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Original Study

On the complex properties of the first equation of the Kadomtsev-Petviashvili hierarchy

Seenith Sivasundaram[1†], Ajay Kumar[2], Ratnesh Kumar Singh[3]

[1]Bethune-Cookman University, Daytona Beach, Florida, 32114-3099, USA

[2]Department of Mathematics, Patna Women's College (Autonomous) Patna University, Patna, Bihar, 800001, India

[3]Department of Science and Technology, Government Engineering College, Buxar, 802101, India

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Abstract

This work studies the first equation of the Kadomtsev-Petviashvili (KP) hierarchy. The sine-Gordon expansion method (SGEM) and the rational SGEM (RSGEM) are applied to the governing model. RSGEM is the developed version of SGEM. New complex travelling wave solutions, logarithmic and complex function properties are obtained. Several simulations such as 2D, 3D and contour surfaces of the obtained results are plotted. Physical meanings of these solutions are also reported. Strain conditions are also extracted.

Keywords: The first equation of the KP hierarchy, SGEM, RSGEM, complex solutions, logarithmic solutions.
AMS 2020 codes: 35A24; 34A34; 34Axx.

1 Introduction

Nowadays, nonlinear models play an important role in the fields of applied science and engineering such as fluid mechanics, ocean, water, marina, propagation of waves and so on. Many analytical and numerical methods used to obtain the solutions of these models have been applied by experts from all over the world. More recently, many important powerful methods have been presented to literature namely the analytical and numerical methods [1–4], the Darboux transformations [5], the TFM [6, 7], the sine-Gordon expansion method [8–10], the $(\frac{G'}{G}, \frac{1}{G})$-expansion method [11], the (m+1/G) expansion method [12], the Lie group analysis [13], the Jacobi elliptic function method [14, 15], the generalized new auxiliary equation method [16], the modified $(\frac{w}{g})$-expansion method [17], the generalized Kudryashov method [18], the auxiliary equation method [19], the generalized exponential rational function method [20], the newly extended direct algebraic technique [21], the generalized bilinear form [22], the Hirota bilinear form [23–25], the transformed rational function method [26], the polynomial expansion method [27], the modified simple equation approach [28], the auto-Bäcklund transformation [29], the generalized bilinear operator [30], the extended homoclinic test approach [31], the bilinear forms [32, 33], the extended tanh-function method [34] and so on [35–53].

†Corresponding author.
Email address: seenithi@gmail.com

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In this work, by using SGEM and RSGEM, we focus on finding some new travelling wave solutions of the first equation of the KP hierarchy [54] given as

$$u_t = \frac{1}{2}u_{xxy} + \frac{1}{2}\partial_x^{-2}(u_{yyy}) + 2u_x\partial_x^{-1}(u_y) + 4uu_y, \tag{1}$$

in which the variable x, y, t are independent variables while the function $u = u(x,y,t)$ is a dependent variable. Wazwaz presented to the literature the multiple soliton solutions to literature [54]. Equation (1) is used to model the shallow water waves with weakly nonlinear restoring forces [55].

The rest of this article is organized as follows. In section 2, we present some important properties of the SGEM and RSGEM. In section 3, we implement the proposed method to observe the logarithmic properties of the governing model. In section 4, a comprehensive conclusion is presented.

2 General properties of approaches

In this part of the paper, we present the general structures of SGEM and RSGEM as follows.

2.1 General properties of SGEM

Considering the sine-Gordon equation [8–10]

$$u_{xx} - u_{tt} = m^2 \sin(u), \tag{2}$$

where $u = u(x,t)$ and m is a real constant. Applying the wave transformation given as $u = u(x,t)= U(\xi)$, $\xi = \mu(x - ct)$ to equation (2)

$$U'' = \frac{m^2}{\mu^2(1-c^2)}\sin(U), \tag{3}$$

where $U = U(\xi)$ and ξ, is the dimension of the travelling wave and c is the velocity of the travelling wave. By applying some calculations, we reach

$$\left((\frac{U}{2})' \right)^2 = \frac{m^2}{\mu^2(1-c^2)}\sin^2(\frac{U}{2}) + K, \tag{4}$$

where K is the integrate constant. Substituting $K = 0, w(\xi) = \frac{U}{2}$ and $a^2 = \frac{m^2}{\mu^2(1-c^2)}$, then equation (4) is converted to the following equation

$$w' = a\sin(w). \tag{5}$$

Putting a = 1 into equation (5) gives

$$w' = \sin(w). \tag{6}$$

Solving equation (6), we obtain the following two important properties as follows

