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Convergence Factors in an Integral Mean

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Abstract : The main theorem proved in this paper gives conditions necessary and sufficient in order that, for some real C_m $(m = 0, 1, \cdots r-1)$

$$I_r f(x)\, g(x) = \sum_{m=0}^{r-1} C_m x^{r-1-m} + o\,(x^{p+q+r}) \text{ whenever}$$

$$I_k f(x) = \sum_{m=0}^{k-1} c_m x^{k-1-m} + o\,(x^{p+k}),$$ where $I_k f(x)$ denotes the k-th iterated integral of $f(x)$ under suitable restrictions on the function f, g, and under the assumptions

$$r, k \in N_0 \quad r \geqslant k \geqslant 0,\ p, q \in R \text{ and } p+q \notin \{-1, -2, \cdots -r\}.$$

1. Introduction

If $f \in L_{loc}$, $k \in N$, let $I_k f(x) = f_k(x)$ denote th k-the iterated integral of $f(x)$. We assume in this paper that $r, k \in N_0$, $r \geqslant k$, $p, q \in R$ and that (a) if $r = k = 0$ then $f, g \in F$ (The space of all real functions defined on $[1, \infty)$).

(b) if $k = 0$, $r \geqslant 1$, then $f \in L_{loc}$, $g \in B_{loc}$ (Space of functions in F bounded and measurable locally),

(c) if $r \geqslant k \geqslant 1$, then $f \in L_{loc}$, $g^{k-1} \in AC_{loc}$ (Space of functions in F absolutely continuous locally),

g^m denoting the m - th derivative of g, g^0 denoting g.

We prove the following theorems; omitting the trivial case $r = k = 0$.

Theorem 1. Let $p + q > -1$.

(a) If $k = 0$, $r \geqslant 1$, a necessary and sufficient condition that $I_r f(x)\, g(x) = 0\,(x^{p+q+r})$ whenever $I_k f(x) = 0\,(x^{p+k})$ is that

$$\text{(i)}_0 \quad 0 \int_1^x t^{-q} |g(t)|\, dt = 0\,(x), \text{ as } x \to \infty.$$

(b) If $r \geqslant k \geqslant 1$, then necessary and sufficient conditions that $I_r f(x)\, g(x) = 0\,(x^{p+q+r})$ whenever $I_k f(x) = 0\,(x^{p+k})$ are:

$$\text{(i)} \quad g(x) = 0\,(x^q) \text{ as } x \to \infty,$$

$$\text{(ii)} \quad \int_1^x t^{k-q}\, |g^k(t)|\, dt = 0\,(x) \text{ as } x \to \infty.$$

We find that the conditions in Theorem 1 are necessary even in the case $p+q \leqslant -1$. We also find that if $p+q < -1$, $p+q \notin \{-1, -2, \cdots -r\}$ and the conditions of Theorem 1 hold, then there exist constants C_m, $m = 0, 1, \cdots r-1$ such that
$$I_r f(x) g(x) = \sum_{m=0}^{r-1} C_m x^{r-1-m} + 0(x^{r+q+r}).$$
We combine these results to give :

Theorem 2

(a) If $r \geqslant k \geqslant 0$, $r \geqslant 1$, $p+q \notin \{-1, -2, \cdots -r\}$, c_m $(m = 0, 1, \cdots k-1)$ are real, $I_k f(x) = \sum_{m=0}^{k-1} c_m x^{k-1-m} + 0(x^{p+k})$, and the conditions of Theorem 1 hold, then there exist constants C_m $(m = 0, 1, \cdots r-1)$ such that
$$I_r f(x) g(x) = \sum_{m}^{r-1} c_m x^{r-1-m} + 0(x^{p+q+r}).$$
(b) If $r \geqslant k \geqslant 0$, $r \geqslant 1$, and there exist constants C_m $(m = 0, 1, \cdots r-1)$ such that
$$I_r f(x) g(x) = \sum_{m=0}^{r-1} C_m x^{r-1-m} + 0(x^{p+q+r})$$
whenever $I_k f(x) = \sum_{m=0}^{k-1} c_m x^{k-1-m} + 0(x^{p+k})$, where c_m $(m = 0, 1, \cdots k-1)$ are real, then the conditions of Theorem 1 are necessary. An empty sum denotes zero.

