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# TEMPORAL CENTRAL LIMIT THEOREM FOR A MULTIDIMENSIONAL ADDING MACHINE

Mordechay B. Levin

Bar-Ilan University, Ramat-Gan, ISRAEL

ABSTRACT. Let $p_1, \dots, p_{s+1}$ be distinct primes and let $T_{p_i}$ be the von Neumann--Kakutani adding machine $(1 \le i \le s)$, $T_{\mathcal{P}}(\mathbf{x}) = \big(T_{p_1}(x_1), \dots, T_{p_s}(x_s)\big)$. Let $y_i \in (0,1)$ be a $p_{s+1}$-rational $(1 \le i \le s)$, $\mathbb{1}_{[0,\mathbf{y})}$ the indicator function of the box $[0,y_1) \times \dots \times [0,y_s)$. In this paper, we prove the following central limit theorem:

$$\frac{\sum_{k=-n}^{n-1} \mathbb{1}_{[0,\mathbf{y})}(T_P^k(\mathbf{x})) - 2ny_1y_2\dots y_s}{\mathcal{H}_N(\mathbf{x}) \log_2^{s/2} N} \xrightarrow{w} \mathcal{N}(0,1),$$

when $n$ is sampled uniformly from $\{1, \dots, N\}$, $\mathcal{H}_N(\mathbf{x}) \in [\upsilon_1, \upsilon_2]$ with some $\upsilon_1, \upsilon_2 > 0$, for almost all $\mathbf{x} \in [0,1)^s$. The main tool in the proof is the $S$-unit theorem and the theorem on linear forms in the logarithm.

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Contents




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# 1. Introduction

Let $\mathcal{P}_N = (\beta_n)_{n=0}^{N-1}$ be an $N$-element point set in the $s$-dimensional unit cube $[0,1)^s$. The local **discrepancy** function of $\mathcal{P}_N$ is defined as

$$D(\mathbf{x},\mathcal{P}_N) = \sum\nolimits_{k=0}^{N-1} \mathbf{1}_{B_{\mathbf{x}}}(\beta_n) - N x_1 \cdots x_s, \tag{1}$$

where $\mathbf{1}_{B_{\mathbf{x}}}(\mathbf{y}) = 1$, if $\mathbf{y} \in B_{\mathbf{x}}$, and $\mathbf{1}_{B_{\mathbf{x}}}(\mathbf{y}) = 0$, if $\mathbf{y} \notin B_{\mathbf{x}}$ with $B_{\mathbf{x}} = [0,x_1) \times \cdots \times [0,x_s)$, $\mathbf{x} = (x_1,\dots,x_s)$. We define the $L_q$ discrepancy of $\mathcal{P}_N$ as

$$D_{s,\infty}(\mathcal{P}_N) = \sup_{0<x_1,\dots,\,x_s\le 1} |D(\mathbf{x},\mathcal{P}_N)|, \qquad D_{s,q}(\mathcal{P}_N) = \|D(\mathbf{x},\mathcal{P}_N)\|_{s,q}\,,$$

$$\|f(\mathbf{x})\|_{s,q} = \Big(\int_{[0,1]^s} |f(\mathbf{x})|^q d\mathbf{x}\Big)^{1/q}.$$

A sequence $(\beta_k)_{k\ge 0}$ is of *low discrepancy* (abbr. l.d.s.) if $D_{s,\infty}\big((\beta_k)_{k=0}^{N-1}\big) = O(\log^s N)$ for $N \to \infty$. An ergodic transformation $T$ of $s$-dimensional torus is of low discrepancy in the orbit of $\mathbf{x}$ if $\big(T^k(\mathbf{x})\big)_{k\ge 0}$ is a l.d.s.

Let $p_1,\dots,p_s$ be distinct primes

$$k = \sum_{j\ge 1} e_{j,i}(k) p_i^{j-1},\ e_{j,i}(k) \in \{0,1,\dots,p_i-1\}, \quad \text{and} \quad \phi_i(k) = \sum_{j\ge 1} e_{j,i}(k) p_i^{-j}.$$

Van der Corput proved that $\big(\phi_1(k)\big)_{k\ge 0}$ is a 1-dimensional l.d.s. The first example of multidimensional l.d.s. was proposed by Halton :

$$H_s(k) = \big(\phi_1(k),\dots,\phi_s(k)\big), \quad k = 0,1,2,\dots$$

For other examples of l.d.s., see, e.g., [6], [22].

Let $q \ge 2$ be an integer, and let $v = .v_1 v_2 \dots v_j \cdots = \sum_{j=1}^{\infty} v_j/q^j$ be the $q$--expansion of $v$. We define von Neumann-Kakutani's $q$-adic adding machine:

$$T_q(v) := (v_k+1)/q^k + \sum\nolimits_{j\ge k+1} v_j/q^j, \qquad T_q^n(v) := T_q\big(T_q^{n-1}(v)\big), \tag{2}$$

$n = 2,3,\dots, T_q^0(v) = v$, where $k = \min\{j \mid v_j \ne q-1\}$.

Let $\mathcal{P} = (p_1,\dots,p_s)$ and let $T_{\mathcal{P}}(\mathbf{x}) = \big(T_{p_1}(x_1),\dots,T_{p_s}(x_s)\big)$. As is known, the sequence $\big(T_{p_i}^k(x)\big)_{n\ge 1}$ coincides for $x = 0$ with the van der Corput sequence in base $p_i$ $(1 \le i \le s)$ (see, e.g., [10, § 2.5]). Hence $T_{\mathcal{P}}^k(0) = H_s(k)$.

It is easy to show that $\big(T^k_{\mathcal{P}}(\mathbf{x})\big)_{k\ge1}$ is the low discrepancy ergodic transformation for all $\mathbf{x}$. Let

$$\mathcal{T}_q(0) := 0, \quad \mathcal{T}_q(v) := \sum_{j=1}^{k-1} \frac{q-1}{q^j} + \frac{v_k-1}{q^k} + \sum_{j\ge k+1} \frac{v_j}{q^j}, \quad \mathcal{T}^n_q(v) := \mathcal{T}_q\big(\mathcal{T}^{n-1}_q(v)\big), \tag{3}$$

$n = 2,3,\dots,\mathcal{T}^0_q(v) = v$, where $k = \min\{j \mid v_j \ne 0\}$.

It is easy to see that $\mathcal{T}^n_q\big(T^n_q(v)\big) = T^n_q\big(\mathcal{T}^n_q(v)\big) = v$ for $\#\{i\ge1|v_i\ne0\} = \infty$ and $\#\{i\ge1|v_i\ne q-1\} = \infty$, $n\in\mathbb{Z}$. Let $T^{-1}_q(v) := \mathcal{T}_q(v)$. Then

$$T^{-n}_q\big(T^n_q(v)\big) = v \quad \text{for all} \quad n\in\mathbb{Z} \qquad \text{with} \quad \#\{i\ge1 \mid v_i\ne0\} = \infty \tag{4}$$

and $\#\{i\ge1 \mid v_i\ne q-1\} = \infty$.

We will consider the probability space $\mathbb{P}_s = \big([0,1)^s, B([0,1)^s), \lambda_s\big)$, where $\lambda_s$ is the $s$-dimensional Lebesgue measure on $[0,1)^s$.

Let $T:[0,1)^s\to[0,1)^s$ be a map, $f:[0,1)^s\to\mathbb{R}$ is a function, and $\mathbf{x}_0\in[0,1)^s$ is a fixed initial condition. We say that the ergodic sums $S_k(f,\mathbf{x}_0) = f(\mathbf{x}_0) + f\big(T(\mathbf{x}_0)\big) + \dots + f\big(T^{k-1}(\mathbf{x}_0)\big)$ satisfy a *temporal central limit theorem* (TCLT) *on the orbit of* $\mathbf{x}_0$, if there exists constants $A_N$ and $B_N$ such that

$$(S_k - A_N)/B_N \overset{w}{\to} \mathcal{N}(0,1),$$

when $k$ is sampled uniformly from $\{1,\dots,N\}$.