$$\sin(w) = \sin(w(\xi)) = \frac{2pe^{\xi}}{p^2e^{2\xi}+1} \downarrow p_{=1} = \mathrm{sech}(\xi), \tag{7}$$

$$\cos(w) = \cos(w(\xi)) = \frac{2pe^{\xi}}{p^2e^{2\xi}+1} \downarrow p_{=1} = \tanh(\xi), \tag{8}$$

where p is the integral constant and nonzero. By considering these two properties, in general cases we can consider the following PDEs as

$$P(u,u_x,u_t,u_{xx},u_{tt},u_{xt},u_{xxx},\cdots)=0. \tag{9}$$

With the help of $u=u(x,t)=U(\xi),\ \xi=\mu(x-ct)$ into equation (9), we find the following ordinary differential equation

$$N(U,U',U'',U^2,\cdots)=0.$$

In this equation, we suppose the following trial solution equation defined by

$$U(\xi)=\sum_{i=1}^{n}\tanh^{i-1}(\xi)[B_i\text{sech}(\xi)+A_i\tanh(\xi)]+A_0. \tag{10}$$

Taking equations (7) and (8) into equation (10), we rewrite it as follows

$$U(\omega)=\sum_{i=1}^{n}\cos^{i-1}(\omega)[B_i\sin(\omega)+A_i\cos(\omega)]+A_0. \tag{11}$$

We determine the value n via balance principle. Taking the coefficients of $\sin^i(\omega)\cos^j(\omega)$ to be all zero, yield a system of equations. Solving this system by software computing program, the values of A_i,B_i,c and μ may be obtained. Then, we can find the wanted solutions for the governing model.

2.2 General properties of RSGEM

In this part, we take into account the nonlinear partial differential equation (NPDE) as following [56, 57]

$$P(\Psi,\Psi_x,\Psi_{xt},\Psi^2,\cdots)=0. \tag{12}$$

Taking as $\Psi=\Psi(x,t)=U(\xi),\ \xi=\mu(x-ct)$, it is converted as

$$N(U,U',U'',U^2,\cdots)=0. \tag{13}$$

Here $U=U(\xi),U'=\frac{dU}{d\xi}$. In equation (13), the trial equation for solution function may be considered as

$$U(\xi)=\frac{\sum_{i=1}^{n}\tanh^{i-1}(\xi)[A_i\text{sech}(\xi)+c_i\tanh(\xi)]+A_0}{\sum_{i=1}^{m}\tanh^{i-1}(\xi)[B_i\text{sech}(\xi)+d_i\tanh(\xi)]+B_0}. \tag{14}$$

Equation (14) may be rewritten with the help of equations (7) and (8) as following

$$U(\omega)=\frac{\sum_{i=1}^{n}\cos^{i-1}(\omega)[A_i\sin(\omega)+c_i\cos(\omega)]+A_0}{\sum_{i=1}^{m}\cos^{i-1}(\omega)[B_i\sin(\omega)+d_i\cos(\omega)]+B_0}, \tag{15}$$

in which the values of n,m will be determined later via balance principle. After putting the necessary derivations of equation (15) into equation (13), we obtain an equation of $\sin^i(\omega)\cos^j(\omega)$. Taking all these terms to zero yields a system of equations. Solving this system by using some computational programs, gives the values of A_i,B_i,c_i,d_i,μ and c. Via these values of parameters A_i,B_i,c_i,d_i,μ and c in equation (14), we obtain the new travelling wave solutions to equation (12).

3 Applications of the schemes

In this part, we apply SGEM and RSGEM to obtain some new analytic solutions to the governing model as follows.

3.1 Application of SGEM

In this subsection, we apply the SGEM to extract some new wave solutions to the equation (1). As a first step for simplicity, we consider the transformation defined as

$$u(x,y,t) = v_{xx}(x,y,t). \tag{16}$$

This transformation converts the equation (1) into the following form

$$v_{xxt} = \frac{1}{2}v_{xxxxy} + \frac{1}{2}v_{yyy} + 2v_{xxx}v_{xy} + 4v_{xx}v_{xxy}. \tag{17}$$

Considering the following wave transformation given as

$$v(x,y,t) = V(\xi), \xi = kx + ly - ct, \tag{18}$$

we find the following nonlinear differential equation

$$\frac{1}{2}lk^4V^{(5)} + \frac{1}{2}l^3V''' + 6k^4lV''V''' + ck^2V''' = 0. \tag{19}$$