Theorem 2 is the analogue for continuous variables of the sequence – sequence problem considered in [1].

2. Auxiliary Results

Lemma 1. Let $k \in N$, $q \in R$ and $g^{k-1} \in AC_{loc}$. Then, each of the following sets of conditions :

(a) $(i)_a$ $\int_1^x t^{-q} |g(t)| dt = 0(x)$ $(ii)_a$ $\int_1^x t^{k-q} |g^k(t)| dt = 0(x)$,

(b) $(i)_b$ $\int_1^x t^{-q} g(t) dt = 0(x)$ $(ii)_b$ $\int_1^x t^{k-q} g^k(t) dt = 0(x)$.

(c) $(i)_c$ $g(x) = 0(x^q)$ $(ii)_c$ $g^{k-1}(x) = 0(x^{q+1-k})$,

implies that $g^j(x) = 0(x^{q-j})$ for $j = 0, 1 \cdots k-1$.

Proof. Clearly, it is enough to prove that $(i)_b$ and $(ii)_b$ imply that
$$g^j(x) = 0(x^{q-j}).$$
Case 1. Let $q > k-1$.

Since $g^{k-1}(x) = g^{k-1}(1) + \int_1^x t^{q-k} [t^{k-q} g^k(t)] dt$, partial integration and $(ii)_b$ give $g^{k-1}(x) = 0(x^{q-k+1})$, and further integration gives the result.

Case 2. $q = k-1$. Integration $(ii)_b$ $n-1$ times where $1 \leqslant n \leqslant k$, we get
$$I_n [x g^k(x)] = 0(x)^n \tag{2.1}$$

Now, $(d/dx)^n [xg^{k-n}(x)] = xg^k(x) + ng^{k-1}(x)$ p. p., and integrating this n times we have

$$xg^{k-n}(x) - nI[g^{k-n}(x)] = 0(x^n), \tag{2·2}$$

by (2·1).

Now, $(i)_b$ and partial integration give $I[g(x)] = 0(x^k)$. (2·3)

Putting $n = k$ in (2·2) and using (2·3), we get

$$g(x) = 0(x^{k-1}). \tag{2·4}$$

Putting $n = k-1$ in (2·2) and using (2·4) we get $g^1(x) = 0(x^{k-2})$ and proceeding similarly we establish the result.

Case 3. $q < k-1$. Partial integration and $(ii)_b$ give, in this case,

$$g^{k-1}(x) = c + 0(x^{q-k+1}) = c + 0(1).$$

Hence $$\int_1^x t^{-q} g(t)\,dt = \frac{cx^{k-q}}{(k-q)(k-1)!} + 0(x^{k-q}),$$

and thus, $(i)_b$ can hold only if $c = 0$, and (2·5) gives $g^{k-1}(x) = 0(x^{q-k+1})$, which is the conclusion if $k = 1$.

If $k \geqslant 2$, since $\int_1^x t^{k-1-q} g^{k-1}(t)\,dt = 0(x)$, in the case $q \geq k-2$, we can use cases 1 and 2 with k replaced by $k-1$ to get

$g^j(x) = 0(x^{q-j})$ for $j = 0, 1, \cdots k-2$. If $q < k-2$, by repetition of the above argument, we get $g^{k-2}(x) = 0(x^{q-k+2})$, and thus

$\int_1^x t^{k-2-q} g^{k-2}(t)\,dt = 0(x)$. We now establish the result by an inductive argument.

Lemma 2. If $d_{ij} = (-1)^j \binom{k}{j} \dfrac{u^{r-1+i-j}}{(r-1+i-j)!}$, $i, j = 0, 1, \cdots k, r \in N$, and D_{ij} is the cofactor of d_{ij} in the determinant $D = \det(d_{ij})$ of order $(k+1) \times (k+1)$, then $D_{ij} = K_{ij}\, u^{k(r-1)+j-i}$, where K_{ij} is independent of

u, $D = Pu^{(k+1)(r-1)}$, $P = \displaystyle\prod_{m=0}^{k} \frac{(-1)^m \binom{R}{m} m!}{(r-1+m)!} \neq 0.$

This result is proved easily by induction with respect to k.

Lemma 3. If $p + k > -1$, $k^1 > k$ and $g^k(x) = 0(x^{p+k})$, then

$g_{k1}(x) = o(x^{p+k1})$. 'o' may be replaced by '0' here

This result is well known.