The ergodic sum $S_k(f,\mathbf{x})$ is said to satisfy an almost sure TCLT on $\mathbb{P}_s$, if for $\lambda_s$-a.e. $\mathbf{x}$, $S_k(f,\mathbf{x})$ satisfy the TCLT. In [12], Dolgopyat and Sarig considered a more general notion of the temporal distributional limit theorem (see also [2], [7], [9], [13], [14], [11], [23]).

The first example of the TCLT was discovered by Beck (see [3–5]). Beck considered the transformation $x\to x+\sqrt{2} \bmod 1$ and the indicator function of an interval $[0,y)$. According to [4, p. 41], the generalizations of this results to the multidimensional case for a Kronecker's lattice

$$\{(n, n\alpha_1+m_1,\dots,n\alpha_{s-1}+m_{s-1}) \mid (n,m_1,\dots,m_{s-1})\in\mathbb{Z}^s\}$$

is very difficult because of the problems connected to Littlewood's conjecture:

$$\liminf_{n\to\infty} \; n\,\|n\alpha\| \; \|n\beta\| = 0$$

for all reals $\alpha,\beta$, where $\|x\| = \min(\{x\}, 1-\{x\})$.

In [20], we proved an $s+1$-parametric spatio-temporal CLT (see definition in [12, p. 12]) for an $s$-dimensional ergodic adding machine (Halton's sequence) with indicator functions. Similar results for another low discrepancy ergodic transformations were obtained in [19] (see also [21]). For another generalization of TCLT from [21] for an ergodic adding machine, see article [1] by Ya. Sinai et al. Note that we have proven spatio-temporal CLT for the one-dimensional case in [21] by direct computation. While for the $s$-dimensional case, we were forced to apply a very deep results of number theory, namely the $S$-unit theorem and the theorem on linear forms in the logarithm (see [19], [20] and this paper). The advantage of this paper is that we obtain only a 1-parameter TCLT for an $s$-dimensional ergodic adding machine instead of an $s+1$-parameter one as in [20].

Let $y_i = \sum_{j\geq 1} y_{i,j}/p_i^j$ be the $p_i$-expansion of $y_i$ $(i = 1,\dots,s)$. We consider the following condition: there are $\kappa_1 > 0$, $\kappa_2 > 0$ such that

$$\liminf_{N\to\infty} \min_{1\leq i\leq s} \#\{j \in [1,N] \mid 1 \leq y_{i,j} \ \text{ and } \ \{y_i p_i^j\} \leq 1-\kappa_1\}/N = \kappa_2 > 0. \tag{5}$$

For example, (5) is true for $q$-rational reels $y_i > 0$, where $(q,p_i) = 1$ $(1 \leq i \leq s)$. It is easy to see that (5) is also true for a.e. $\mathbf{y} = (y_1,\dots,y_s) \in [0,1)^s$. Let

$$\mathscr{D}_{\mathbf{y},\mathbf{x}}(L) = D\Big(\mathbf{y}, \big(T^k_{\mathcal{P}}(\mathbf{x})\big)_{k=-L}^{L-1}\Big),$$

$$\dot{\mathcal{H}}_N(\mathbf{x}) = \int_{[0,1]} \mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])d\theta, \qquad \ddot{\mathcal{H}}_N(\mathbf{x}) = \Big(\int_{[0,1]} \mathscr{D}^2_{\mathbf{y},\mathbf{x}}([\theta N])d\theta\Big)^{1/2}. \tag{6}$$

In this paper, we prove the existence of an almost sure one-parameter TCLT for the discrepancy function in the multidimensional case:

**Theorem 1.1.** *Let $s \geq 1$, $\theta$ be a uniformly distributed random variable in $[0,1]$, $\mathbf{y}$ satisfy (5). Then*

$$\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])/\ddot{\mathcal{H}}_N(\mathbf{x}) \xrightarrow{w} \mathcal{N}(0,1), \qquad \dot{\mathcal{H}}_N(\mathbf{x})/\ddot{\mathcal{H}}_N(\mathbf{x}) \xrightarrow{N\to\infty} 0,$$

*where $(\log_2 N)^{-s/2}\ddot{\mathcal{H}}_N(\mathbf{x}) \in [v_1, v_2]$ with some $v_2 > v_1 > 0$ and for $N \geq N_0(\mathbf{x})$ with some $N_0(\mathbf{x})$* for *a.s.* $\mathbf{x} \in [0,1)^s$.

Now, we describe the structure of the paper. In Lemma 2.1, we get a simple estimate of Fourier series of the discrepancy function $\mathscr{D}_{\mathbf{y},\mathbf{x}}(L)$. In Lemma 2.2 and Lemma 2.4, we minimise the number of terms in the expression of $\mathscr{D}_{\mathbf{y},\mathbf{x}}(L)$. The main tool of the proof is the $S$-unit theorem and the theorem on linear forms in logarithm (see Section 3). In Section 4 and Section 5, we compute the upper and the lower bounds of the variance of $\mathscr{D}_{\mathbf{y},\mathbf{x}}(L)$. In Section 6, we calculate the four moment and Lévy's conditional variance of $\mathscr{D}_{\mathbf{y},\mathbf{x}}(L)$. In order to prove the Theorem 1.1, we use the martingale CLT (see Theorem 7.2). We construct a martingale approximation to $\mathscr{D}_{\mathbf{y},\mathbf{x}}(L)$ in Section 7. In order to apply Theorem 7.2, in Lemma 6.1, we compute the sum of four moment of parts of $\mathscr{D}_{\mathbf{y},\mathbf{x}}(L)$ and in Lemma 6.2, we find Lévy's conditional variance of $\mathscr{D}_{\mathbf{y},\mathbf{x}}(L)$. On the whole, the proof of the Theorem 1.1 follows the lines of [19].

## 2. Discrepancy estimates

### 2.1. Preliminary

First of all, we need Fourier's series decomposition of the discrepancy function. The argument is pretty standard (see, e.g., [22, pp. 29–35]). We closely follow [20, pp. 4–7].

We will use the notation $A \ll B$ equal to $A = O(B)$. Let

$$\Delta(\mathfrak{T}) = \begin{cases} 1, & \text{if } \mathfrak{T} \text{ is true,} \\ 0, & \text{otherwise,} \end{cases} \qquad \delta_M(a) = \begin{cases} 1, & \text{if } a \equiv 0 \bmod M, \\ 0, & \text{otherwise.} \end{cases}$$

Let $[y]$ be the integer part of $y$,

$$I_M = \big[-[(M-1)/2], [M/2]\big] \cap \mathbb{Z}, \qquad I_M^* = I_M \setminus \{0\}. \tag{7}$$

Note that the integers of the interval $I_M$ are a complete set of residues $\bmod M$, $M \geq 1$. By [18, Lemma 2, p. 2], we have

$$\delta_M(a) = \frac{1}{M} \sum_{k \in I_M} e\left(\frac{ak}{M}\right), \qquad \text{where} \quad e(x) = \exp(2\pi i x). \tag{8}$$

By [18, Lemma 1, p. 1], we get

$$\left| \frac{1}{M} \sum_{k=0}^{M-1} e(k\alpha) \right| \leq \min\left(1, \frac{1}{2M\langle\!\langle \alpha \rangle\!\rangle}\right), \quad \text{where} \quad \langle\!\langle \alpha \rangle\!\rangle = \min(\{\alpha\}, 1 - \{\alpha\}). \tag{9}$$

Let $m$ be an integer, $\bar{m} = \max(1, |m|) \leq M/2$. From [18, p.2], we obtain for $R \leq M$:

$$\left| \frac{1}{M} \sum_{k=0}^{R-1} e\left(\frac{mk}{M}\right) \right| \leq \min\left(1, \left| \frac{e(mR/M) - 1}{M(e(m/M) - 1)} \right| \right) \leq \frac{|\sin(\pi m R/M)|}{\bar{m}} \leq \frac{1}{\bar{m}}. \tag{10}$$

Let $x_i = 0.x_{i,1}x_{i,2}\ldots = \sum_{j\geq 1} x_{i,j} p_i^{-j}$, with $x_{i,j} \in \{0, 1, \ldots, p_i - 1\}$, $i = 1, \ldots, s$. We define the truncation

$$[x_i]_r = \sum_{1 \leq j \leq r} x_{i,j} p_i^{-j} \quad \text{with} \quad r \geq 1.$$

If $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$, then the truncation $[\mathbf{x}]_{\mathbf{r}}$ is defined coordinatewise, that is, $[\mathbf{x}]_{\mathbf{r}} = ([x_1]_{r_1}, \ldots, [x_s]_{r_s})$, where $\mathbf{r} = (r_1, \ldots, r_s)$.

