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EXTREMAL PROPERTIES OF LINEAR DYNAMIC SYSTEMS CONTROLLED BY DIRAC'S IMPULSE

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The paper concerns the properties of linear dynamical systems described by linear differential equations, excited by the Dirac delta function. A differential equation of the form $a_n x^{(n)}(t)+\dots+a_1 x^{'}(t)+a_0 x(t) = b_m u^{(m)}(t)+\dots+b_1 u^{'}(t)+b_0 u(t)$ is considered with $a_i, b_j > 0$. In the paper we assume that the polynomials $M_n(s) = a_n s^n + \dots + a_1 s + a_0$ and $L_m(s) = b_m s^m + \dots + b_1 s + b_0$ partly interlace. The solution of the above equation is denoted by $x(t, L_m, M_n)$. It is proved that the function $x(t, L_m, M_n)$ is nonnegative for $t \in (0, \infty)$, and does not have more than one local extremum in the interval $(0, \infty)$ (Theorems 1, 3 and 4). Besides, certain relationships are proved which occur between local extrema of the function $x(t, L_m, M_n)$, depending on the degree of the polynomial $M_n(s)$ or $L_m(s)$ (Theorems 5 and 6).

Keywords: extremal properties, Dirac's impulse, linear systems, transfer function.

1. Introduction

One of the most important quality measures of dynamic systems is the extremal value in the transient process. In automatics it is the maximal value of the error. In electrical systems, especially in long lines, it is the value of the overvoltage. In economic systems it is the determination of the maximal profit (Kaczorek, 2002; 2018). In the article we consider the conditions of positive solutions, and in particular their maximal value (Górecki, 2018).

It is proved that in the systems described by the differential equation

$$\sum_{i=0}^{n} a_i x^{(i)}(t) = \delta(t)$$

controlled by Dirac's impulse $\delta(t)$ the solutions of $x(t)$ are positive and have only one maximal value.

We will use the solutions in the operator form,

$$X(s) = \frac{L(s)}{M(s)}\delta(s),$$

or in the time domain,

$$x(t) = \sum_{i=1}^{n} \frac{L(s_i)}{M^{(1)}(s_i)} e^{s_i t}.$$

2. Notation

Consider real polynomials

$$\begin{aligned} M_n(s) &= s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n \\ &= (s - s_1)(s - s_2)\dots(s - s_n), \\ L_m(s) &= s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m \\ &= (s - z_1)(s - z_2)\dots(s - z_m), \end{aligned} \tag{1}$$

where

$$n \geq 2, \quad 1 \leq m \leq n-1, \quad L_0(s) = 1.$$

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Definition 1. We say that the polynomials $M_n(s)$, $L_m(s)$ *interlace* (Górecki and Zaczyk, 2013) if $1 \le m \le n-1$ and $s_n < z_{n-1} < s_{n-1} < z_{n-2} < \cdots < s_2 < z_1 < s_1 < 0$. If $1 \le m < n-1$ and

$$s_n < s_{n-1} < s_{n-2} < \cdots < s_{m+2} < s_{m+1}$$
$$< z_m < s_m < z_{m-1} < \cdots < s_2 < z_1 < s_1 < 0,$$

then the polynomials $M_n(s)$, $L_m(s)$ *partly interlace*. Additionally we assume that the polynomials $M_n(s)$, $L_0(s) = 1$ partly interlace.

Write

$$N = \{1, 2, \dots, n\},$$
$$f^{(k)}(s) = \frac{\mathrm{d}^k f(s)}{\mathrm{d}s^k}, \quad k = 1, 2, \dots,$$
$$f'(s) = f^{(1)}(s), \quad f''(s) = f^{(2)}(s),$$

where $\deg w(s)$ is the degree of a polynomial $w(s)$.

For the polynomials $M_n(s)$, $L_m(s)$, which partly interlace, we use the following notation:

$$M_{n(j)}(s) = \frac{M_n(s)}{s - s_j} \quad \text{for } j \in \mathbb{N},$$
$$M_{n(0)}(s) = M_n(s),$$
$$A_i(j) = \frac{L_m(s_i)}{M'_{n(j)}(s_i)} \quad \text{for } i \in \mathbb{N}, \ j \in \{0, 1, \dots, n\},$$
$$x(t, L_m, M_{n(j)}) = \sum_{i=1, i \ne j}^{n} A_i(j) e^{s_i t}, \tag{2}$$

for $i \in \mathbb{N}$, $j \in \{0, 1, 2, \dots, n\}$.

