

Accurate determination of a balanced axisymmetric vortex in a compressible atmosphere

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ABSTRACT

We give a simple method to calculate without approximation the balanced density field of an axisymmetric vortex in a compressible atmosphere in various coordinate systems given the tangential wind speed as a function of radius and height and the vertical density profile at large radius. The method is generally applicable, but the example considered is relevant to tropical cyclones. The exact solution is used to investigate the accuracy of making the anelastic approximation in a tropical cyclone, i.e. the neglect of the radial variation of density when calculating the gradient wind.

We show that the core of a baroclinic vortex with tangential wind speed decreasing with height is positively buoyant in terms of density differences compared at constant height, but at some levels may be interpreted as cold-cored or warm-cored depending on the surfaces along which the temperature deviation is measured. However, it is everywhere warm-cored if the potential temperature deviation is considered. In contrast, a barotropic vortex in a stably-stratified atmosphere is cold-cored at all levels when viewed in terms of the temperature deviation at constant height or constant σ , but warm-cored when viewed in terms of the potential temperature deviation along these surfaces. The calculations provide a possible explanation for the observed reduction in surface air temperature in the inner core of tropical cyclones.

1. Introduction

An important problem relating to the initialization of tropical cyclones in numerical models, either for prediction or research purposes, is to determine the balanced density field consistent with a prescribed vortical wind distribution and a prescribed vertical profile of density at some large radius from the vortex centre. The problem is complicated because many numerical models are formulated with σ as the vertical coordinate, requiring that the density field be specified on σ -surfaces. Here $\sigma = (p - p_t)/(p_s - p_t)$, where p_s is the surface pressure and p_t is the pressure at the top of the domain, normally assumed to be a constant. The problem has been examined by various authors over the years and a list of references for the earlier studies as well as a critique of these is given by Wang (1995). In the same paper, Wang proposed a general method for accomplishing the initialization.

Frequently, the initial vortex is taken to be axisymmetric with the tangential wind distribution specified as a function of radius and p or σ , but it could just as well be specified as a function of radius and height. For this case we outline a much easier and unapproximated method to obtain the balanced density field on p -surfaces and σ -surfaces as well as on surfaces of constant height. The method is based on the exact form of the thermal wind equation derived by Smith et al. (2005).

The reduced pressure in the centre of a balanced cyclonic vortex is associated with a reduced density. Thus, if the tangential wind speed decreases with height as in a tropical cyclone, the vortex has a warm core when viewed in pressure coordinates (i.e. the temperature increases radially inwards along the isobars, which in turn become lower). The questions then arise: does the temperature increase with decreasing radius at constant height or constant σ and, in particular, how large is the temperature gradient at the surface? The latter question is important because observations have shown a reduction in the sea–air temperature contrast in the inner-core region of a tropical cyclone on the order of 5°C (e.g. Korelov et al., 1990; Pudov, 1992). This difference has been attributed primarily to a reduction of the air temperature in the region of strong wind speeds caused by the evaporation of sea spray (see, for example, Fairhall et al., 1994, and references therein). While not discounting the importance of this mechanism, the results presented below offer an alternative and simpler explanation.

2. Unapproximated thermal wind equation

2.1. Height–radius coordinates (r, z)

The gradient wind equation and hydrostatic equation for an axisymmetric vortex may be written in vector form as

$$\left(\frac{\partial p}{\partial r}, \frac{\partial p}{\partial z} \right) = \rho(C, -g), \quad (1)$$

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where

$$C = \frac{v^2}{r} + fv \quad (2)$$

is the sum of the centrifugal and Coriolis forces per unit mass, respectively, v is the tangential wind speed, r is the radius, z is the height, ρ is the density, p is the pressure, f is the Coriolis parameter, and g is the acceleration due to gravity. Eliminating the pressure from the two components of eq. (1) by cross-differentiation gives the first-order partial differential equation

$$\frac{\partial}{\partial r} \ln \rho + \frac{C}{g} \frac{\partial}{\partial z} \ln \rho = -\frac{1}{g} \frac{\partial C}{\partial z}, \quad (3)$$

which is the unapproximated form of the thermal wind equation. The characteristics of this equation satisfy

$$\frac{dz}{dr} = \frac{C}{g} \quad (4)$$

and along these characteristics

$$\frac{d}{dr} \ln \rho = -\frac{1}{g} \frac{\partial C}{\partial z}. \quad (5)$$

