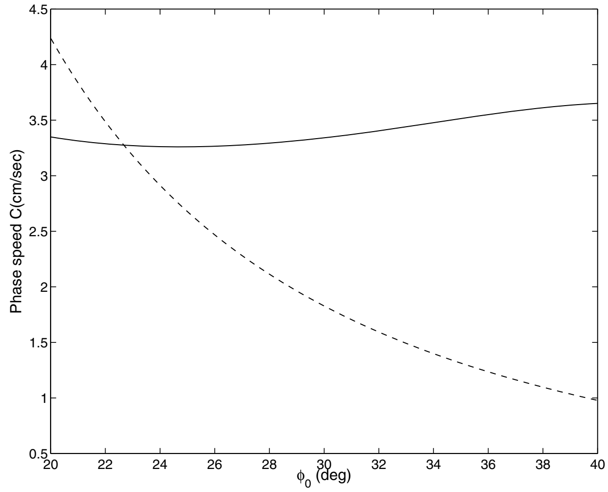

Fig 4. The dependence of the phase speed C on the central latitude φ_0 for the Airy-type approximation given by eq. (28) (solid line) and for the harmonic approximation given by eq. (27) (dashed line). The value of k in both expressions is 0 and in eq. (27), l is also 0. The southern boundary, φ_s, is located at 20°. The two graphs show that between φ_0 values of 23° and 40°, the trapped wave theory yields larger values of C than the harmonic wave theory and that the difference between the two increases with φ_0.

In the application of these dispersion relations to observations, eq. (27) is usually applied in its long-wave (non-dispersive) limit form $C \approx -\beta R_D^2$ (i.e. with $(k^2 + l^2) = 0$) in which case, one only needs to specify the central latitude, φ_0, which, for want of information, is set to the latitude where the observations are carried out. The corresponding long-wave limit form of the trapped waves' dispersion relation is obtained by setting $k^2 = 0$ and selecting $n = 0$ for the non-harmonic meridional mode. Therefore, the application of eq. (28) requires the specification of both φ_0 (as in harmonic waves) and φ_s (the latitude where V vanishes), even in the long-wave limit. Accordingly, long trapped waves are expected to become relevant to oceanic observations taken poleward of a zonal boundary (that provides the required value of φ_s) such as in the Mediterranean Sea off the North African coast or poleward of the South Australian coast. As in the case of harmonic waves, the value of φ_0 in eq. (28) is set to the latitude of observation. The solid curve in Fig. 4 displays the dependence of the phase speed of trapped waves (eq. 28 with $k^2 = 0$) on the latitude of observation (i.e. φ_0), for fixed $\varphi_s = 20°$. The dotted curve in this figure shows the dependence of the long-wave harmonic phase speed, $-\beta R_D^2$ (see eq. 27) on φ_0. It is clear from these two curves that trapped waves move westwards significantly faster than harmonic waves, sufficiently far from φ_s (i.e., at $\varphi_0 - \varphi_s \geq 3°$) while closer to φ_s, harmonic waves have somewhat larger phase speed. As explained above, the mathematical reason is quite obvious: at sufficiently small $\varphi_0 - \varphi_s$, the term $(2b)^{2/3}\zeta_n - 2(\varphi_s - \varphi_0)\cot\varphi_0 (= E_n - 1)$ in the denominator of eq. (28) becomes positive, so the phase speed becomes smaller

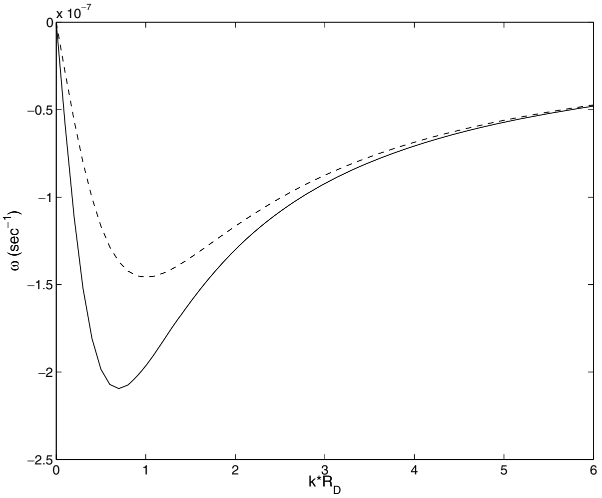

Fig 5. The dispersion relation, frequency as a function of kR_D, for the Airy-type Rossby wave solution (solid curve) and for the harmonic Rossby wave solution (dashed curve). The value of φ_0 is 50° whereas the value of φ_s in the trapped wave curve is 30°. The values of l in the harmonic wave curve and n in the trapped wave curve are both 0. In the long-wave limit, $kR_D \ll 1$, the phase speed of trapped waves is about twice that of harmonic waves whereas in the short-wave limit, $kR_D \gg 1$, the two phase speeds are identical.

than βR_D^2. The sharp variation of the phase speed of harmonic waves with the radius of deformation (determined by φ_0) is thus mitigated in trapped waves by the change in E_n with the increase in $\varphi_0 - \varphi_s$. The theoretical expectations regarding the increase in phase speed of trapped waves should be verified in future studies in which altimetry observations of SSH patterns are analysed on Hovmoller diagrams in locations such as those indicated above.

The difference between the dispersion relations of trapped waves of the present theory and harmonic waves of the classical theory is summarized in Fig. 5. The larger slope (by a factor of 2) of the frequency–wavenumber curve for trapped waves (solid curve) compared with that of harmonic waves (dotted curve) in the long-wave limit, $kR_D \ll 1$, is the main result of the present theory. The faster phase propagation of trapped waves in the new theory is accompanied by a shift in the latitude where the $V(y)$ attains its maximum value, towards the southern boundary whereas in the corresponding harmonic wave theory (for the first, $n = 0$, mode), this maximum is located half-way between the southern and northern boundaries.

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