Instead of $A_i(0)$ we shall write A_i, and $x(t, L_m, M_n)$ instead of $x(t, L_m, M_{n(0)})$.

We see that

$$\left.\begin{array}{r} M_n(s) = M_{n(j)}(s)\,(s - s_j) \quad \text{for } j \in \mathbb{N}, \\ M'(s_i) = M'_{n(j)}(s_i)\,(s_i - s_j) \ \text{ for } i \ne j, \\ \deg M_{n(j)}(s) = n - 1. \end{array}\right\} \tag{3}$$

In this paper we prove some properties of the function $x(t, L_m, M_n)$, in particular, the extremal ones.

3. Statement of results

Lemma 1. *If $n \ge 2$, $s_i \in \mathbb{R}\backslash\{0\}$ for $i \in \mathbb{N}$, $z_i \in \mathbb{R}\backslash\{0\}$ for $i = 1, 2, \dots, m$, $s_i \ne s_j$, $z_i \ne z_j$ then*

$$x'(t, L_m, M_n) = s_j x(t, L_m, M_n) + x(t, L_m, M_{n(j)}) \quad \textit{for } j \in \mathbb{N}, \tag{4}$$

$$x(t, L_m, M_n) = x'(t, L_{m-1}, M_n) - z_m x(t, L_{m-1}, M_n), \tag{5}$$

for $m \in \{1, 2, \dots, n-1\}$ and for every $t \in \mathbb{R}$, where the functions $x(t, L_m, M_n)$, $x(t, L_m, M_{n(j)})$, $x(t, L_{m-1}, M_n)$ are determined by (2).

Proof. From (2) we have

$$x(t, L_m, M_n) = \sum_{i=1}^{n} \frac{L_m(s_i) e^{s_i t}}{M'_n(s_i)},$$
$$x(t, L_m, M_n) e^{-s_j t} = \sum_{i=1}^{n} \frac{L_m(s_i) e^{(s_i - s_j)t}}{M'_n(s_i)}.$$

We can differentiate both the sides of the last relations with respect to t and obtain

$$x'(t, L_m, M_n) e^{-s_j t} - s_j x(t, L_m, M_n) e^{-s_j t} = \sum_{i=1, i \ne j}^{n} (s_i - s_j) \frac{L_m(s_i)}{M'_n(s_i)} e^{(s_i - s_j)t}.$$

From this and (3) we have, for $j \in \mathbb{N}$,

$$\begin{aligned} x'(t, L_m, M_n) - s_j x(t, L_m, M_n) &= \sum_{i=1, i \ne j}^{n} \frac{L_m(s_i)}{M'_{n(j)}(s_i)} e^{s_i t} \\ &= x(t, L_m, M_{n(j)}). \end{aligned}$$

This means that the relation (4) is true.

From the definition of the polynomial $L_m(s)$ and (2) we have

$$\begin{aligned} L_m(s_i) &= (s_i - z_1)(s_i - z_2) \dots (s_i - z_m) \\ &= L_{m-1}(s_i)(s_i - z_m), \\ x(t, L_m, M_n) &= \sum_{i=1}^{n} \frac{L_m(s_i) e^{s_i t}}{M'_n(s_i)} \\ &= \sum_{i=1}^{n} \frac{L_{m-1}(s_i)}{M'_n(s_i)} (s_i - z_m) e^{s_i t} \\ &= \sum_{i=1}^{n} \frac{L_{m-1}(s_i)}{M'_n(s_i)} s_i e^{s_i t} \\ &\quad - z_m \sum_{i=1}^{n} \frac{L_{m-1}(s_i)}{M'_n(s_i)} e^{s_i t} \\ &= x'(t, L_{m-1}, M_n) - z_m x(t, L_{m-1}, M_n). \end{aligned}$$

This completes the proof. ■

Remark 1. If $s_2 < s_1 < 0$, then the function

$$x_2(t) = \frac{e^{s_1 t} - e^{s_2 t}}{s_1 - s_2} \tag{6}$$

has a unique local extremum-maximum in the interval $(0, \infty)$ for

$$t_2 = \frac{\ln \frac{s_2}{s_1}}{s_1 - s_2}.$$

The proof is straightforward.