Integrating equation (19) with respect to ξ and getting to the zero of integral constant, we get

$$\frac{1}{2}lk^4V^{(4)} + \frac{1}{2}l^3V'' + 3k^4l(V'')^2 + ck^2V'' = 0. \tag{20}$$

For simplicity, we take follows

$$V'' = W, V^{(4)} = W'', \tag{21}$$

which produces

$$lk^4W'' + 6k^4lW^2 + (2ck^2 + l^3)W = 0. \tag{22}$$

With the balance in equation (22), we have

$$M = 2. \tag{23}$$

This produces the following solution form

$$W(w) = B_1\sin(w) + A_1\cos(w) + B_2Cos(w)\sin(w) + A_2\cos^2(w) + A_0. \tag{24}$$

For the second derivation of equation (24), is obtained

$$\begin{aligned} W''(w) = {} & B_1\cos^2(w)\sin(w) - B_1\sin^3(w) - 2A_1\sin^2(w)\cos(w) + \\ & B_2Cos^3\sin(w) - 5B_2\sin^3(w)\cos(w) - 4A_2\cos^2(w)\sin^2(w) + 2A_2\sin^4(w), \end{aligned} \tag{25}$$

where either A_2 or B_2 may be zero, however, both A_2 and B_2 cannot be zero, simultaneously. By substituting equations (24) and (25) into equation (22), we extract the following results.

Case-1 After selecting as $A_0 = \frac{1}{2},\ A_1 = 0,\ A_2 = -\frac{1}{2},\ B_1 = 0,\ B_2 = \frac{i}{2},\ c = \frac{-l(k^4+l^2)}{2k^2}$, we obtain the dark optical solitary solutions to the equation (17) as

$$u_1(x,y,t) = \frac{1}{2}\left(-2i\arctan\left[\tanh[\frac{1}{2}(\frac{l(k^4+l^2)t}{2k^2} + kx + ly)]\right] + \ln\left[\cosh[\frac{l(k^4+l^2)t}{2k^2} + kx + ly]\right]\right), \tag{26}$$

which is $k \neq 0, l \neq 0$ as strain conditions for validity of equation (26).

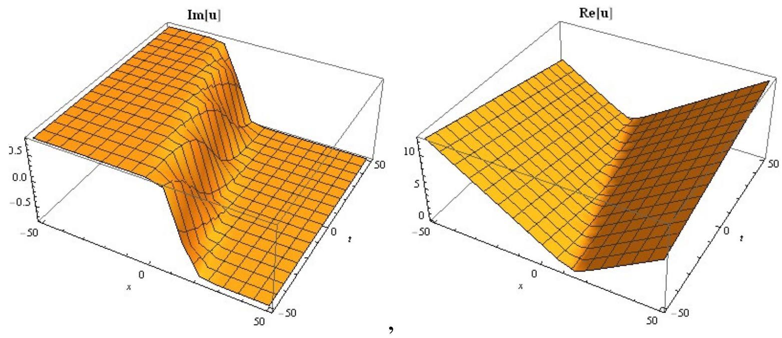

Fig. 1 3D graphs of imaginary and real part of equation (26)

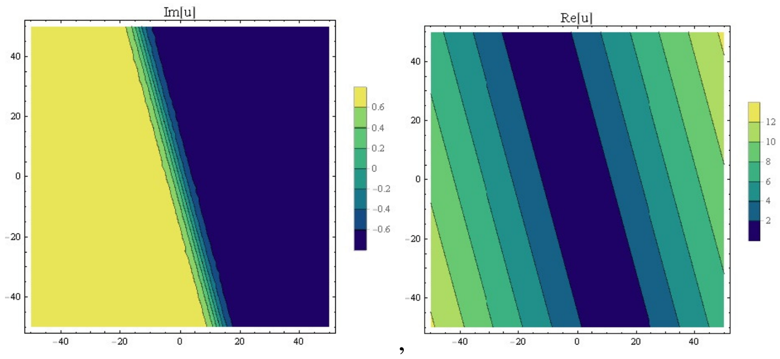

Fig. 2 Contour graphs of imaginary and real part of equation (26)

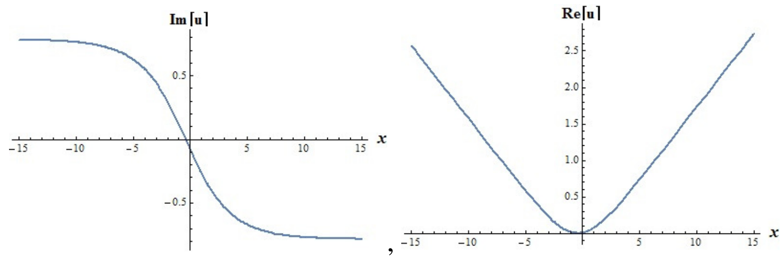

Fig. 3 2D graphs of imaginary and real part of equation (26)