Lemma 4. If $p + k > -1$, $p + q > -1$ and $g_k(x) = o(x^{p+k})$, then $I_k x^q g(x) = o(x^{p+q+k})$. 'o' may be replaced by '0' here.

of [2] (Lemma 1) and [3] (Lemma 4).

Lemma 5. Let $n \in N$ and let S_0^n be the subset of F such that every $s \in S_0^n$ satisfies $s^n \in AC_{loc}$ and $s(t) = o(1)$ as $t \to \infty$. Suppose $a(x, t) = 0$ for $t > x$, $a(x, t) \in L[1, x]$ for every fixed $x \geq 1$, and $\int_1^T |a(x, t)| dt = o(1)$ as $x \to \infty$ for every $T > 1$. Then, in order that $v(x) = \int_1^x a(x, t) s(t) dt = o(1)$ as $x \to \infty$ whenever $s \in S_0^n$, $s(1) = s^1(1) = \cdots = s^n(1) = o$ it is necessary that

$$\int_1^x |a(x, t)| dt = O(1) \text{ as } x \to \infty.$$

See [4] (Lemma 8).

Lemma 6. Suppose $a(x, t)$ is bounded and measurable on $[1, x]$ for every $x > 1$ and $a(x, t) = 0$ for $t > x$. Then in order that $v(x) = \int_1^x a(x, t) s(t) dt = o(1)$ as $x \to \infty$ whenever $s \in L$ and $s(t) = o(1)$ as $t \to \infty$, it is necessary that $\int_1^x |a(x, t)| dt = O(1)$ as $x \to \infty$.

3. Proofs of Theorems

Proof of Theorem 1. Necessity.

By repeated partial integration, when $r \geq k \geq 1$, we have

$$I_r f(x) g(x) = \int_1^x \frac{(x-t)^{r-1}}{(r-1)!} f(t) g(t) dt$$

$$= (-1)^k \int_1^x f_k(t) (D_t)^k G_r(x, t) dt + \delta_{rk} f_k(x) g(x), \tag{3.1}$$

where $D_t = \frac{\partial}{\partial t}$, $G_r(x, t) = \frac{(x-t)^{r-1}}{(r-1)!} g(t)$, and δ_{rk} is the Kronecker delta.

When $k = 0$, $r \geq 1$, the formula still holds.

Define $a(x, t) = (-1)^k t^{p+k} x^{-p-q-r} (D_t)^k G_r(x, t)$, $s(t) = t^{-p-k} f_k(t)$

and $v(x) = \int_1^x a(x, t) s(t) dt$.

Then, (3.1) gives

$$x^{-p-q-r} I_r f(x) g(x) = v(x) + \delta_{rk} x^{-p-q-r} f_k(x) g(x). \tag{3.2}$$

Since conditions $(i)_0$, (i) and (ii) are independent of r, $r \geq \max(1, k)$, it is enough, by Lemma 3, to consider the case $r \geq k + 1$ here, and then (3.2) gives:

$v(x) = o(1)$ as $x \to \infty$ whenever $s \in S_0^{k-1}$ when $k \geq 1$, and

$v(x) = o(1)$ whenever $s \in L$ and $s(t) = o(1)$, when $k = 0$, where S_0^{k-} is defined in Lemma 5.

Clearly, $a(x, t)$ satisfies the conditions of Lemma 5 and 6, and hence we get the necessary condition $\int_1^x |a(x, t)| dt = O(1)$, which gives $\int_1^x t^{p+k} |(D_t)^k G_r(x, t)| dt = O(x^{p+q+r})$, $k \geq 0$, and by Lemma 3, r may be replaced by $r + i$, $i = 0, 1, \cdots$ in this equation.

Thus,

$$\int_1^x t^{p+k} |D_t)^k G_{r+i}(x, t)| dt = O(x^{p+q+r+i}). \tag{3.3}$$

Now ${}_t(D_t)^k G_{r+i}(x, t) = \sum_{j=0}^{k} \dfrac{(-1)^j \binom{k}{j} (x-t)^{r+i-j-1}}{(r+i-j-1)!} g^{k-j}(t)$, $i = 0, 1, \cdots k$,

and solving these equations and using Lemma 2, we get

$$P(x-t)^{(k+1)(r-1)} g^{k-j}(t) = \sum_{i=0}^{k} D_{ij} (D_t)^k G_{r+i}(x, t)$$

$= \sum_{i=0}^{k} K_{ij} (x-t)^{k(r-1)+j-i} (D^t)^k G_{r+i}(x, t)$, for $j = 0, 1, \cdots k$, where K_{ij} is independent of x and t, $P \neq 0$.