It is also known (Osiowski, 1965) that for the polynomials $M_n(s)$, $L_0(s) = 1$ we have the following initial conditions:

$$\left.\begin{array}{c} x\,(0, L_0, M_n) = 0, \\ x^{(k)}\,(0, L_0, M_n) = 0 \quad (k = 1, 2, \dots, n-2), \\ x^{(n-1)}(0, L_0, M_n) = 1. \end{array}\right\} \quad (7)$$

4. Extremal properties of the function $x(t, L_m, M_n)$

Let us consider the polynomials $M_n(s)$, $L_m(s)$ (1) and the function $x(t, L_m, M_n)$, which is determined by the formula (2). We investigate the properties of this function.

Theorem 1. *If*

$$\deg M_n(s) = n \geq 2,$$

$$\deg L_m(s) = m = n - 1,$$

and the polynomials $M_n(s)$, $L_m(s)$ interlace, then the function $x(t, L_m, M_n)$, defined by the formula (2)*, satisfies the following inequalities for each $t \geq 0$:*

$$x(t, L_m, M_n) = \sum_{i=1}^{n} A_i e^{s_i t} > 0\ ,$$

$$x'(t, L_m, M_n) < 0.$$

Proof. From the assumption that the polynomials $M_n(s), L_m(s)$ interlace, it follows that their coefficients are positive. Therefore $M_n(0) > 0, L_m(0) > 0$ and

$$M_n'(s_1) > 0,\quad M_n'(s_2) < 0,\quad M_n'(s_3) > 0,\ \ M_n'(s_4) < 0,\ \dots$$

$$L_m(s_1) > 0,\ \ L_m(s_2) < 0,\quad L_m(s_3) > 0,\ \ L_m(s_4) < 0,\ \dots.$$

Thus

$$A_i = \frac{L_m(s_i)}{M_n'(s_i)} > 0 \quad \text{for } i \in \mathbb{N},$$

$$x(0, L_m, M_n) > 0, \quad \lim_{t\to\infty} x(t, L_m, M_n) = 0\ ,$$

$$x'(t, L_m, M_n) < 0 \quad \text{for} \quad t \in \mathbb{R}.$$

This completes the proof. ■

Theorem 2. *If* $\deg M_n(s) = n \geq 2$, $\ L_0(s) = 1$ *and*

$$s_n < s_{n-1} < s_{n-2} < \dots < s_2 < s_1 < 0,$$

then the function $x(t, L_0, M_n)$, determined by (2)*, for each $t > 0$ satisfies*

$$x(t, L_0, M_n) > 0. \qquad (8)$$

Proof. The proof will be by induction on n. For $n = 2$ the inequality (8) follows from Remark 1, because for $n = 2$ the function $x(t, L_0, M_2) = x_2(t)$, where $x_2(t)$ is determined by the formula (6). We assume that the inequality (8) is true for $n - 1$. From this assumption it is evident that

$$x(t, L_0, M_{n(j)}) > 0 \qquad (9)$$

for each $t > 0$, $\ j \in N$, because $\deg M_{n(j)}(s) = n - 1$.

From Lemma 1 we have

$$\begin{aligned} x'(t, L_0, M_n) &= s_j x(t, L_0, M_n) \\ &\quad + x(t, L_0, M_{n(j)}) \quad \text{for } j \in \mathbb{N}. \end{aligned} \qquad (10)$$

We will prove the inequality (8) by *reductio ad absurdum*; we assume that there exists $t_1 \in (0, \infty)$ such that $x(t_1, L_0, M_n) \leq 0$.

Consider the case where $x(t_1, L_0, M_n) < 0$. Then (7) and the relation

$$\lim_{t\to\infty} x(t, L_0, M_n) = 0$$

imply that there exists $t_2 \in (0, \infty)$ such that

$$x'(t_2, L_0, M_n) = 0, \quad x(t_2, L_0, M_n) < 0.$$

Hence and from (10) it follows that $x(t_2, L_0, M_{n(j)}) < 0$. This is a contradiction to (9). Similarly, we proceed in the case where $x(t_1, L_0, M_n) = 0$. Therefore the inequality (8) is true. ■

Now we give some generalization of the last theorem.