Case-2 When $A_0 = \frac{1}{3}$, $A_1 = 0$, $A_2 = -\frac{1}{2}$, $B_1 = 0$, $B_2 = -\frac{i}{2}$, $c = \frac{l(k^4-l^2)}{2k^2}$, we obtain the following hyperbolic function solution to the equation (17) as

$$\begin{aligned} u_2(x,y,t) = &-\frac{1}{12}\left(-\frac{l(k^4-l^2)t}{2k^2}+kx+ly\right)+i\arctan\left[\tanh[\frac{1}{2}(-\frac{l(k^4-l^2)t}{2k^2}+kx+ly)]\right] \\ &+\frac{1}{2}\ln\left[\cosh(\frac{l(k^4-l^2)t}{2k^2}-kx-ly)\right], \end{aligned} \quad (27)$$

which is $k \neq 0, l \neq 0, k^4 \neq l^2$ as strain conditions for validity of equation (27).

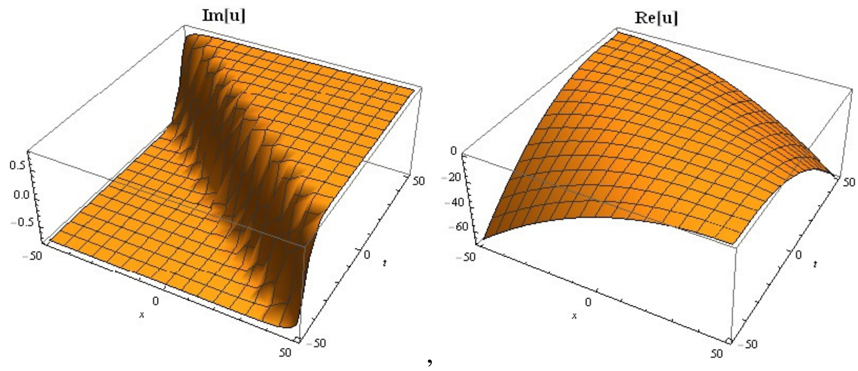

Fig. 4 3D graphs of imaginary and real part of equation (27)

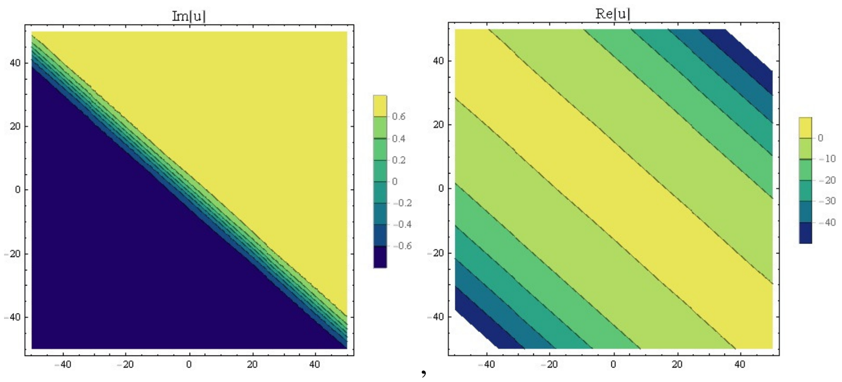

Fig. 5 Contour graphs of imaginary and real part of equation (27)

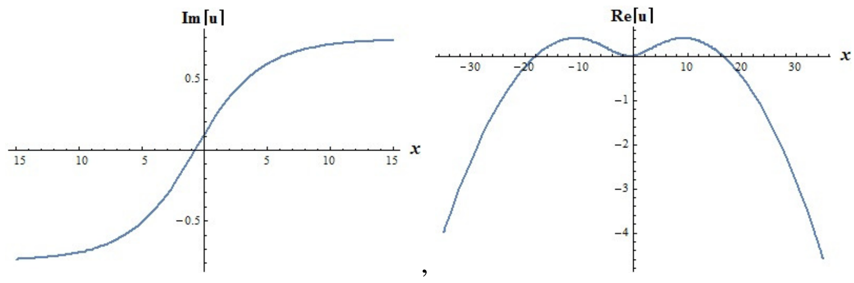

Fig. 6 2D graphs of imaginary and real part of equation (27)

Case-3 When $A_0 = 1,\ A_1 = 0,\ A_2 = -1,\ B_1 = 0,\ B_2 = 0,\ c = -\frac{(4lk^4+l^3)}{2k^2}$, we thus obtain the logarithmic solution to the equation (17) given as

$$u_3(x,y,t) = \ln\left[\cosh[\frac{(4lk^4+l^3)t}{2k^2} + kx + ly]\right], \tag{28}$$

in where is $k \neq 0, l \neq 0$ and real constants.