Let

$$\mathcal{X}_{i,r}(x_i) = \sum_{1 \leq j \leq r} x_{i,j} p_i^{j-1} \in [0, p_i^{r_i}). \tag{11}$$

It is easy to see that

$$[u_i]_r = [z_i]_r \iff \mathcal{X}_{i,r}(u_i) \equiv \mathcal{X}_{i,r}(z_i) \bmod p_i^r \quad \text{for} \quad u_i, z_i \in [0,1). \tag{12}$$

By (2), we have

$$\mathcal{X}_{i,r}\big(T_{p_i}(x_i)\big) \equiv \mathcal{X}_{i,r}(x_i) + 1 \bmod p_i^r.$$

Hence

$$\mathcal{X}_{i,r}\big(T_{p_i}^k(x_i)\big) \equiv \mathcal{X}_{i,r}(x_i) + k \bmod p_i^r, \qquad k = 0,1,\dots \tag{13}$$

In view of (3), we have that $T_{p_i}^{-k}\big(T_{p_i}^k(x_i)\big) = T_{p_i}^k\big(T_{p_i}^{-k}(x_i)\big) = x_i$ for all $k \in \mathbb{Z}$ for a.e. $x_i \in [0,1)$. Therefore,

$$\mathcal{X}_{i,r}(x_i) = \mathcal{X}_{i,r}\Big(T_{p_i}^k\big(T_{p_i}^{-k}(x_i)\big)\Big) \equiv \mathcal{X}_{i,r}\big(T_{p_i}^{-k}(x_i)\big) + k \bmod p_i^r,$$

and

$$\mathcal{X}_{i,r}\big(T_{p_i}^{-k}(x_i)\big) \equiv \mathcal{X}_{i,r}(x_i) - k \bmod p_i^r, \qquad k = 0,1,\dots$$

We have that (13) is also true for $k = -1,-2,\dots$ for a.e. $x_i \in [0,1)$. According to (12), we get

$$[T_{p_i}^k(x_i)]_r = [y_i]_r \iff \mathcal{X}_{i,r}\big(T_{p_i}^k(x_i)\big) \equiv \mathcal{X}_{i,r}(y_i) \pmod{p_i^r}.$$

In view of (13), we obtain

$$[T_{p_i}^k(x_i)]_r = [y_i]_r \iff k \equiv \mathcal{X}_{i,r}(y_i) - \mathcal{X}_{i,r}(x_i) \pmod{p_i^r}, \quad 1 \le i \le s \tag{14}$$

for all $k \in \mathbb{Z}$ for a.e. $x_i \in [0,1)$.

Let $p_0 = p_1 p_2 \cdots p_s$, $P_{\mathbf{r}} = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ and let $M_{i,\mathbf{r}}$ be the unique integer satisfying the two conditions

$$M_{i,\mathbf{r}} \equiv \big(P_{\mathbf{r}}/p_i^{r_i}\big)^{-1} \pmod{p_i^{r_i}}, \quad M_{i,\mathbf{r}} \in [0, p_i^{r_i}), \qquad 1 \le i \le s. \tag{15}$$

We define $\mathcal{X}_{\mathbf{r},\mathbf{x}} \in [0, P_{\mathbf{r}})$ as follows

$$\mathcal{X}_{\mathbf{r},\mathbf{x}} \equiv \sum_{i=1}^{s} M_{i,\mathbf{r}} P_{\mathbf{r}} p_i^{-r_i} \mathcal{X}_{i,r_i}(x_i) \equiv \sum_{i=1}^{s} M_{i,\mathbf{r}} P_{\mathbf{r}} p_i^{-r_i} \sum_{1 \le j \le r_i} x_{i,j} p_i^{j-1} \bmod P_{\mathbf{r}}. \tag{16}$$

Applying (14) and the Chinese Remainder Theorem, we get

$$\begin{aligned} &[T_{\mathcal{P}}^k(\mathbf{x})]_{\mathbf{r}} = [\mathbf{y}]_{\mathbf{r}} \iff [T_{p_i}^k(x_i)]_{r_i} = [y_i]_{r_i}, \\ &1 \le i \le s \iff k \equiv \mathcal{X}_{i,r}(y_i) - \mathcal{X}_{i,r}(x_i) \pmod{p_i^r}, \\ &1 \le i \le s \iff k \equiv \mathcal{X}_{\mathbf{r},\mathbf{y}} - \mathcal{X}_{\mathbf{r},\mathbf{x}} \pmod{P_{\mathbf{r}}} \end{aligned} \tag{17}$$

for all $k \in \mathbb{Z}$ for a.e. $\mathbf{x} \in [0,1)^s$.

Let

$$n = [\log_2 N] + 1, \quad \theta \in [0,1), \quad L = [\theta N].$$

From [22, pp. 29, 30] and (6), we get

$$\mathcal{D}(L) := \mathscr{D}_{[\mathbf{y}]_n,\mathbf{x}}(L) = \big(\mathscr{D}_{\mathbf{y},\mathbf{x}}(L)\big) + \epsilon s, \quad |\epsilon| \le 1. \tag{18}$$

Let

$$\mathcal{X}_{\mathbf{r},\mathbf{y},\mathbf{b}} \equiv \sum_{i=1}^{s} M_{i,\mathbf{r}} P_{\mathbf{r}} p_i^{-r_i} y_{i,r_i,b_i} \bmod P_{\mathbf{r}}, \quad \mathcal{X}_{\mathbf{r},\mathbf{y},\mathbf{b}} \in [0, P_{\mathbf{r}}),$$

with

$$y_{i,r_i,b_i} = \sum_{1\le j<r_i} y_{i,j} p_i^{j-1} + b_i p_i^{r_i-1}, \qquad 1 \le i \le s. \tag{19}$$

By (17), we derive

$$[T^k_{\mathcal{P}}(\mathbf{x})]_{\mathbf{r}} = (y_{1,r_1,b_1}, \dots, y_{s,r_s,b_s}) \iff k \equiv \mathcal{X}_{\mathbf{r},\mathbf{y},\mathbf{b}} - \mathcal{X}_{\mathbf{r},\mathbf{x}} \pmod{P_{\mathbf{r}}}.$$

Similarly to [22, pp. 37–39], we obtain from (1), (6), (17) and (19)

$$\mathcal{D}(L) = \sum_{k=-L}^{L-1} \sum_{r_1,\dots,r_s=1}^{n} \sum_{b_1=0}^{y_{1,r_1}-1} \cdots \sum_{b_s=0}^{y_{s,r_s}-1} \delta_{P_{\mathbf{r}}}(k - \mathcal{X}_{\mathbf{r},\mathbf{y},\mathbf{b}} + \mathcal{X}_{\mathbf{r},\mathbf{x}}) - 2L[y_1]_n \cdots [y_s]_n. \tag{20}$$