Theorem 3. *If* $\deg M_n(s) = n \geq 2$, $\ \deg L_m(s) = m \in \{0, 1, 2, \dots, n-1\}$ *and the polynomials $M_n(s)$ and $L_m(s)$ partly interlace, then the function $x(t, L_m, M_n)$, determined by* (2)*, for each t >0 satisfies the inequality*

$$x(t, L_m, M_n) > 0. \qquad (11)$$

Proof. For $m = n - 1$ the inequality (11) follows from Theorem 1. For $m \in \{0, 1, 2, \dots, n-2\}$ the proof will be by induction on m. For $m = 0$ the inequality (11) follows from Theorem 2. We assume that the inequality (11) is true for $m - 1 \in \{0, 1, 2, \dots, n-2\}$, and from this we deduce that it is true for $m \in \{0, 1, 2, \dots, n-2\}$.

The polynomials $L_{m-1}(s)$ and $L_m(s)$ have the form

$$L_{m-1}(s) = (s - z_1)(s - z_2)\dots(s - z_{m-1}),$$

$$L_m(s) = L_{m-1}(s)(s - z_m).$$

Hence, from (4) and (5) we have

$$x(t, L_m, M_n) = x'(t, L_{m-1}, M_n) - z_m x(t, L_{m-1}, M_n)\ ,$$

$$\begin{aligned} x'(t, L_{m-1}, M_n) &= s_j x(t, L_{m-1}, M_n) \\ &\quad + x(t, L_{m-1}, M_{n(j)}) \quad \text{for } j \in \mathbb{N}. \end{aligned}$$

Therefore

$$\begin{aligned}x(t,L_m,M_n) &= s_j x(t,L_{m-1},M_n) + x(t,L_{m-1},M_{n(j)})\\ &\quad - z_m x(t,L_{m-1},M_n)\\ &= (s_j - z_m)x(t,L_{m-1},M_n)\\ &\quad + x(t,L_{m-1},M_{n(j)}).\end{aligned}$$

For $j = m$ the polynomials $L_{m-1}(s),\ M_n(s)$ and $L_{m-1}(s),\ M_{n(j)}(s)$ partly interlace and $(s_m - z_m) > 0$. From this and the inductive hypothesis, we have

$$x(t,L_{m-1},M_n) > 0\,, \quad x(t,L_{m-1},M_{n(m)}) > 0$$

for each $t > 0$. Therefore the inequality (11) is true.

From the assumption that the polynomials $L_m(s)$ and $M_n(s)$ $(m = 0,1,2,\dots,n-1)$ partly interlace it follows that

$$\lim_{t\to\infty} x(t,L_m,M_n) = 0\,.$$

From (7) and the work of Osiowski (1965) we have $x(0,L_m,M_n) = 0$ $(m = 0,1,\dots,n-2)$ and Theorem 3 implies that $x(t,L_m,M_n) > 0$, $(m = 0,1,\dots,n-1)$, for each $t > 0$. Therefore, the following holds true. ■

Remark 2. If for $m \in \{0,1,2,\dots,n-2\}$ the polynomials $L_m(s)$ and $M_n(s)$ partly interlace, then the function $x(t,L_m,M_n)$ has a local maximum in the interval $(0,\infty)$.

Now we will prove that this local maximum is unique in the interval $(0,\infty)$.

Theorem 4. *If* $\deg M_n(s) = n \geq 2$, $\deg L_m(s) = m \in \{0,1,2,\dots,n-2)$ *and the polynomials* $L_m(s)$ *and* $M_n(s)$ *partly interlace, then the function* $x(t,L_m,M_n)$*, determined by* (2)*, has only one extremum in* $(0,\infty)$*—a local maximum. Moreover this function for* $m = n-1$ *has no local extremum in the interval* $(0,\infty)$.

Proof. From Theorem 1 it follows that the function $x(t,L_{m-1},M_n)$ has no local extrema in $(0,\infty)$. From Remark 2 we have that for $m \in \{0,1,2,\dots,n-2)$ the function $x(t,L_m,M_n)$ has a local extremum, namely, a local maximum in $(0,\infty)$. We will prove that this extremum is unique. The proof will be by induction on n.

For $n=2$ and $m \in \{0,1,2,\dots,n-2\}$ we have $m=0$. For $n=2$ and $m=0$ the conclusion of Theorem 4 follows from Remark 1. We assume that the thesis of Theorem 4 is true for $n-1$, and we prove that it is true for n.