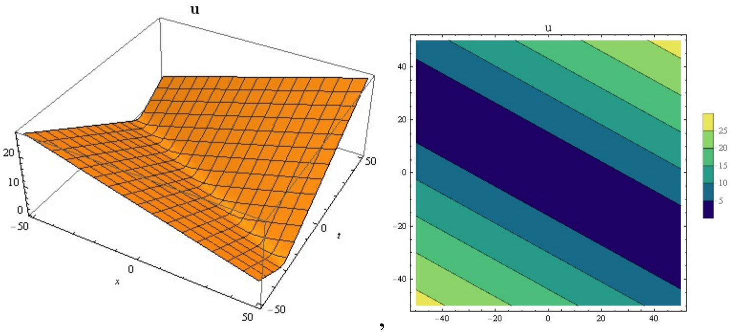

Fig. 7 The 3D and contour surfaces of equation (28)

Case-4 If we select as $A_0 = \frac{1}{3}$, $A_1 = 0$, $A_2 = -1$, $B_1 = 0$, $B_2 = 0$, $l = \frac{-4(-3)^{\frac{1}{3}}k^4 + (-1)^{\frac{2}{3}}\left(-9ck^2+\sqrt{3}\sqrt{k^4(27c^2-64k^8)}\right)^{\frac{2}{3}}}{(3)^{\frac{2}{3}}\left(-9ck^2+\sqrt{3}\sqrt{k^4(27c^2-64k^8)}\right)^{\frac{1}{3}}}$, we obtain the dark soliton solutions of equation (17) as following

$$
\begin{aligned}
u_4(x,y,t) = \frac{-1}{3}&\left(-ct + kx + \frac{(-4(-3)^{\frac{1}{3}}k^4 + (-1)^{\frac{2}{3}}\left(-9ck^2+\sqrt{3}\sqrt{k^4(27c^2-64k^8)}\right)^{\frac{2}{3}})y}{(3)^{\frac{2}{3}}\left(-9ck^2+\sqrt{3}\sqrt{k^4(27c^2-64k^8)}\right)^{\frac{1}{3}}}\right)^2 \\
&+ ln[\cosh\left[ct - kx - \frac{(-4(-3)^{\frac{1}{3}}k^4 + (-1)^{\frac{2}{3}}\left(-9ck^2+\sqrt{3}\sqrt{k^4(27c^2-64k^8)}\right)^{\frac{2}{3}})y}{(3)^{\frac{2}{3}}\left(-9ck^2+\sqrt{3}\sqrt{k^4(27c^2-64k^8)}\right)^{\frac{1}{3}}}]\right],
\end{aligned} \tag{29}
$$

where $k \neq 0, c \neq 0, 27c^2 \neq 64k^8$ as strain conditions for validity of equation (29).

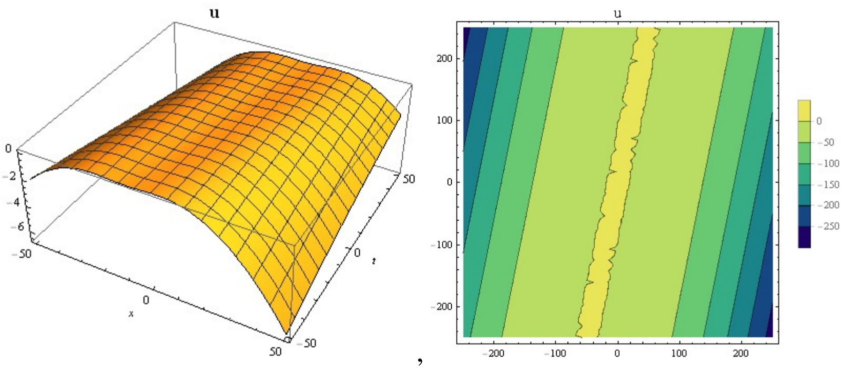

Fig. 8 The 3D and contour surfaces of equation (29)