According to eq. (4), displacements (dr, dz) along characteristics satisfy $(dr, dz)(C, -g) = 0$, i.e. they are at right angles to the pressure gradient (from eq. 1). Thus, the characteristics of eq. (3) are the isobaric surfaces. Equation (5) tells us how $\ln \rho$ varies along these characteristics and shows that the variation is proportional, *inter alia*, to $\partial v / \partial z$. If $\partial v / \partial z > 0$, then $\ln \rho$ (and therefore ρ) decrease with radius and temperature, T , increases with radius along isobar surfaces. In this case, a cyclonic vortex is cold-cored and an anticyclonic vortex is warm-cored. The reverse is true when $\partial v / \partial z < 0$.

We are now able to address the problem of initializing tropical cyclones in numerical models raised in Section 1. The problem is to determine the balanced density field consistent with a prescribed initial tangential wind distribution $v(r, z)$ and a prescribed ambient density profile $\rho_0(R, z)$ at some large radius R . Equations (4) and (5) provide an accurate way to accomplish this task. We refer the reader to Fig. 1a. Typically one has a set of grid points in the radial and vertical direction. Consider a grid point P with coordinates (r_p, z_p) . We can integrate eqs. (4) and (5) radially outwards to radius R to find the height of the isobaric surface, z_R , at this radius and the change in $\ln \rho$ between P and the point (R, z_R) . Because $p(R, z_R)$ and $\ln \rho$ are known [$p(R, z_R)$ is obtained by integrating the hydrostatic equation at radius R], we can immediately determine ρ and p at the point P , and then, of course, the temperature at this point using the ideal gas equation.

2.2. Pressure coordinates (r, p)

In pressure coordinates the analogous equation to eq. (1) is

$$\left(\frac{\partial \phi}{\partial r}, \frac{\partial \phi}{\partial p} \right) = (C, -\alpha), \quad (6)$$

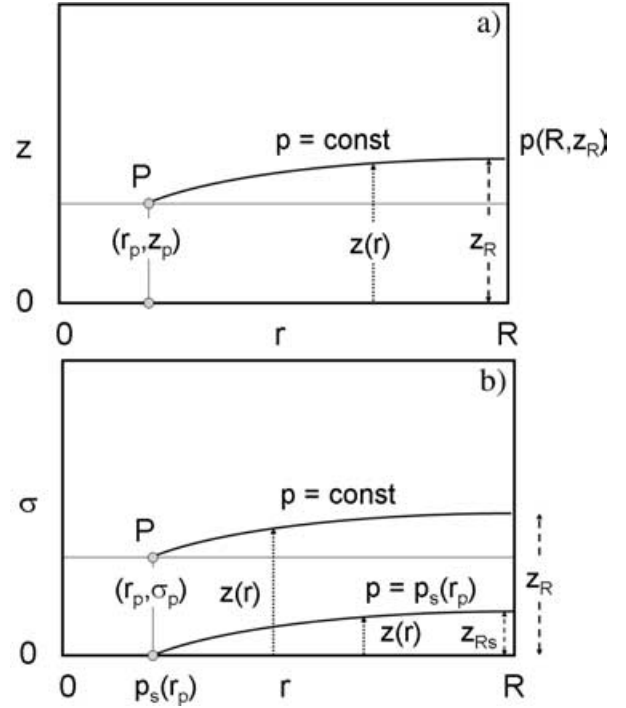


Fig. 1. Illustration relating to the calculation of the pressure and density at a grid point, P , in (a) height coordinates and (b) σ coordinates. The curved lines indicate isobaric surfaces through P and through a point at the surface below P . These surfaces, which satisfy eq. (4), are characteristics of eq. (3) and we can integrate eq. (4) to find the height of these surfaces at $r = R$, where the vertical pressure distribution is known.

where ϕ is the geopotential ($= gz$) and α is the specific volume ($= 1/\rho$). Cross-differentiation to eliminate ϕ then leads to

$$\frac{\partial \alpha}{\partial r} = -\frac{\partial C}{\partial p}. \quad (7)$$