We see that for $m \in \{0,1,\dots,(n-1)-2\} = \{0,1,\dots,n-3\}$ and for $j \in \{n-2,n-1\}$ $\deg M_{n(j)}(s) = n-1$ the polynomials $L_m(s),\ M_{n(j)}(s)$ partly interlace. Hence and from the induction assumption it follows that for $m \in \{0,1,\dots,n-3\}$, $j \in \{n-2,n-1\}$ the function $x(t,L_m,M_{n(j)})$ has one local extremum in $(0,\infty)$ only. We will prove the thesis of Theorem 4 by *reductio ad absurdum*. We assume that $\deg M_n(s) = n > 2$, $m \in \{0,1,\dots,n-2\}$ and the function $x(t,L_m,M_n)$ has two local extrema in $(0,\infty)$; this means that there are numbers $t_1,t_2 \in (0,\infty)$, $t_1 < t_2$ and such that

$$\left.\begin{aligned} x'(t_1,L_m,M_n) = 0, \quad x''(t_1,L_m,M_n) \neq 0,\\ x'(t_2,L_m,M_n) = 0, \quad x''(t_2,L_m,M_n) \neq 0.\end{aligned}\right\} \tag{12}$$

Without loss of generality we can assume that in the intervals $(0,t_1)$, (t_1,t_2) this function has no local extremum. From the work of Osiowski (1965) and Theorem 3 we have

$$x(0,L_m,M_n) = 0, \quad x(t,L_m,M_n) > 0 \ \text{ for } t > 0.$$

Hence

$$x''(t_1,L_m,M_n) < 0 \ \text{ and } \ x''(t_2,L_m,M_n) > 0.$$

From the relations

$$x'(t_2,L_m,M_n) = 0, \quad x''(t_2,L_m,M_n) > 0,$$

$$\lim_{t\to\infty} x(t,L_m,M_n) = 0\,,$$

it follows that there exists a number $t_3 > t_2$ such that

$$x'(t_3,L_m,M_n) = 0, \quad x''(t_3,L_m,M_n) < 0;$$

this means that the function $x(t,L_m,M_n)$ has local extrema at three points $t_1 < t_2 < t_3$. From Lemma 1 we have

$$x''(t,L_m,M_n) = s_j x'(t,L_m,M_n) + x'(t,L_m,M_{n(j)})$$

for $j \in \{n-2,n-1\}$. Hence from (12) we obtain

$$x''(t_1,L_m,M_n) = x'(t_1,L_m,M_{n(j)}) < 0,$$

$$x''(t_2,L_m,M_n) = x'(t_2,L_m,M_{n(j)}) > 0,$$

$$x''(t_3,L_m,M_n) = x'(t_3,L_m,M_{n(j)}) < 0,$$

for $j \in \{n-2,n-1\}$. It follows that there are numbers $\overline{t_1} \in (t_1,t_2)$ and $\overline{t_2} \in (t_2,t_3)$ such that

$$x'(\overline{t_1},L_m,M_{n(j)}) = 0, \quad x'(\overline{t_2},L_m,M_{n(j)}) = 0$$

for $j \in \{n-2,n-1\}$, and $x'(t,L_m,M_{n(j)})$ at the points $\overline{t_1},\overline{t_2}$ changes its sign. This means that the function $x(t,L_m,M_{n(j)})$ has two local extrema in the interval $(0,\infty)$. But this contradicts the induction hypothesis. Therefore Theorem 4 holds true. ■

5. Comparison of the extremal values of the function $x(t, L_m, M_n)$ with respect to $\deg M_n(s)$ and $\deg L_m(s)$

Consider the polynomials $M_n(s)$, $L_m(s)$, given by (1), where $\deg M_n(s) = n \geq 3$, $\deg L_m(s) = m \in \{0, 1, \dots, n-3\}$. Specifically, consider the polynomials

$$\left.\begin{array}{c} M_1(s) = (s - s_1), \\ M_i(s) = M_{i-1}(s)(s - s_i), \\ (i = 2, 3, \dots, n) \end{array}\right\} \quad (13)$$

and the function $x(t, L_m, M_i)$ $(i = m+1, m+2, \dots, n)$, determined by (2). We give the extremal properties of the function $x(t, L_m, M_i)$ $(i = m+2, m+3, \dots, n)$.