Hence, since $k(r-1) + j - i \geqslant 0$, we have

$$\begin{aligned}
&(\tfrac{1}{2}x)^{(k+1)(r-1)} \int_1^{x/2} t^{k+p} \,|\, g^{k-j}(t) \,|\, dt \\
&\leqslant \int_1^{x/2} t^{k+p} (x-t)^{(k+1)(r-1)} \,|\, g^{k-j}(t) \,|\, dt \\
&< \int_1^x t^{k+p} (x-t)^{(k+1)(r-1)} \,|\, g^{k-j}(t) \,|\, dt \\
&\leqslant \sum_{i=0}^{k} \left| \frac{K_{ij}}{P} \right| x^{k(r-1)+j-i} \int_1^x t^{p+k} \,|\, (D_t^k) G_{r+i}(x, t) \,|\, dt \\
&= O(x^{(k+1)(r-1)+p+q+1+j}) \text{ for } j = 0, 1, \cdots k, \text{ by } (3\cdot3), (4\cdot4)
\end{aligned}$$

Now, $(3\cdot4)$ gives

$\int_1^x t^{k+p} \,|\, g^{k-j}(t) \,|\, dt = O(x^{p+q+1+j})$, $j = 0, 1, \cdots k$, and by Lemma 4,

this gives $\qquad \int_1^x t^{k-q-j} \,|\, g^{k-j}(t) \,|\, dt = O(x)$. $\qquad (3\cdot5)$.

If $k \geqslant 1$, taking $j = 0$ and $j = k$ in $(3\cdot5)$ and applying Lemma 1, we get the necessary conditions (i) and (ii).

If $k = 0$, we get the single condition $(\mathrm{i})_0$ from $(3\cdot5)$.

Sufficiency. By expanding $(x-t)^{r-1}$ and applying Leibniz's theorem,

we get $\qquad (-1)^k (D_t)^k G_r(x, t) = \sum_{m=0}^{r-1} A_m x^{r-1-m} \sum_{i=0}^{k} B_i t^{m-i} g^{k-i}(t)$ when $k \geqslant 1$

and $\qquad (-1)^k (D_t)^k G_r(x, t) = \sum_{m=0}^{r-1} A_m x^{r-1-m} t^m g(t)$ when $k = 0$, where

$$A_m = \frac{(-1)^{m+k} \binom{r-1}{m}}{(r-1)!}, \quad B_i = \binom{k}{i} m(m-1) \cdots (m-i+1) \text{ for } i \geqslant 1, B_0 = 0.$$

Substituting in $(3\cdot1)$, we get, when $k \geqslant 1$, $I_r f(x) g(x)$

$$= \sum_{m=0}^{r-1} A_m x^{r-1-m} \sum_{i=0}^{k} B_i \int_1^x t^{m-i} f_k(t) g^{k-i}(t)\, dt + \delta_{rk} f_k(x) g(x), \qquad (3\cdot6)$$

and $\qquad I_r f(x) g(x) = \sum_{m=0}^{r-1} A_m x^{r-1-m} \int_1^x t^m f(t) g(t)\, dt$ when $k = 0$. $\qquad (3\cdot7)$

If $k = 0$ and $(\mathrm{i})_0$ holds, Lemma 4 gives

$\int_1^x t^{p+m} \,|\, g(t) \,|\, dt = O(x^{p+q+m+1})$ for $m = 0, 1, \cdots r-1$, and since $f(t) = o(t^p)$,

$$\int_1^x t^m f(t) g(t)\, dt = \int_1^x o(t^{p+m}) \,|\, g(t) \,|\, dt = O(x^{p+q+m+1}), \text{ and hence}$$

(3·7) gives $I_r f(x)\, g(x) = o(x^{p+q+r})$, the required result.