Let

$$\mathbb{D}_{\mathbf{r},L} = \sum_{b_1=0}^{y_{1,r_1}-1} \cdots \sum_{b_s=0}^{y_{s,r_s}-1} \sum_{k=-L}^{L-1} \big(\delta_{P_{\mathbf{r}}}(k - \mathcal{X}_{\mathbf{r},\mathbf{y},\mathbf{b}} + \mathcal{X}_{\mathbf{r},\mathbf{x}}) - 1/P_{\mathbf{r}}\big). \tag{21}$$

It is easy to see that $|\mathbb{D}_{\mathbf{r},L}| \le y_{1,r_1} \cdots y_{s,r_s} < p_0$ and

$$\mathcal{D}(L) \; = \sum_{r_1,\dots,r_s=1}^{n} \mathbb{D}_{\mathbf{r},L}. \tag{22}$$

Let $W_0 = 50 p_0 s^2 \log_2 n$,

$$U = \{(r_1, \dots, r_s) \in [1,n]^s \mid \max_i (r_i) > W_0\}$$

and

$$\tilde{\mathcal{D}}(L) = \sum_{\mathbf{r}\in U} \mathbb{D}_{\mathbf{r},L}. \tag{23}$$

From (20) – (23),we derive

$$|\mathcal{D}(L) - \tilde{\mathcal{D}}(L)| \; \le \; p_0 W_0^s \; = \; p_0^{s+1} (50 s^2)^s \log_2^s n \; \ll \; \log_2^s n.$$

Using (18), we obtain

$$\mathscr{D}_{\mathbf{y},\mathbf{x}}(L) = \tilde{\mathcal{D}}(L) + O(\log_2^s n). \tag{24}$$

### 2.2. Truncated Fourier's series of the discrepancy $\mathbb{D}_{\mathbf{r},L}$

For the sake of completeness, we repeat here [20, Lemma 1].

**Lemma 2.1.** *With the notations as above, we have*

$$\mathbb{D}_{\mathbf{r},L} = \sum_{m \in I^*_{P_\mathbf{r}}} \varphi_{\mathbf{r},L,m}\, \psi_\mathbf{r}(m,\mathbf{y})\, e\Big(\frac{m}{P_\mathbf{r}}(\mathcal{X}_{\mathbf{r},\mathbf{x}} - \mathcal{X}_{\mathbf{r},\mathbf{y}})\Big),$$

$$\varphi_{\mathbf{r},L,m} = \frac{2i\sin(2\pi mL/P_\mathbf{r})}{P_\mathbf{r}(e(m/P_\mathbf{r})-1)},$$

$$|\varphi_{\mathbf{r},L,m}| \le \frac{|\sin(2\pi mL/P_\mathbf{r})|}{\bar{m}} \le \frac{1}{\bar{m}},$$

$$\psi_\mathbf{r}(m,\mathbf{y}) = \prod_{i=1}^{s} \dot{\psi}(i, \{-mM_{i,\mathbf{r}}/p_i\}p_i, y_{i,r_i}),$$

$$\dot{\psi}(i,m',y_{i,r_i}) = \sum_{0 \le b < y_{i,r_i}} e\big(m'(b-y_{i,r_i})/p_i\big),$$

$$\dot{\psi}(i,m',y_{i,r_i}) = \frac{1-e(-m'y_{i,r_i}/p_i)}{e(m'/p_i)-1} \tag{25}$$

*for* $\{m'/p_i\} \ne 0, \quad \dot{\psi}(i,0,y_{i,r_i}) = y_{i,r_i}, \quad \dot{\psi}(i,m',0) = 0, \quad |\dot{\psi}(i,m',y_{i,r_i})| \le p_i.$

P r o o f. Using (7), (8) and (21), we obtain

$$\begin{aligned}
\mathbb{D}_{\mathbf{r},L} &= \sum_{b_1=0}^{y_{1,r_1}-1} \cdots \sum_{b_s=0}^{y_{s,r_s}-1} \sum_{k=-L}^{L-1} \frac{1}{P_\mathbf{r}} \sum_{m \in I^*_{P_\mathbf{r}}} e\Big(\frac{m}{P_\mathbf{r}}(k - \mathcal{X}_{\mathbf{r},\mathbf{y},\mathbf{b}} + \mathcal{X}_{\mathbf{r},\mathbf{x}})\Big)\\
&= \sum_{b_1=0}^{y_{1,r_1}-1} \cdots \sum_{b_s=0}^{y_{s,r_s}-1} \sum_{m \in I^*_{P_\mathbf{r}}} \frac{e(mL/P_\mathbf{r}) - e(-mL/P_\mathbf{r})}{P_\mathbf{r}(e(m/P_\mathbf{r})-1)}\, e\Big(\frac{m}{P_\mathbf{r}}(\mathcal{X}_{\mathbf{r},\mathbf{x}} - \mathcal{X}_{\mathbf{r},\mathbf{y},\mathbf{b}})\Big)\\
&= \sum_{m \in I^*_{P_\mathbf{r}}} e\Big(\frac{m}{P_\mathbf{r}}(\mathcal{X}_{\mathbf{r},\mathbf{x}} - \mathcal{X}_{\mathbf{r},\mathbf{y}})\Big) \varphi_{\mathbf{r},L,m} \sum_{b_1=0}^{y_{1,r_1}-1} \cdots \sum_{b_s=0}^{y_{s,r_s}-1} e\Big(\frac{m}{P_\mathbf{r}}(\mathcal{X}_{\mathbf{r},\mathbf{y}} - \mathcal{X}_{\mathbf{r},\mathbf{y},\mathbf{b}})\Big).
\end{aligned}$$

According to (16) and (19), we get

$$\mathcal{X}_{\mathbf{r},\mathbf{y},\mathbf{b}} \equiv \mathcal{X}_{\mathbf{r},\mathbf{y}} + \sum_{i=1}^{s} M_{i,\mathbf{r}} P_\mathbf{r}(b_i - y_{i,r_i})/p_i \bmod P_\mathbf{r}.$$

Therefore

$$\mathbb{D}_{\mathbf{r},L} = \sum_{m \in I^*_{P_\mathbf{r}}} e\left(\frac{m}{P_\mathbf{r}}(\mathcal{X}_{\mathbf{r},\mathbf{x}} - \mathcal{X}_{\mathbf{r},\mathbf{y}})\right) \varphi_{\mathbf{r},L,m}\, \varpi,$$

with

$$\varpi = \sum_{b_1=0}^{y_{1,r_1}-1} \cdots \sum_{b_s=0}^{y_{s,r_s}-1} e\left(-m \sum_{i=1}^{s} M_{i,\mathbf{r}}(b_i - y_{i,r_i})/p_i\right).$$

Taking into account that

$$\sum_{0 \le b < y_{i,r_i}} e\big(m'(b - y_{i,r_i})/p_i\big) = \dot{\psi}(i, m', y_{i,r_i}), \tag{26}$$

we have

$$\varpi = \prod_{i=1}^{s} \dot{\psi}(i, \{-mM_{i,\mathbf{r}}/p_i\}p_i, y_{i,r_i}) = \psi_\mathbf{r}(m, \mathbf{y}).$$

Hence

$$\mathbb{D}_{\mathbf{r},L} = \sum_{m \in I^*_{P_\mathbf{r}}} \varphi_{\mathbf{r},L,m}\, \psi_\mathbf{r}(m,\mathbf{y})\, e\left(\frac{m}{P_\mathbf{r}}(\mathcal{X}_{\mathbf{r},\mathbf{x}} - \mathcal{X}_{\mathbf{r},\mathbf{y}})\right).$$