As expected, this is a simpler equation to solve for the density (actually its inverse) than eq. (3), a reflection of the fact that, as shown above, pressure coordinates are the characteristic coordinates for this problem. To complete the specification of the vortex we may integrate the first component of eq. (6) radially to obtain the height of the isobaric surfaces in physical space. The procedure is similar to that of the previous section. Again we assume that $v(r, z)$ is prescribed for all r and z and that $\rho = \rho_0(R, z)$ at some large radius R . In principle, for each grid point P with coordinates (r, p) , we can integrate the first component of eq. (6) together with eq. (7) radially outwards to radius R to find the geopotential height difference and the difference in α between the point P and the point (r, p) . It should be remembered that in integrating the first of these equations, $v = v(r, \phi/g)$. Moreover, because we know v as a function of r and z rather than r and p , it is preferable to reformulate eq. (7) as

$$\frac{\partial}{\partial r} \ln \rho = -\frac{1}{g} \frac{\partial C}{\partial z}, \quad (8)$$

which, of course, is basically eq. (5). Because ρ and ϕ are known at (R, p) from an integration of the second component of eq. (6), we can infer their values at the point (r, p) from the differences in ϕ and $\ln \rho$ obtained by integrating the first component of eq. (6) and eq. (8) as described above. If $\phi(r, p) > 0$, the possible intersection of a given isobaric surface with the surface $z = 0$ can be determined by continuing the inward integration of the first component of eq. (6) until either $r = 0$ or $\phi(r, p) = 0$, the latter point corresponding with the surface. If it happens that for the chosen point (r, p) , $\phi(r, p) < 0$, the point (r, p) already lies below the surface. Clearly we need to monitor the sign of ϕ during the inward integration.

2.3. Sigma coordinates (r, σ)

Most tropical cyclone models are formulated in σ -coordinates, where $\sigma = (p - p_t)/(p_s - p_t)$, p_s is the surface pressure and p_t is the pressure at the top of the domain, normally assumed to be a constant. Now the equation analogous to eq. (1) is

$$\left(\frac{\partial \phi}{\partial r}, \frac{\partial \phi}{\partial \sigma} \right) = \left[C - \sigma \alpha \frac{\partial p_s}{\partial r}, -\alpha(p_s - p_t) \right] \quad (9)$$

and cross-differentiation leads to

$$\frac{\partial \alpha}{\partial r} - \sigma \frac{\partial}{\partial r} \ln(p_s - p_t) \frac{\partial \alpha}{\partial \sigma} = -\frac{1}{p_s - p_t} \frac{\partial C}{\partial \sigma}. \quad (10)$$

The characteristics of this equation satisfy

$$\frac{d\sigma}{dr} = -\sigma \frac{\partial}{\partial r} \ln(p_s - p_t), \quad (11)$$

the solution of which is

$$\sigma(p_s - p_t) = \text{constant}. \quad (12)$$

It follows from the definition of σ , not surprisingly, that these are again just the isobaric surfaces. Equation (9) tells us that along these characteristics

$$\frac{d\alpha}{dr} = -\frac{1}{(p_s - p_t)} \frac{\partial C}{\partial \sigma}. \quad (13)$$

To find the balanced density field of a vortex in this case, the first inclination is to proceed as before using eqs. (9)–(13). However, it turns out to be best to carry out the actual analysis in height coordinates using the methodology of Section 2.1. Now, we start from a grid point P with coordinates (r_p, σ_p) in σ -coordinates. We refer the reader to Fig. 1b. First we must find the surface pressure at this radius, $p_s(r_p)$. We do this by integrating eq. (5) radially outwards along the isobaric surface from $(r_p, \sigma = 1)$, i.e. $(r, 0)$ in z -coordinates, to find the height of this surface, z_{Rs} , at $r = R$. We can determine the pressure at this radius as before from an integration of the hydrostatic equation and this pressure is equal also to $p_s(r)$. Knowledge of $p_s(r)$ enables us to determine the pressure, p , at P from the knowledge of σ there and the definition of σ . The next step is to find the height, z_R , at which this isobaric surface intersects the boundary $r = R$ and the density of air at that point of intersection from our complete

knowledge of conditions at $r = R$. Then we integrate eqs. (4) and (5) radially inwards to radius r to find the height of the pressure surface at the point P as well as the density at that point. This procedure can be repeated for all grid points.