Theorem 5. *Let the polynomials $M_n(s)$, $L_m(s)$ have the form* (1) *and the polynomial $M_{n-1}(s)$ have the form* (13)*, where $\deg M_n(s) = n \geq 3$, $\deg L_m(s) = m \in \{0, 1, \dots, n-3\}$. If the polynomials $L_m(s)$ and $M_n(s)$ partly interlace, then there exist only two numbers $t_{n-1}, t_n \in (0, \infty)$, such that*

$$\left.\begin{array}{c} \max\limits_{t>0} x(t, L_m, M_n) = x(t_n, L_m, M_n), \\ \max\limits_{t>0} x(t, L_m, M_{n-1}) = x(t_{n-1}, L_m, M_{n-1}) \end{array}\right\} \quad (14)$$

and

$$\left.\begin{array}{c} t_{n-1} < t_n, \\ x(t_n, L_m, M_n) < \dfrac{1}{|s_n|} x(t_{n-1}, L_m, M_{n-1}). \end{array}\right\} \quad (15)$$

This means that

$$\max_{t>0} x(t, L_m, M_n) < \frac{1}{|s_n|} \max_{t>0} x(t, L_m, M_{n-1}).$$

Proof. It is evident that, if the polynomials $L_m(s)$ and $M_n(s)$ partly interlace, then so do $L_m(s)$ and $M_{n-1}(s)$. Hence, by Theorem 1, the function $x(t, L_m, M_{n+1})$ in the interval $(0, \infty)$ has no local extrema. From Theorem 4 it follows that the function $x(t, L_m, M_i)$ $(i = m+2, m+3, \dots, n)$ in the interval $(0, \infty)$ has precisely one local extremum, which means that there are precisely two numbers $t_{n-1}, t_n \in (0, \infty)$, which satisfy the relation (14). Moreover,

$$\begin{aligned} x(0, L_m, M_n) &= \lim_{t\to\infty} x(t, L_m, M_n) \\ &= \lim_{t\to\infty} x(t, L_m, M_{n-1}) \\ &= x(0, L_m, M_{n-1}) = 0 \end{aligned}$$

and, by Theorem 3, we have

$$x(t, L_m, M_n) > 0, \quad x(t, L_m, M_{n-1}) > 0$$

for $t > 0$.

Hence and from (14) we get

$$\left.\begin{array}{rll} x'(t, L_m, M_n) > 0 & \text{for} & 0 < t < t_n, \\ x'(t, L_m, M_n) = 0 & \text{for} & t = t_n, \\ x'(t, L_m, M_n) < 0 & \text{for} & t > t_n, \\ x'(t, L_m, M_{n-1}) > 0 & \text{for} & 0 < t < t_{n-1}, \\ x'(t, L_m, M_{n-1}) = 0 & \text{for} & t = t_{n-1}, \\ x'(t, L_m, M_{n-1}) < 0 & \text{for} & t > t_{n-1}. \end{array}\right\} \quad (16)$$

From Lemma 1 we have

$$\begin{aligned} x''(t, L_m, M_n) = s_n x'(t, L_m, M_n) \\ + x'(t, L_m, M_{n-1}), \end{aligned}$$

$$\begin{aligned} x''(t_n, L_m, M_n) = s_n x'(t_n, L_m, M_n) \\ + x'(t_n, L_m, M_{n-1}), \end{aligned}$$

and from (16) it follows that

$$\begin{gathered} x'(t_n, L_m, M_n) = 0, \\ x''(t_n, L_m, M_{n-1}) < 0. \end{gathered}$$

Hence $x'(t_n, L_m, M_{n-1}) < 0$ and $x'(t, L_m, M_{n-1}) < 0$ only for $t > t_{n-1}$. Therefore $t_n > t_{n-1}$.

Now we prove (15). From Lemma 1 we obtain

$$\begin{aligned} x'(t_n, L_m, M_n) = s_n x(t_n, L_m, M_n) \\ + x(t_n, L_m, M_{n-1}), \end{aligned}$$

and from (16) it follows that

$$x'(t_n, L_m, M_n) = 0.$$

This means that

$$x(t_n, L_m, M_n) = \frac{1}{|s_n|} x(t_n, L_m, M_{n-1}).$$

From this and the inequality $t_{n-1} < t_n$ we have

$$x(t_n, L_m, M_n) < \frac{1}{|s_n|} x(t_{n-1}, L_m, M_{n-1}),$$

and this means that the inequalities (15) are true. ■

Let the polynomials $M_n(s)$, $L_m(s)$ have the form (1) and the polynomial $M_{n-1}(s)$ have the form (13). Consider the polynomial $\overline{M}_n(s) = M_{n-1}(s)(s - \overline{s})$, with $\overline{s}$ satisfying the inequality $\overline{s} < s_n$. It can be seen that if the polynomials $L_m(s)$ and $M_n(s)$ satisfy the assumptions of Theorem 5, then the polynomials $L_m(s)$ and $\overline{M}_n(s)$ satisfy these assumptions, too.