In view of (26), we get $|\dot{\psi}(i, m', y_{i,r_i})| \le p_i$, $|\psi_\mathbf{r}(m', \mathbf{y})| \le p_0$. If $y_{i,r_i} = 0$, then $\dot{\psi}(i, m', y_{i,r_i}) = 0$. Let $y_{i,r_i} \ne 0$. If $\{m'/p_i\} = 0$, then $\dot{\psi}(i, m', y_{i,r_i}) = y_{i,r_i} \ge 1$. Bearing in mind that $\sin(\pi x/2) \ge x$ for $x \in [0,1]$, we obtain from (10) and (25) that

$$|\varphi_{\mathbf{r},L,m}| = \left|\frac{2\sin(2\pi mL/P_\mathbf{r})}{P_\mathbf{r}(e(m/P_\mathbf{r}) - 1)}\right| = \frac{|\sin(2\pi mL/P_\mathbf{r})|}{|P_\mathbf{r}\sin(\pi m/P_\mathbf{r})|} \le \frac{|\sin(2\pi mL/P_\mathbf{r})|}{2\bar{m}} \le \frac{1}{2\bar{m}}.$$

Hence Lemma 2.1 is proved. □

**Lemma 2.2.** *Let* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$,

$$\dot{\mathbb{D}}_{\mathbf{r},L} := \sum_{\substack{\alpha_i \in [0, 10s\log_2 n] \\ 1 \le i \le s}} \sum_{\substack{mP_{\boldsymbol{\alpha}} \in I^*_{P_\mathbf{r}} \\ (m,p_0)=1}} \varphi_{\mathbf{r},L,mP_{\boldsymbol{\alpha}}}\, \psi_\mathbf{r}(mP_{\boldsymbol{\alpha}}, \mathbf{y})\, e\left(\frac{mP_{\boldsymbol{\alpha}}}{P_\mathbf{r}}(\mathcal{X}_{\mathbf{r},\mathbf{x}} - \mathcal{X}_{\mathbf{r},\mathbf{y}})\right),$$

$$\dot{\mathcal{D}}(L) := \sum_{\mathbf{r} \in U} \dot{\mathbb{D}}_{\mathbf{r},L}. \tag{27}$$

*Then*

$$\mathbb{D}_{\mathbf{r},L} = \dot{\mathbb{D}}_{\mathbf{r},L} + O(n^{-9s}), \quad \mathbb{D}_{\mathbf{r},L} \ll n \qquad \text{and} \qquad \tilde{\mathcal{D}}(L) = \dot{\mathcal{D}}(L) + O(n^{-8s}),$$

P r o o f. Let $m = m'P_{\boldsymbol{\alpha}}$, with $(m', p_0) = 1$. Using Lemma 2.1, we get

$$\mathbb{D}_{\mathbf{r},L} = \sum_{\substack{\alpha_i \ge 0 \\ 1 \le i \le s}} \sum_{\substack{m'P_{\boldsymbol{\alpha}} \in I^*_{P_\mathbf{r}} \\ (m',p_0)=1}} \varphi_{\mathbf{r},L,m'P_{\boldsymbol{\alpha}}}\, \psi_\mathbf{r}(m'P_{\boldsymbol{\alpha}}, \mathbf{y}) e\left(\frac{m'P_{\boldsymbol{\alpha}}}{P_\mathbf{r}}(\mathcal{X}_{\mathbf{r},\mathbf{x}} - \mathcal{X}_{\mathbf{r},\mathbf{y}})\right).$$

Hence

$$|\mathbb{D}_{\mathbf{r},L} - \dot{\mathbb{D}}_{\mathbf{r},L}| \le \sum_{\substack{\max_i \alpha_i > 10s\log_2 n \\ \alpha_i \ge 0, 1\le i\le s}} \sum_{\substack{m'P_{\boldsymbol{\alpha}} \in I^*_{P_{\mathbf{r}}} \\ (m',p_0)=1}} \left|\varphi_{\mathbf{r},L,m'P_{\boldsymbol{\alpha}}}\ \psi_{\mathbf{r}}(m'P_{\boldsymbol{\alpha}},\mathbf{y})\right|$$

$$\le p_0 \sum_{\substack{\max_i \alpha_i > 10s\log_2 n \\ \alpha_i \ge 0, 1\le i\le s}} \sum_{\substack{m'P_{\boldsymbol{\alpha}} \in I^*_{P_{\mathbf{r}}} \\ (m',p_0)=1}} (\bar{m}'P_{\boldsymbol{\alpha}})^{-1} \ll n \sum_{\substack{\max_i \alpha_i > 10s\log_2 n \\ \alpha_i \ge 0, 1\le i\le s}} P_{-\boldsymbol{\alpha}} \ll n^{-9s}.$$

It is easy to see that $\mathbb{D}_{\mathbf{r},L} \ll \sum_{|m|\le p_0^n} 1/\bar{m} \ll n$. Bearing in mind that $\#U \le n^s$, we obtain from (23) and (27) the assertion of Lemma 2.2. □

We consider expectations in probability spaces $\mathbb{P}_1$ and $\mathbb{P}_s$:

$$\mathbf{E}_{\boldsymbol{\theta}} f_1 = \mathbf{E}_{\boldsymbol{\theta}}(f_1) = \int_{[0,1)} f_1(\theta)d\theta, \quad \mathbf{E}_x f_2 = \mathbf{E}_x(f_2) = \int_{[0,1)^s} f_2(\mathbf{x})d\mathbf{x}. \tag{28}$$

It is easy to see that

$$\mathbf{E}_{\boldsymbol{\theta}} f([\theta N]) = \frac{1}{N}\sum_{k=0}^{N-1} f(k). \tag{29}$$

In the following Lemma 2.3, we calculate the expectation of the essential part of the discrepancy function (see (27)).

## 2.3. First moment estimates

**Lemma 2.3.** *Let $\mu \in [2,8]$ be integer, $|m_j| \le n^{10s}$, $0 \le \alpha_{j,i} \le 10s\log_2 n$, $r_{j,i} \ge 0$, $\max_i r_{j,i} \ge W_0$, $r_{0,i} = \max_{1\le j\le\mu}(r_{j,i})$, $\mathbf{r}_0 = (r_{0,1},\dots,r_{0,s})$, $i=1,\dots,s$, $j=1,\dots,\mu$,*

$$\omega_1 = \sum_{j=1}^{\mu} m_j(\mathcal{X}_{\mathbf{r}_j,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_j,\mathbf{y}})/P_{\mathbf{r}_j} \quad \text{and} \quad \ddot{\omega}_{\boldsymbol{\alpha}} = \sum_{j=1}^{\mu} m_j P_{\boldsymbol{\alpha}_j}(\mathcal{X}_{\mathbf{r}_j,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_j,\mathbf{y}})/P_{\mathbf{r}_j}.$$

*Then*

$$\mathbf{E}_x e(\omega_1) = \delta_{P_{\mathbf{r}_0}}\left(\sum_{j=1}^{\mu} m_j P_{\mathbf{r}_0-\mathbf{r}_j}\right), \quad \mathbf{E}_x e(\ddot{\omega}_{\boldsymbol{\alpha}}) = \Delta\left(\sum_{j=1}^{\mu} m_j P_{-\mathbf{r}_j+\boldsymbol{\alpha}_j} = 0\right)$$

*and*

$$|m_j/P_{\mathbf{r}_j-\boldsymbol{\alpha}_j}| < 1/32, \quad 1\le j\le\mu. \tag{30}$$