Case-5 When we choose other coefficients as $A_0 = \frac{1}{3},\ A_1 = 0,\ A_2 = -\frac{1}{2},\ B_1 = 0,\ B_2 = \frac{i}{2},\ c = \frac{l(k^4-l^2)}{2k^2}$, we acquire the mixed dark-bright logarithmic function solution to the equation (17) as

$$\begin{aligned} u_5(x,y,t) = &-\frac{1}{12}\left(-\frac{l(k^4-l^2)t}{2k^2}+kx+ly\right)^2 - i\arctan\left[\tanh[\frac{1}{2}(-\frac{l(k^4-l^2)t}{2k^2}+kx+ly)]\right] \\ &+\frac{1}{2}ln\left[\cosh[\frac{l(k^4-l^2)t}{2k^2}-kx-ly]\right], \end{aligned} \tag{30}$$

which is $k \neq 0, l \neq 0, k^4 \neq l^2$ as strain conditions for validity of equation (30).

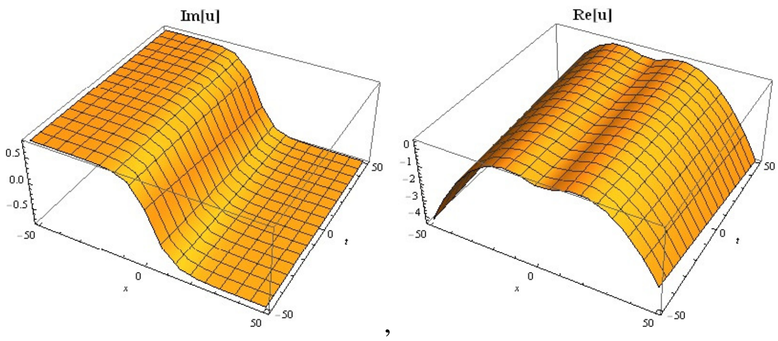

Fig. 9 3D graphs of imaginary and real part of equation (30)

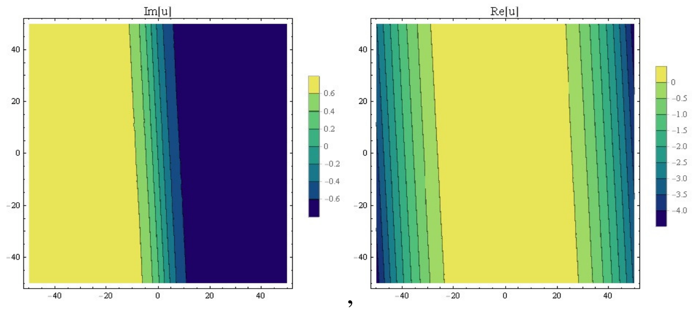

Fig. 10 Contour graphs of imaginary and real part of equation (30)

3.2 Application of RSGEM

In this subsection of the paper, especially, we take as $n = m$. Via the balance rule in equation (22) gives $n = m = 2$. Getting $n = m = 2$ in equation (14), we write the test function of solution as

$$W(\xi) = \frac{A_0 + A_1\tanh(\xi) + c_1\tanh(\xi) + \cos(\xi)[A_2\text{sech}(\xi) + c_2\tanh(\xi)]}{B_0 + B_1\text{sech}(\xi) + d_1\tanh(\xi) + \tanh(\xi)[B_2\text{sech}(\xi) + d_2\tanh(\xi)]}. \tag{31}$$

When we consider equations (7) and (8), we rewrite equation (31) as following

$$W(w) = \frac{A_0 + A_1\sin(w) + c_1\cos(w) + \cos(w)[A_2\sin(w) + c_2\cos(w)]}{B_0 + B_1\sin(w) + d_1\cos(w) + \cos(w)[B_2\sin(w) + d_2\cos(w)]}. \tag{32}$$

Substituting equation (32) and its second derivative into equation (22), we obtain an equation of $\sin^i(w)\cos^i(w)$. Getting all coefficients of these terms to zero, we gain a system of equations. Solving this system produces the values of parameters A_i, B_i, c_1, d_1, k, l and c which results in many entirely new travelling wave solutions to the equation (17).

Case-1 When $B_2 = d_2 = 0, k = 1, l = 2, A_0 = -c_2, A_1 = -\frac{id_1}{2}, A_2 = ic_2, B_0 = -2c_2, B_1 = -id_1, c_1 = 0, c = -5$ into equation (31), it yields the following entirely new complex function solution to the governing model as

$$u_6(x,y,t) = -\frac{1}{2}i\big(x+2y+5t+2\arctan\left[\coth\left[\frac{1}{2}(x+2y+5t)\right]\right]+i\ln\left[x+2y+5t\right]\big). \tag{33}$$

Selecting suitable values in equation (33), we can observe the simulations of equation (33) via Figures (11,12,13) as follows.