From Theorem 5 we have the following.

Corollary 1. *If the polynomials $L_m(s)$ and $M_n(s)$ have the form* (1) *and satisfy the assumptions of Theorem 5, then for every number $\overline{s} < s_n$*

$$\max_{t>0} x\left(t, L_m, \overline{M}_n\right) < \frac{1}{|\overline{s}|} \max_{t>0} x(t, L_m, M_{n-1})$$

$$\lim_{s_n \to -\infty} \left[\max_{t>0} x(t, L_m, M_n)\right] = 0\ .$$

Consider the polynomials $M_2(s)$, $M_3(s), \ldots, M_n(s)$ of the form (13)*, the polynomial $L_0(s) = 1$ and the functions $x(t, L_0, M_i)$ $(i = 2, 3, \ldots, n)$ of the form* (2)*. It is easy to see that, if the polynomials $L_0(s)$ and $M_n(s)$ satisfy the assumptions of Theorem 5, then the polynomials $L_0(s)$ and $M_i(s)$ $(i = 3, 4, \ldots, n)$ satisfy these assumptions, too. Hence, by Theorem 5, it follows that there exist numbers $t_i \in (0, \infty)$ $(i = 2, 3, \ldots, n)$ such that*

$$\max_{t>0} x(t, L_0, M_i) = x(t_i, L_0, M_i) \qquad (i = 2, 3, \ldots, n). \tag{17}$$

Hence, by Theorem 5 and Remark 1, we have the following result:

Corollary 2. *If the polynomials $L_0(s)$ and $M_n(s)$ have the form* (1) *and satisfy the assumptions of Theorem 5, and the numbers t_i $(i = 2, 3, \ldots, n)$ are defined by the relation* (17)*, then*

$$t_2 < t_3 < \cdots < t_n,$$

where $t_2 = \ln(s_2/s_1)/(s_1 - s_2)$ is determined in Remark 1.

Theorem 6. *Let the polynomials $M_n(s)$, $L_m(s)$,*

$$L_{m-1}(s) = \frac{L_m(s)}{s - z_m}$$

have the form (1)*, where $\deg M_n(s) = n \geq 2$, $\deg L_m(s) = m \in \{1, 2, \ldots, n-2\}$. If the polynomials $L_m(s)$ and $M_n(s)$ partly interlace, then there exist precisely two numbers $T_{m-1}, T_m \in (0, \infty)$ such that*

$$\max_{t>0} x(t, L_m, M_n) = x(T_m, L_m, M_n)\ ,$$

$$\max_{t>0} x(t, L_{m-1}, M_n) = x(T_{m-1}, L_m, M_n)\ ,$$

and

$$T_m < T_{m-1},$$

$$x(T_{m-1}, L_{m-1}, M_n) < \frac{1}{|z_m|} x(T_m, L_m, M_n),$$

which means that

$$\max_{t>0} x(t, L_{m-1}, M_n) < \frac{1}{|z_m|} \max_{t>0} x(t, L_m, M_n).$$

Proof. The proof is analogous to that of Theorem 5, when using the relation (5) instead of (4) from Lemma 1. ■

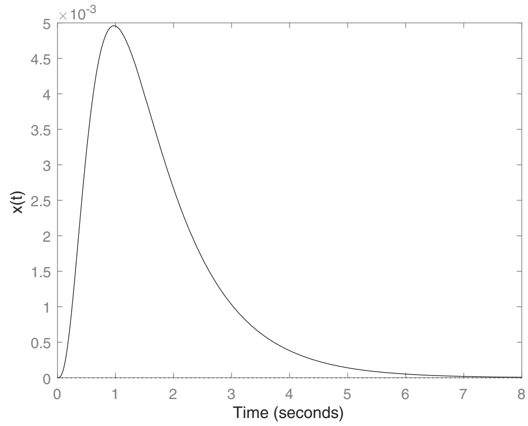

Fig. 1. Time response of the system for $s_1 = -1$, $s_2 = -3$, $s_3 = -5$, $s_4 = -7$.