P r o o f. By (8), (11), (15), (16) and (28), we obtain

$$\mathcal{X}_{\mathbf{r},\mathbf{x}}/P_{\mathbf{r}} = \sum_{i=1}^{s} M_{i,\mathbf{r}} \sum_{\nu=1}^{r_i} \frac{x_{i,\nu}p_i^{\nu-1}}{p_i^{r_i}} \equiv \sum_{i=1}^{s} \frac{M_{i,\mathbf{r}}}{p_i^{r_i}} \sum_{\nu=1}^{r_{0,i}} x_{i,\nu}p_i^{\nu-1} \equiv \sum_{i=1}^{s} \mathcal{X}_{i,r_{0,i}}(x_i)\frac{M_{i,\mathbf{r}}}{p_i^{r_i}} \mod 1$$

and

$$\mathbf{E}_x e(m\mathcal{X}_{\mathbf{r},\mathbf{x}}/P_{\mathbf{r}}) = \mathbf{E}_x e\left(m\sum_{i=1}^{s} M_{i,\mathbf{r}} \sum_{\nu=1}^{r_i} x_{i,\nu}p_i^{\nu-1}/p_i^{r_i}\right) = \prod_{i=1}^{s} \delta_{p_i^{r_i}}(mM_{i,\mathbf{r}}) = \delta_{P_{\mathbf{r}}}(m).$$

Hence

$$\begin{aligned}\omega_1 &= \sum_{j=1}^{\mu} m_j(\mathcal{X}_{\mathbf{r}_j,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_j,\mathbf{y}})/P_{\mathbf{r}_j} \equiv \sum_{j=1}^{\mu} m_j \sum_{i=1}^{s} M_{i,\mathbf{r}_j}\big(\mathcal{X}_{i,r_{j,i}}(x_i) - \mathcal{X}_{i,r_{j,i}}(y_i)\big)/p_i^{r_{j,i}} \\ &\equiv \sum_{j=1}^{\mu} m_j \sum_{i=1}^{s} M_{i,\mathbf{r}_j}\big(\mathcal{X}_{i,r_{0,i}}(x_i) - \mathcal{X}_{i,r_{0,i}}(y_i)\big)/p_i^{r_{j,i}} \\ &\equiv \sum_{i=1}^{s}\big(\mathcal{X}_{i,r_{0,i}}(x_i) - \mathcal{X}_{i,r_{0,i}}(y_i)\big)/p_i^{r_{0,i}} \sum_{j=1}^{\mu} m_j M_{i,\mathbf{r}_j} p_i^{r_{0,i}-r_{j,i}} \mod 1.\end{aligned}$$

Let

$$\omega_2 = e\left(-\sum_{1\le i\le s} \mathcal{X}_{i,r_{0,i}}(y_i)/p_i^{r_{0,i}} \sum_{1\le j\le \mu} m_j M_{i,\mathbf{r}_j} p_i^{r_{0,i}-r_{j,i}}\right).$$

We have

$$\mathbf{E}_x e(\omega_1) = \omega_2 \prod_{i=1}^{s} \delta_{p_i^{r_{0,i}}}\left(\sum_{j=1}^{\mu} m_j M_{i,\mathbf{r}_j} p_i^{r_{0,i}-r_{j,i}}\right).$$

Bearing in mind that $\omega_2 = 1$for $\sum_{1\le j\le\mu} m_j M_{i,\mathbf{r}_j} p_i^{r_{0,i}-r_{j,i}} \equiv 0 \mod p_i^{r_{0,i}}$ $(i = 1, \dots, s)$, we obtain

$$\begin{aligned}\mathbf{E}_x e(\omega_1) &= \prod_{i=1}^{s} \delta_{p_i^{r_{0,i}}}\left(\sum_{j=1}^{\mu} m_j M_{i,\mathbf{r}_j} p_i^{r_{0,i}-r_{j,i}}\right) \\ &= \prod_{i=1}^{s} \delta_{p_i^{r_{0,i}}}\left(\prod_{\substack{1\le\nu\le s\\ \nu\ne i}} p_\nu^{r_{0,\nu}} \sum_{j=1}^{\mu} m_j M_{i,\mathbf{r}_j} p_i^{r_{0,i}-r_{j,i}}\right).\end{aligned}$$

From(15), we get

$$M_{i,\mathbf{r}_j} \equiv \prod_{1\le\nu\le s,\nu\ne i} p_\nu^{-r_{j,\nu}} \mod p_i^{r_{j,i}}$$

and

$$p_i^{r_{0,i}-r_{j,i}} \prod_{1\le\nu\le s,\nu\ne i} p_\nu^{r_{0,\nu}} M_{i,\mathbf{r}_j} \equiv \prod_{1\le\nu\le s} p_\nu^{r_{0,\nu}-r_{j,\nu}} \mod p_i^{r_{0,i}}.$$

Therefore,

$$\begin{aligned}\mathbf{E}_x e(\omega_1) &= \prod_{i=1}^{s} \delta_{p_i^{r_{0,i}}}\left(\sum_{j=1}^{\mu} m_j p_i^{r_{0,i}-r_{j,i}} \prod_{1\le\nu\le s,\nu\ne i} p_\nu^{r_{0,\nu}} M_{i,\mathbf{r}_j}\right) \\ &= \prod_{i=1}^{s} \delta_{p_i^{r_{0,i}}}\left(\sum_{j=1}^{\mu} m_j \prod_{1\le\nu\le s} p_\nu^{r_{0,\nu}-r_{j,\nu}}\right) = \delta_{P_{\mathbf{r}_0}}\left(\sum_{j=1}^{\mu} m_j \prod_{1\le\nu\le s} p_\nu^{r_{0,\nu}-r_{j,\nu}}\right) \\ &= \delta_{P_{\mathbf{r}_0}}\left(\sum_{j=1}^{\mu} m_j P_{\mathbf{r}_0-\mathbf{r}_j}\right).\end{aligned}$$

Let us consider $\ddot{\omega}_{\boldsymbol{\alpha}}$. By (30), we get

$$\mathbf{E}_x e(\omega_{\boldsymbol{\alpha}}) = \delta_{P_{\mathbf{r}_0}}\left(\sum_{j=1}^{\mu} m_j P_{\mathbf{r}_0-\mathbf{r}_j+\boldsymbol{\alpha}}\right).$$

It is easy to see that

$$\sum_{j=1}^{\mu} m_j P_{\mathbf{r}_0-\mathbf{r}_j+\boldsymbol{\alpha}} \equiv 0 \bmod P_{\mathbf{r}_0} \Leftrightarrow \sum_{j=1}^{\mu} m_j P_{-\mathbf{r}_j+\boldsymbol{\alpha}} \equiv 0 \bmod 1$$

$$\Leftrightarrow \sum_{j=1}^{\mu} m_j P_{\ddot{\mathbf{r}}_0-\mathbf{r}_j+\boldsymbol{\alpha}} \equiv 0 \bmod P_{\ddot{\mathbf{r}}_0},$$

where $\ddot{\mathbf{r}}_0 = (\ddot{r}_{0,1},\dots,\ddot{r}_{0,s})$, $\ddot{r}_{0,i} = \max_{1\le j\le\mu}(0, \ r_{j,i} - \alpha_{i,j})$, $i = 1,\dots,s$. Hence,

$$\mathbf{E}_x e(\omega_{\boldsymbol{\alpha}}) = \delta_{P_{\ddot{\mathbf{r}}_0}}\left(\sum_{j=1}^{\mu} m_j P_{\ddot{\mathbf{r}}_0-\mathbf{r}_j+\boldsymbol{\alpha}}\right).$$

We have that the expression for $\mathbf{E}_x e(\ddot{\omega}_{\boldsymbol{\alpha}})$ in(30) follows from the estimate

$$|m_j P_{\ddot{\mathbf{r}}_0-\mathbf{r}_j+\boldsymbol{\alpha}_j}| < P_{\ddot{\mathbf{r}}_0}/32, \quad 1\le j\le \mu\le 8.$$

Now we consider the last estimate for $|m_j| \le n^{10s}$, $0 \le \alpha_{j,i} \le 10s\log_2 n$, $\max_i r_{j,i} \ge W_0$, $i = 1,\dots,s$, $j = 1,\dots,\mu$.

$$\left|\sum_{1\le j\le\mu} m_j/P_{\mathbf{r}_j-\boldsymbol{\alpha}_j}\right| \le 8n^{10s}\max_j \prod_{i=1}^{s} p_i^{-r_{j,i}+\alpha_{j,i}} \le 8n^{10s}\prod_{i=1}^{s} p_i^{10s\log_2 n} 2^{-W_0}$$

$$\le 8n^{10s} P_{[10s\log_2 n]+1} P_{-[50s^2\log_2 n]+1} < 1/32.$$

Hence Lemma 2.3 is proved. □

We can apply our main tools, Theorem 3.1 and Theorem 3.3, only if $m$ is small enough. The purpose of Lemma 2.4 is to show that the part of the discrepancy for large $m$ is negligible. This will allow us to apply Theorem 3.1 and Theorem 3.3 in the future.