Remark-1 Equation (33) contains algebraic independent variables under the natural logarithmic function. This is also used to symbolise the physical meanings of the the first equation of the nonlinear Kadomtsev-Petviashvili hierarchy. In the solution of (33), it may be seen that it contains a specific critical symmetry point. This is one of the new properties of the first equation of the nonlinear Kadomtsev-Petviashvili hierarchy.

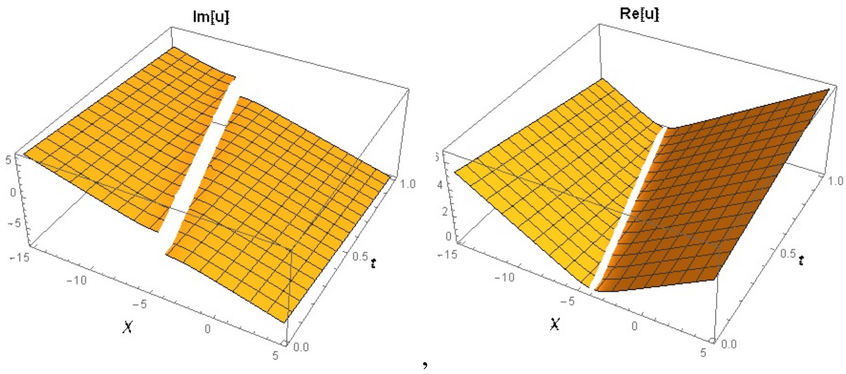

Fig. 11 The 3D surfaces of equation (33)

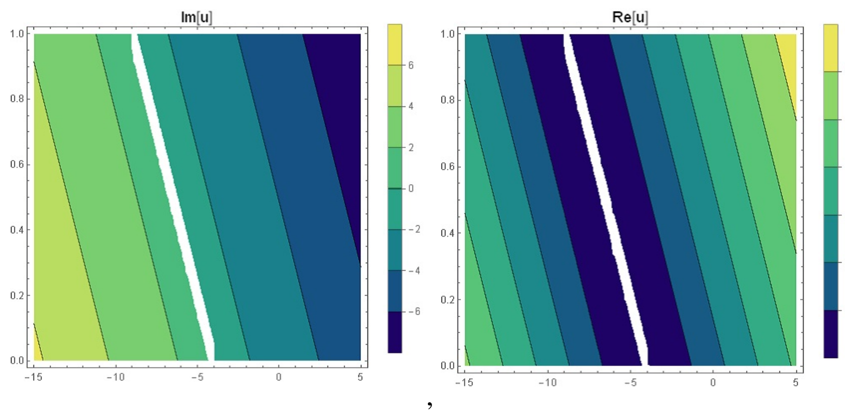

Fig. 12 The contour surface of equation (33)

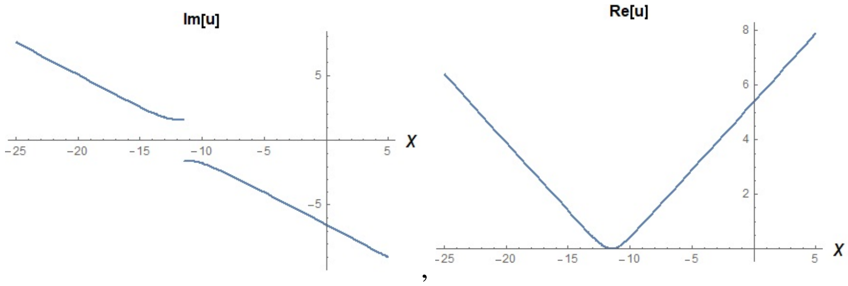

Fig. 13 The 2D surface of equation (33)

Case-2 If $B_2 = d_2 = 0, k = 1, l = 2, A_0 = -\frac{2c_2}{3}, A_1 = \frac{id_1}{3}, A_2 = -ic_2, B_0 = -2c_2, B_1 = id_1, c_1 = \frac{-d_1}{6}, c = -3$ into equation (31), it yields the following new complex solution to the governing model as

$$u_7 = -\frac{1}{12}(x+2y+3t)(x+2y-6i+3t) + i\arctan\left[\coth\left(\frac{1}{2}(x+2y+3t)\right)\right] + \frac{1}{2}\ln\left[\cosh(x+2y+3t)\right]. \quad (34)$$

Various simulations of equation (34) may be seen via Figures (14,15,16) as follows.