2.4. Height–radius coordinates, anelastic approximation

The ability to construct an exact solution to the axisymmetric problem enables us to investigate the accuracy of approximate solutions, which might be more amenable to generalization to the non-axisymmetric case. One such approximation is the anelastic approximation in which the density is assumed to be a function of height only when calculating the inertia of the air (Ogura and Phillips, 1962). Then, eq. (1) takes the form

$$\left(\frac{\partial p}{\partial r}, \frac{\partial p}{\partial z} \right) = (\rho_0 C, -\rho g), \quad (14)$$

and elimination of the pressure then gives

$$\frac{\partial \rho}{\partial r} = -\frac{1}{g} \frac{\partial}{\partial z} (\rho_0 C). \quad (15)$$

Given $v(r, z)$ and $\rho_0(z)$, the first component of eq. (14) can be integrated with respect to r at constant z to give the pressure $p(r, z)$ and eq. (14) can be integrated to give the density $\rho(r, z)$. The accuracy of this procedure is investigated below.

3. Discussion

Figure 2a shows an example of a calculation of the density field for a tangential wind field with scales appropriate to a moderately intense tropical cyclone, while Fig. 2b shows the height of the isobaric and σ surfaces in this case (here $p_t = 100$ mb). The radial integration of eqs. (4) and (5) is accomplished using a standard fourth-order Runge–Kutta procedure (see, for example, Press et al., 1992, Section 16.1). The vortex has a maximum tangential wind speed of 40 m s^{-1} at a radius of 40 km and the circulation decays with height,¹ i.e. $\partial v / \partial z \leq 0$. Both the density and pressure surfaces dip down near the vortex axis so that the inner core of the vortex is positively buoyant in the conventional sense, i.e. $-g[\rho(r, z) - \rho_0(z)]/\rho(r, z) > 0$ (see Smith et al., 2005).

It is well known that tropical cyclones are warm-cored vortices, but it is not obvious from a casual inspection of the pressure and density fields in Fig. 2 that a balanced vortex with the tangential wind speed distribution in Fig. 2a is warm-cored, or whether this vortex has a radial temperature gradient at the surface. Of course, the temperature structure is easy to calculate using the gas equation, $p = \rho RT$, where R is the specific gas constant for air. As shown in Fig. 3, the temperature deviation from that in the environment depends on the coordinate surfaces that are used to calculate it. It follows immediately from eq. (5) that if $\partial v / \partial z < 0$, ρ decreases with decreasing radius along surfaces

¹In reality, the maximum tangential wind speed moves radially outwards with height, but this detail is unimportant for the present illustrations.

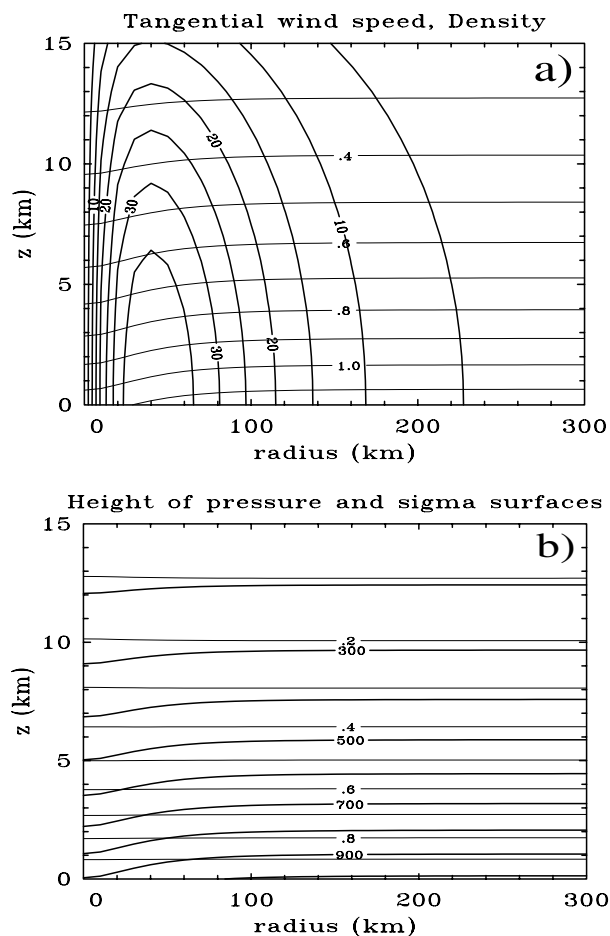


Fig. 2. (a) Isotachs of tangential wind speed v (thick solid lines, contour interval 5 m s^{-1}) and isopleths of constant density ρ (thin solid lines, contour interval 0.1 kg m^{-3}) in a vertical plane through the axis of a balanced, axisymmetric, tropical-cyclone-scale vortex. (b) Surfaces of constant pressure p (thick solid lines, contour interval 100 mb) and constant σ (thin solid lines, contour interval 0.1).