If $k \geqslant 1$, (i) and (ii) hold, by Lemma 1 we get

$\int_1^x t^{p+k+m-i}\, |g^{k-i}(t)|\, dt = O(x^{p+q+m+1})$, and since $f_k(x) = o(x^{p+k})$,

$\int_1^x t^{m-i} f_k(t)\, g^{k-i}(t)\, dt = o(x^{p+q+m+1})$, for $i = 0, 1, \cdots k$, and substituting in (3·6) we get

$$I_r f(x)\, g(x) = o(x^{p+q+r}) + o(x^{p+q+k}) = o(x^{p+q+r}).$$

Proof of Theorem 2.

(a) We may take, without loss of generality, $c_m = 0$ for $m = 0, 1, \cdots k-1$, since $f(x)$ may be replaced by $h(x) = f(x) - \sum_{m=0}^{k-1} c'_m e^{-(m+1)x}$, so that $h_k(x) = f_k(x) - \sum_{m=0}^{k-1} c_m x^{k-1-m} = o(x^{p+k})$, (the constants c'_m being chosen suitably), while the form of the conclusion is unchanged. It is also enough to take $p+q < -1$, $p+q \notin \{-1, -2, \cdots -r\}$.

If $k = 0$ and $(i)_0$ holds, as in the sufficiency part of Theorem 1, we get $\int_1^x t^m f(t)\, g(t)\, dt = o(x^{p+q+m+1}) + a_m$ for $m = 0, 1, \cdots r-1$, and substituting in (3·7) we get

$$I_r f(x)\, g(x) = \sum_{m=0}^{r-1} C_m x^{r-1-m} + o(x^{p+q+r}), \quad \text{where } C_m = A_m a_m.$$

If $k \geqslant 1$ and (i) and (ii) hold, as before we have

$\int_1^x t^{m-i} f_k(t)\, g^{k-i}(t)\, dt = o(x^{p+q+m+1}) + a_{mi}$, $i = 0, 1, \cdots k$ and $m = 0, 1, \cdots r-1$. Substituting in (3·6) we get the conclusion with $C_m = A_m \sum_{i=0}^{k} B_i a_{mi}$. Clearly, every $C_m = 0$ when $p+q > -1$.

(b) We may take every $c_m = 0$, $p+q \leqslant -1$. Let n be a positive integer such that $p+q+n > -1$.

By (a) above, with $f(x), g(x), p, q, k$ replaced by $f(x)g(x), x^n, p+q, n, r$ respectively, we see that $I_r f(x)\, g(x) = \sum_{m=0}^{r-1} C_m x^{r-1-m} + o(x^{p+q+r})$ implies $I_r x^n f(x)\, g(x) = o(x^{(p+q)+n+r})$, since $(p+q)+n > -1$, i.e. $I_r f(x) \,.\, x^n g(x) = o(x^{p+(q+n)+r})$ whenever $f_k(x) = o(x^{p+k})$, and hence by Theorem 1, with $g(x), q$ replaced by $x^n g(x), q+n$, we get the necessary conditions:

$$\int_1^x t^{-(q+n)}\, |t^n g(t)|\, dt = O(x) \text{ when } k = 0 \qquad (3\cdot8)$$

and

$$x^{-(q+n)}\, x^n g(x) = O(1), \qquad (3\cdot9)$$

$$\int_1^x t^{k-(q+n)}\, |(d/dt)^k [t^n g(t)]|\, dt = O(x) \text{ when } k \geqslant 1 \qquad (3\cdot10)$$

(3·8) and (3·9) are the same as $(i)_0$ and (i) respectively

By Lemma 1, (3·9) and (3·10) imply, for $j = 0, 1, \cdots k$,

$$\int_1^x t^{j-q-n} \left| (d/dt)^j [t^n g(t)] \right| dt = O(x). \tag{3·11}$$

But, $$g^k(t) = (d/dt)^k [t^{-n} \cdot t^n g(t)] = \sum_{j=0}^{k} d_j \, t^{-n-k+j} (d/dt)^j [t^n g(t)].$$

where $$d_j = \binom{k}{j} (-n)(-n-1) \cdots (-n-k+j+1).$$

Hence $$\int_1^x t^{k-q} |g^k(t)| \, dt \leqslant \sum_{j=0}^{k} |d_j| \int_1^x t^{j-q-n} \left| \left(\frac{d}{dt}\right)^j [t^n g(t)] \right| dt$$
$$= O(x), \text{ by (3·11)}.$$

This completes the proof.

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