**Lemma 2.4.** *Let*

$$\ddot{\mathbb{D}}_{\mathbf{r},L} = \sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\le i\le s}} \sum_{\substack{0<|m|\le n^{10s}\\ (m,p_0)=1}} \varphi_{\mathbf{r},L,mP_{\boldsymbol{\alpha}}}\psi_{\mathbf{r}}(mP_{\boldsymbol{\alpha}},\mathbf{y})\, e\left(\frac{mP_{\boldsymbol{\alpha}}}{P_{\mathbf{r}}}(\mathcal{X}_{\mathbf{r},\mathbf{x}}-\mathcal{X}_{\mathbf{r},\mathbf{y}})\right),$$

$$\ddot{\mathcal{D}}(L) = \sum_{\mathbf{r}\in U}\ddot{\mathbb{D}}_{\mathbf{r},L}. \qquad (31)$$

*Then there exists $N_1(\mathbf{x})$ such that*

$$\mathbf{E}_{\boldsymbol{\theta}}\Big(\big(\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])\big) - \ddot{\mathcal{D}}([\theta N])\Big)^2 \le \log_2^{2s+1} n,$$

*with $N > N_1(\mathbf{x})$ for a.s. $\mathbf{x}\in[0,1)^s$.*

P r o o f. By (24) and Lemma 2.2, we get

$$\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N]) = \tilde{\mathcal{D}}([\theta N]) + O(\log_2^s n), \quad \tilde{\mathcal{D}}([\theta N]) = \dot{\mathcal{D}}([\theta N]) + O(n^{-8s}).$$

It is easy to see that

$$\begin{aligned}\mathbf{E}_{\boldsymbol{\theta}}\big(\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N]) - \ddot{\mathcal{D}}([\theta N])\big)^2 &\le 2\mathbf{E}_{\boldsymbol{\theta}}\big(\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N]) - \dot{\mathcal{D}}([\theta N])\big)^2 \\ &\quad + 2\mathbf{E}_{\boldsymbol{\theta}}\big(\dot{\mathcal{D}}([\theta N]) - \ddot{\mathcal{D}}([\theta N])\big)^2 \\ &\le 2\mathbf{E}_{\boldsymbol{\theta}}\big(\dot{\mathcal{D}}([\theta N]) - \ddot{\mathcal{D}}([\theta N])\big)^2 + \log_2^{2s} n.\end{aligned}$$

Therefore, in order to prove Lemma 2.4 it is enough to verify that

$$\mathbf{E}_{\boldsymbol{\theta}}|\dot{\mathcal{D}}([\theta N]) - \ddot{\mathcal{D}}([\theta N])|^2 \ll 1 \qquad \text{for } a.s.\ \mathbf{x} \in [0,1)^s. \tag{32}$$

Applying the Borel-Cantelli lemma, we get that (32) results from the following inequality

$$\mathrm{Prob}\Big(\mathbf{x} \in [0,1)^s \mid \mathbf{E}_{\boldsymbol{\theta}}\big(\dot{\mathcal{D}}([\theta N]) - \ddot{\mathcal{D}}([\theta N])\big)^2 > 1\Big) \ll 1/n^2.$$

By Chebyshev's inequality, it is enough to prove that

$$\mathbf{E}_x\mathbf{E}_{\boldsymbol{\theta}}\big|\dot{\mathcal{D}}([\theta N]) - \ddot{\mathcal{D}}([\theta N])\big|^2 \ll 1/n^2. \tag{33}$$

Let us consider (23), (27) and (31). We have $\max_i r_i > W_0 = 50s^2 p_0 \log_2 n$ and

$$P_{\mathbf{r}} \ge \max_i p_i^{r_i} \ge 2^{50p_0 s^2 \log_2 n} = n^{50p_0 s^2} > n^{20s(1+\log_2 p_0)} > 4n^{10s} p_0^{10s\log_2 n}.$$

We obtain that, if $|m| \le n^{10s}$, then $mP_{\boldsymbol{\alpha}} \in I_{P_{\mathbf{r}}}$ for all $\boldsymbol{\alpha}$ with $\alpha_i \in [0, 10s\log_2 n]$. Hence

$$\mathbf{E}_x\mathbf{E}_{\boldsymbol{\theta}}\big|\dot{\mathcal{D}}([\theta N]) - \ddot{\mathcal{D}}([\theta N])\big|^2 = \sum_{\mathbf{r}_1,\mathbf{r}_2 \in U} Z_{\mathbf{r}_1,\mathbf{r}_2},$$

with

$$\begin{aligned}Z_{\mathbf{r}_1,\mathbf{r}_2} &:= \sum_{\substack{\alpha_{j,i} \in [0,10s\log_2 n] \\ 1\le i\le s, 1\le j\le 2}} \\ &\times \sum_{\substack{|m_j| > n^{10s},\ m_j \in I^*_{P_{\mathbf{r}}} \\ (m_j,p_0)=1, j=1,2}} \mathbf{E}_{\boldsymbol{\theta}}\Big(\varphi_{\mathbf{r}_1,[\theta N],m_1P_{\boldsymbol{\alpha}_1}} \overline{\varphi_{\mathbf{r}_2,[\theta N],m_2P_{\boldsymbol{\alpha}_2}}}\Big)\psi_{\mathbf{r}_1}(m_1P_{\boldsymbol{\alpha}_1},\mathbf{y})\overline{\psi_{\mathbf{r}_1}(m_2P_{\boldsymbol{\alpha}_2},\mathbf{y})} \\ &\times \mathbf{E}_x e\left(\frac{m_1P_{\boldsymbol{\alpha}_1}}{P_{\mathbf{r}_1}}(\mathcal{X}_{\mathbf{r}_1,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_1,\mathbf{y}}) - \frac{m_2P_{\boldsymbol{\alpha}_2}}{P_{\mathbf{r}_2}}(\mathcal{X}_{\mathbf{r}_2,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_2,\mathbf{y}})\right).\end{aligned} \tag{34}$$