of constant pressure, whereupon the temperature and potential temperature must both increase inwards and the vortex must be warm-cored in this sense. For this reason, we chose not to plot this temperature deviation in pressure coordinates, but rather as a function of radius and height (Fig. 3a) to enable comparison with other cases. We see that the temperature perturbation is everywhere positive with a maximum on the axis at a height between 8 and 9 km. It is less obvious from the equations what the behaviour should be at the surface, but for the chosen sounding and tangential wind speed distribution, Fig. 3a shows that the radial gradient of temperature deviation is negative at the surface. In this case the magnitude of the surface temperature deviation at the centre is small (less than 1°C), but remember that the temperature deviation in this panel is the difference along the isobaric surface through the point in question. The situation is quite different if one compares temperature along surfaces of

constant z or constant σ as indicated in Figs. 3b and c, respectively. In these two cases, which are similar on account of the fact that the σ surfaces are close to horizontal (see Fig. 2b), the vortex would be judged as cold-cored in the lower troposphere and warm-cored above. In these two coordinate systems,² the radial temperature gradient at the surface is positive and now there is a reduction of surface air temperature at the vortex centre of the order of 5°C , which is comparable to the typical observed difference between the surface air temperature and the sea surface temperature in the inner core of tropical cyclones as noted above.

The situation is more orderly if we examine the potential temperature deviation. Because this quantity is positive when defined along isobaric surfaces, it must be even more positive when defined in z coordinates because the isobaric surfaces rise with increasing radius and the potential temperature at large radii typically increases with height. The same is true of the deviation defined along σ surfaces because these are very nearly horizontal and it follows that the vortex may be regarded consistently as warm-cored in all three coordinate systems. As an example, Fig. 4a shows the potential temperature deviation calculated at constant z . The maximum deviation lies on the vortex axis at a height between 9 and 10 km and there is a negative radial gradient at the surface.

While attention has been focused so far on the situation relevant to tropical cyclones, one obvious question at this point is: what is the situation for a barotropic vortex in a vertically stratified compressible atmosphere? In a barotropic vortex the tangential velocity is independent of height, whereupon the right-hand side of eq. (5) is zero. It follows that the isobaric surfaces are also surfaces of constant density, constant temperature and constant potential temperature. Such a vortex must be cold-cored at all levels when viewed in height coordinates or σ coordinates, because the isobaric surfaces dip down as the axis is approached. In contrast, it is warm-cored in terms of potential temperature if the atmosphere is stably stratified. These features help us to understand the low-level structures in the baroclinic case because the vertical gradient of tangential wind speed is small at these levels for the chosen wind distribution (indeed it is zero at the surface).

As discussed in Section 1, some observations have shown a reduction in the air temperature compared with that of the sea in the inner-core region of tropical cyclones of up to 5°C . The foregoing calculation shows that a temperature difference of this magnitude is to be expected simply as part of the balanced structure of a tropical-cyclone-like vortex (see, for example, Fig. 3b), a factor that does not appear to have been considered previously. Even though the gradient-wind balance assumption breaks down in the tropical-cyclone boundary layer, the overall constraint

²Some care should be exercised in comparing the details of the plots in height and σ coordinates as the vertical extent of the domain is a little different.