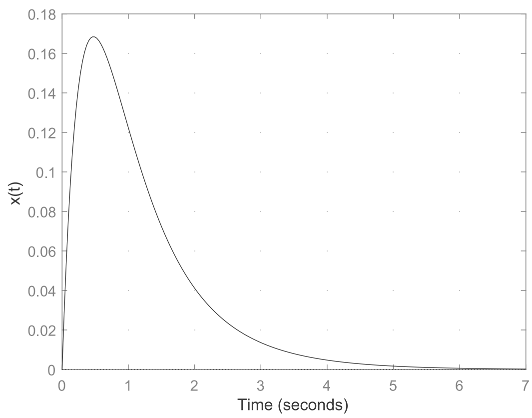

Fig. 2. Time response of the system for $s_1 = -1$, $s_2 = -2$, $s_3 = -3$, $z_1 = -1.5$.

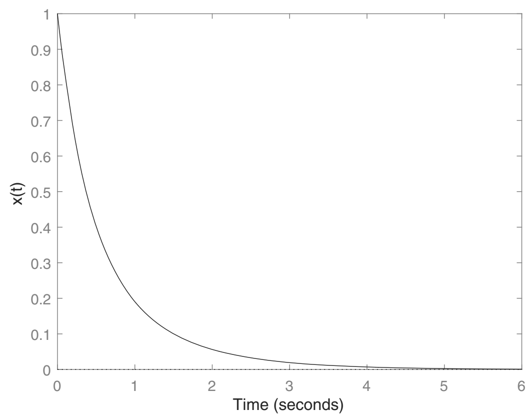

Fig. 3. Time response of the system for $s_1 = -1$, $s_2 = -2$, $s_3 = -3$, $z_1 = -1.5$, $z_2 = -2.5$.

6. Numerical examples

In Fig. 1 the time response of the system is shown for $s_1 = -1,\ s_2 = -3,\ s_3 = -5,\ s_4 = -7$. The numerical solution yields $\tau_e = (0, 0, 0.973)$ and $x_e = 0.0049587176$.

In Fig. 2 the time response of the system is shown for $s_1 = -1,\ s_2 = -2,\ s_3 = -3,\ z_1 = -1.5$. The numerical solution yields $\tau_e = 0.473$ and $x_e = 0.16846125$.

In Fig. 3 the time response of the system is shown for $s_1 = -1,\ s_2 = -2,\ s_3 = -3,\ z_1 = -1.5,\ z_2 = -2.5$. No extremum exists.

7. Conclusion

In the paper, certain properties of solutions to differential equations of the form

$$\begin{aligned} &a_n x^{(n)}(t) + \cdots + a_1 x'(t) + a_0 x(t) \\ &= b_m u^{(m)}(t) + \cdots + b_1 u'(t) + b_0 u(t) \end{aligned} \tag{18}$$

were proved, under the assumption that $a_i, b_j > 0$. These equations are mathematical models of linear dynamical systems excited by Dirac's impulse $\delta(t)$. The solution of Eqn. (18) is denoted by $x(t, L_m, M_n)$, where the polynomials $L_m(s)$ and $M_n(s)$ have the form (1), $\deg\ L_m(s) = m$ and $\deg\ M_n(s) = n$. Throughout the paper, it was assumed that the polynomials $M_n(s)$ and $L_m(s)$ partly interlace. It was proved that, if $\deg M_n(s) = n \geq 2$ and $\deg L_m(s) = m = n - 1$, then the function $x(t, L_m, M_n)$ is positive and decreasing for every $t \geq 0$ (Theorem 1). In the case where $\deg M_n(s) = n \geq 2$ and $\deg L_m(s) = m \in \{0, 1, \ldots, n-1\}$, we have $x(t, L_m, M_n) > 0$ for every $t > 0$ (Theorem 3). Besides, it was proved that, in the case where $\deg M_n(s) = n \geq 2$ and $\deg\ L_m(s) = m \in \{0, 1, \ldots, n-2\}$, the function considered has exactly one local extremum in the interval $(0, \infty)$ (Theorem 4). In Theorem 5, certain relationships between the extrema of the function $x(t, L_m, M_n)$ were shown, depending on the degree of the polynomial $M_n(s)$, and in Theorem 6, analogous relationships were proved depending on the degree of the polynomial $L_m(s)$.

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