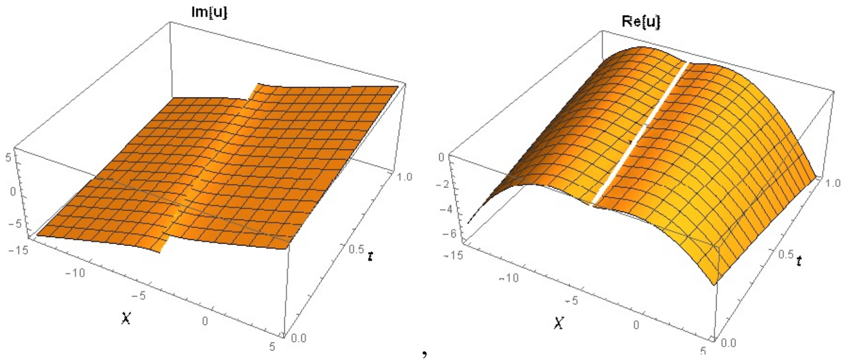

Fig. 14 The 3D surfaces of equation (34)

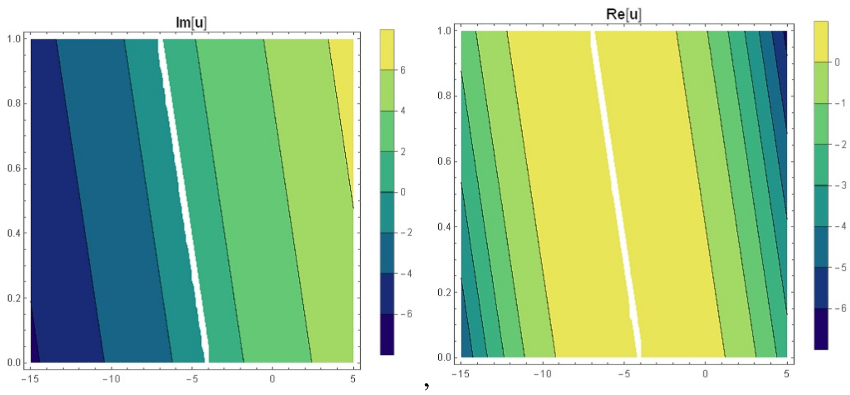

Fig. 15 The contour surface of equation (34)

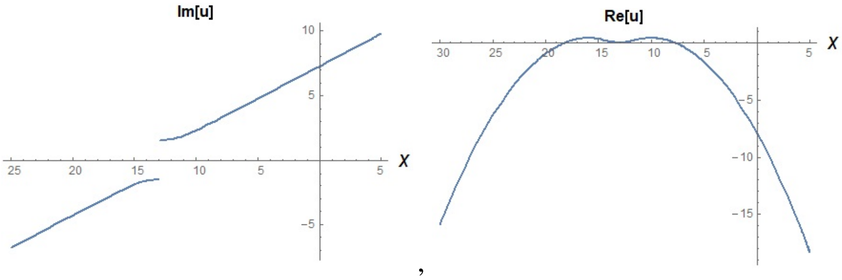

Fig. 16 The 2D surface of equation (34)

4 Conclusion

In this paper, we have successfully applied SGEM and RSGEM to the first equation of the KP hierarchy. Many new complex, logarithmic and hyperbolic function solutions have been obtained by considering different values of coefficients of solutions by proposed schemes. Parametric conditions for validity of solutions have also been reported. Finally, we have plotted the surfaces of various solutions in 2D, 3D and contour sides by Figures (1-16) with the help of some powerful computation programs. The proposed schemes supply many new coefficients which produce new properties of the model when we compare these results with existing papers. It is observed that the recently developed scheme RSGEM produces more complicated solutions to the governing model. So, it is estimated that the solutions obtained in this paper may be helpful to better understand the deeper properties of the first equation of the KP hierarchy. Moreover, these methods are also applied to the other nonlinear models in the fields of real world problems in wave distributions.

5 Declarations

5.1 Conflict of interest:

According to the authors of this paper, there are no conflicts of interest to report regarding the article that is being presented.

5.2 Author's contributions:

S.S-Conceptualization, Methodology, Supervision. A.K.-Formal analysis, Writing-Review and Editing. R.K.S.-Resources, Writing-Original Draft, Methodology, Validation. All authors read and approved the final submitted version of this manuscript.

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5.5 Data availability statement:

All data that support the findings of this study are included within the article.

5.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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