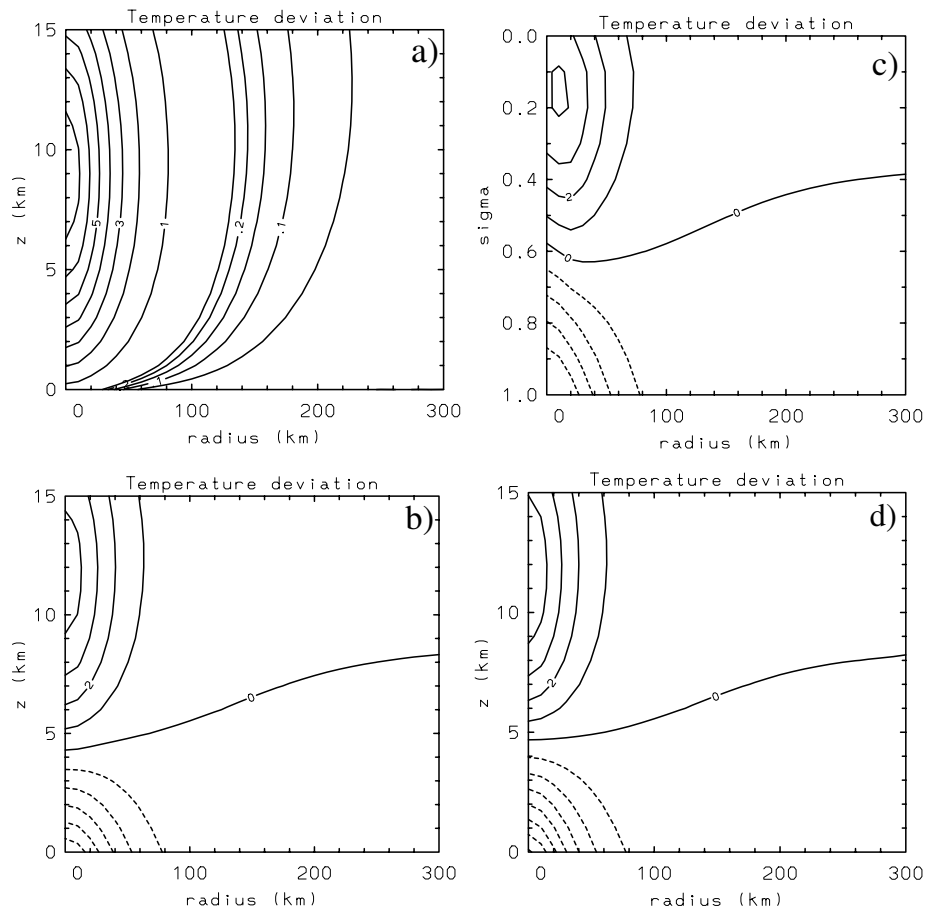


Fig. 3. Perturbation temperature along (a) the isobaric surfaces, (b) the level surfaces, (c) constant σ levels and (d) along level surfaces, but with the anelastic approximation. Contour interval is 1 K. Additional contours from 0.05 to 0.25 K with an interval of 0.05 K are included in (a) to highlight the radial temperature gradient at the surface.

above the boundary layer may remain important in determining the thermal structure of the core region.

A final question that we address is: how good is the anelastic approximation as a basis for computing the balanced pressure, density and temperature fields for a prescribed tangential wind distribution? Figure 3d shows the temperature deviation at constant height for an anelastic calculation following the method outlined in Section 2.4, which should be compared with Fig. 3b. These fields look very close and indicate that the method outlined is rather accurate. Specifically, the differences are largest at low levels near the axis, where the anelastic approximation is too cool by a maximum of 1.35 K at the surface. Above a height of about 6 km the approximation is very slightly too warm by a maximum of 0.35 K. Figure 4b shows the difference in the potential temperature deviation calculated along level surfaces with and without the anelastic approximation. For this field the maximum differences are similar in magnitude, the anelastic approximation being too cool near the axis below about 4 km, with a maximum discrepancy of 1.02 K at the surface, and too warm above, with a maximum discrepancy of 0.82 K at a height of

9–10 km. Less accurate results were obtained with the anelastic approximation by using a second-order finite difference form of the second component of eq. (14) to determine ρ once the pressure field has been determined from a radial integration of the first component.

The foregoing result that the anelastic approximation is accurate in the axisymmetric case suggests that it may be accurate quite generally, the assumption of which should simplify the calculation of the balanced density field of a prescribed asymmetric vortex.

4. Conclusions

We have presented a simple method to calculate without approximation the balanced density field of an axisymmetric vortex given the tangential wind speed as a function of height and the vertical density profile at large radius. Further, we have shown how to obtain a relatively accurate approximation to this solution by making the anelastic approximation.

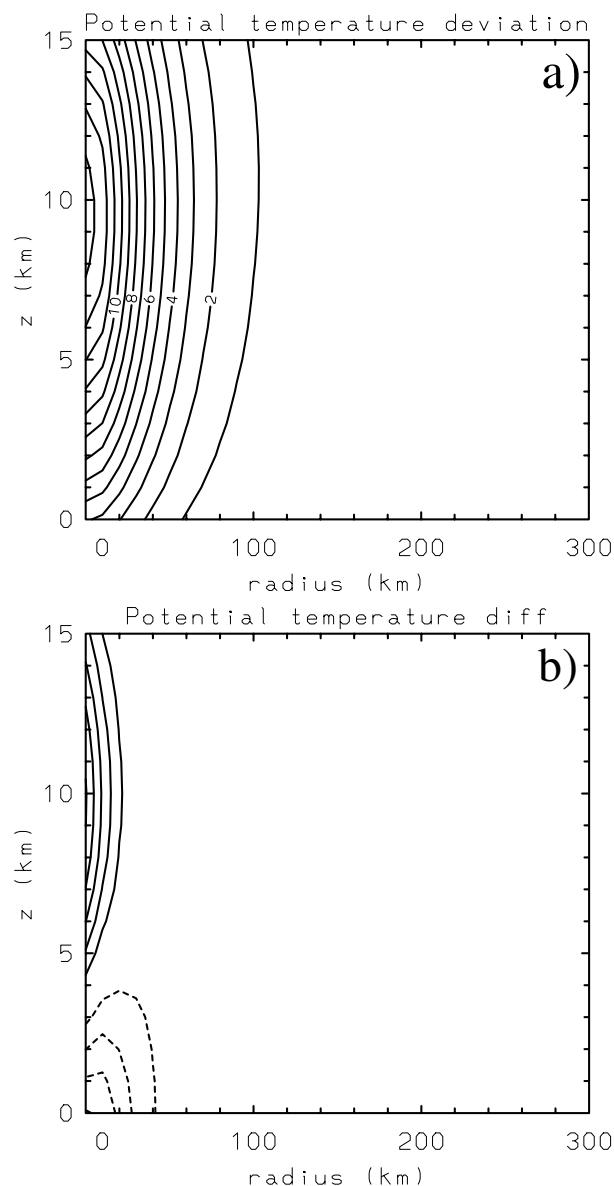


Fig. 4. Perturbation potential temperature calculated at constant z for the exact calculation in Section 2.1 (a). Contour interval is 1 K. (b) The differences that arise when the calculation is carried out using the anelastic approximation. Contour interval is 0.2 K.

We used the method to calculate the density, temperature and potential temperature fields for a prescribed tangential wind field with scales appropriate to a moderately intense tropical cyclone.

When viewed in height coordinates or σ coordinates, the vortex is cold-cored at low levels in the sense that the temperature deviation along these surfaces is negative near the vortex axis. However, it is warm-cored at upper levels. In contrast, the vortex is warm-cored at all levels when viewed in pressure coordinates and/or in terms of the potential temperature deviation. The balanced vortex has a positive radial gradient of temperature at the surface. A barotropic vortex in a stably-stratified atmosphere is cold-cored at all levels when viewed in terms of the temperature deviation at constant height or constant σ , but warm-cored when viewed in terms of the potential temperature deviation along these surfaces.

The low-level cold-core structure of a balanced, tropical-cyclone-like vortex may explain, at least in part, the increased sea–air temperature differences that are observed in a cyclone’s inner core.

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References

- Fairhall, C. W., Kepert, J. D. and Holland, G. J. 1994. The effect of sea spray on surface energy transports over the ocean. *Atmos. Ocean Syst.* **2**, 1213–142.
- Korelov, V. SW., Petrichenko, S. A. and Pudov, V. D. 1990. Heat and moisture exchange between the ocean and atmosphere in Tropical Storms Tess and Skip (English translation). *Sov. Meteorol. Hydrol.* **3**, 92–94.
- Ogura, A. and Phillips, N. A. 1962. Scale analysis of deep and shallow convection in the atmosphere. *J. Atmos. Sci.* **19**, 173–179.
- Press, W. H., Teukolsky, S. A., Vetterling, W. T. and Flannery, B. P. 1992. *Numerical Recipes in Fortran: The Art of Scientific Computing*. Cambridge University Press, Cambridge, 2nd edition.
- Pudov, V. D. 1992. The ocean response to the cyclone’s influence and its possible role in their track formation. *ICSU/WMO International Symposium on Tropical Cyclone Disasters*. WMO, 367–376.
- Smith, R. K., Montgomery, M. T. and Zhu, H. 2005. Buoyancy in tropical cyclones and other rapidly rotating vortices. *Dyn. Atmos. Oceans*, **40**, 189–208.
- Wang, Y. 1995. An inverse balance equation in sigma coordinates for model initialization. *Mon. Wea. Rev.* **123**, 482–488.