Using Lemma 2.1 and Lemma 2.3, we obtain

$$Z_{\mathbf{r}_1,\mathbf{r}_2} = \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, 1\le j\le 2}} \sum_{\substack{|m_j|>n^{10s},\ m_j\in I^*_{P_{\mathbf{r}}}\\ (m_j,p_0)=1, j=1,2}} \mathbf{E}_{\boldsymbol{\theta}}\Big(\varphi_{\mathbf{r}_1,[\theta N],m_1P_{\boldsymbol{\alpha}_1}}\overline{\varphi_{\mathbf{r}_2,[\theta N],m_2P_{\boldsymbol{\alpha}_2}}}\Big)$$

$$\times\ \psi_{\mathbf{r}_1}(m_1P_{\boldsymbol{\alpha}_1},\mathbf{y})\overline{\psi_{\mathbf{r}_1}(m_2P_{\boldsymbol{\alpha}_2},\mathbf{y})}\delta_{P_{\mathbf{r}_0}}(P_{\mathbf{r}_0-\mathbf{r}_1+\boldsymbol{\alpha}_1}m_1 - P_{\mathbf{r}_0-\mathbf{r}_2+\boldsymbol{\alpha}_2}m_2)$$

$$\ll \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, 1\le j\le 2}} \sum_{\substack{p_0^n\ge|m_j|>n^{10s}\\ (m_j,p_0)=1, j=1,2}} (\bar{m}_1\bar{m}_2P_{\boldsymbol{\alpha}_1}P_{\boldsymbol{\alpha}_2})^{-1}$$

$$\times\,\delta_{P_{\mathbf{r}_0}}(P_{\mathbf{r}_0-\mathbf{r}_1+\boldsymbol{\alpha}_1}m_1 - P_{\mathbf{r}_0-\mathbf{r}_2+\boldsymbol{\alpha}_2}m_2), \tag{35}$$

with

$$\mathbf{r}_0 = (r_{0,1},\dots,r_{0,s}), \quad r_{0,i} = \max(r_{1,i},r_{2,i}), \quad i=1,\dots,s.$$

By (34) and (35), we get that (33) go after the following inequality

$$\varpi_{\mathbf{r}_1,\mathbf{r}_2} := \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, 1\le j\le 2}} \sum_{\substack{p_0^n\ge|m_j|>n^{10s}\\ (m_j,p_0)=1, j=1,2}} (\bar{m}_1\bar{m}_2P_{\boldsymbol{\alpha}_1}P_{\boldsymbol{\alpha}_2})^{-1}$$

$$\times\,\delta_{P_{\mathbf{r}_0}}(P_{\mathbf{r}_0-\mathbf{r}_1+\boldsymbol{\alpha}_1}m_1 - P_{\mathbf{r}_0-\mathbf{r}_2+\boldsymbol{\alpha}_2}m_2) \ll n^{-2s-2}. \tag{36}$$

From 23, we have that there exists $i_0\in[1,s]$ such that $r_{1,i_0}\ge W_0 = 50s^2p_0\times\log_2 n$. Let $\rho_j = r_{j,i_0}-\alpha_{j,i_0}$, $\rho_{j_1} = \max_{j=1,2}\rho_j$ with $j_1\in\{1,2\}$. Bearing in mind that $m_1,m_2$ satisfy the congruence

$$P_{\mathbf{r}_0-\mathbf{r}_1+\boldsymbol{\alpha}_1}m_1 - P_{\mathbf{r}_0-\mathbf{r}_2+\boldsymbol{\alpha}_2}m_2 \equiv 0 \bmod P_{\mathbf{r}_0} \quad \text{and that} \quad \max_{j,i}\alpha_{j,i}\le 10s\log_2 n,$$

we get

$$\rho_{j_1}\ge 40s^2\log_2 n \quad \text{and} \quad m_{j_1}\equiv m_{j_2}p_{i_0}^{\rho_{j_1}-\rho_{j_2}}A_0 \bmod p_{i_0}^{\rho_{j_1}}, \quad \text{with} \quad j_2\in\{1,2\}\setminus\{j_1\}$$

for some integer $A_0$ with $(A_0,p_{i_0})=1$. Therefore,

$$\varpi_{\mathbf{r}_1,\mathbf{r}_2} \le \sum_{n^{10s}<|m_{j_2}|\le p_0^n}\frac{1}{\bar{m}_{j_2}} \sum_{n^{10s}<|m_{j_1}|\le p_0^n}\frac{\delta_{p_{i_0}^{[40s^2\log_2 n]}}(m_{j_1}-m_{j_2}p_{i_0}^{\rho_{j_1}-\rho_{j_2}}A_0)}{\bar{m}_{j_1}}$$

$$\ll \sum_{n^{10s}<|m_{j_2}|\le p_0^n}\frac{1}{\bar{m}_{j_2}} \sum_{0\le\ell\le p_0^n}\frac{1}{n^{10s}+\ell p_{i_0}^{40s^2\log_2 n}}$$

$$\ll n^{-9s}.$$

Hence (36), (33) and Lemma 2.4 are proved. □

## 3. Diophantine inequalities

We consider the following simple variant of the `S-unit theorem` (see [16, Theorem 1.1, p. 808]): Let $\beta_1,\dots,\beta_d \in \mathbb{Q}$, $\beta_i \neq 0$, $i = 1,\dots,d$. We consider the equation

$$\beta_1 P_{\mathbf{r}_1} + \dots + \beta_d P_{\mathbf{r}_d} = 1, \qquad P_{\mathbf{r}_j} = \prod_{i=1}^{s} p_i^{r_{j,i}}, \;\; r_{j,i} \in \mathbb{Z}. \tag{37}$$

A solution $(\mathbf{r}_1,\dots,\mathbf{r}_d)$ of (37) is called `non-degenerate` if $\sum_{i\in J}\beta_i P_{\mathbf{r}_i} \neq 0$ for every nonempty subset $J$ of $\{1,\dots,d\}$.

**Theorem 3.1.** *The number $A(\beta_1,\dots,\beta_d)$ of non-degenerate solutions of equation (37) satisfies the estimate*

$$A(\beta_1,\dots,\beta_d) \le \exp\big((6d)^{3d}s\big).$$

**Corollary 3.2.** *Let $\boldsymbol{\alpha}_j \in \mathbb{Z}^s$, $|m_j| > 0$, $1 \le j \le 2d$, $d = 2,4$. Then*

$$\sum_{\substack{\mathbf{r}_j\in\mathbb{U}\\ 1\le j\le 2d}} \Delta\Bigg(\sum_{1\le j\le 2d} m_j P_{-\mathbf{r}_j+\boldsymbol{\alpha}_j} = 0\Bigg) \ll \#\mathbb{U}^d.$$

**Linear forms in logarithms (see [8, Theorem 2.15, p. 42]).**
Write $\Lambda$ for the linear form in logarithms,

$$\Lambda = b_1 \ln\alpha_1 + \dots + b_{\tilde{k}} \ln\alpha_{\tilde{k}},$$

where $b_1,\dots,b_{\tilde{k}}$ are integers, $|b_i| \le B$ $(i = 1,\dots,\tilde{k})$, $B \ge e$. We shall assume that $\alpha_1,\dots,\alpha_{\tilde{k}}$ are non-zero algebraic numbers with heights at most $A_1,\dots,A_{\tilde{k}}$ (all $\ge e$), respectively.

**Theorem 3.3.** *If $\Lambda \neq 0$, then*

$$|\Lambda| > \exp\big(-(16\tilde{k}d)^{2(\tilde{k}+2)} \ln A_1 \cdots \ln A_{\tilde{k}} \ln B\big),$$

*where $d$ denote the degrees of $\mathbb{Q}(\alpha_1,\dots,\alpha_{\tilde{k}})$.*

**Corollary 3.4.** *Let $|r_i| \le 2n$, $0 < |m_j| \le n^{10s}$, $|m_1/m_2|P_{\mathbf{r}} > 1$. Then*

$$|m_1/m_2 P_{\mathbf{r}} - 1| > \exp(-C_1 \ln^2 n), \quad \text{with} \quad C_1 = 20s(16(s+1))^{2(s+3)} \ln^s p_0.$$

P r o o f. Taking into account that $\exp(x) - 1 \ge x$ for $x \ge 0$, we get

$$||m_1/m_2|P_{\mathbf{r}} - 1| \ge \ln(|m_1/m_2|P_{\mathbf{r}}) = r_1\ln(p_1) + \dots + r_s\ln(p_s) + \ln(|m_1/m_2|).$$

Using Theorem 3.3 with

$$\begin{array}{ccc} \tilde{k} = s+1, & b_i = r_i, & \alpha_i = A_i = p_i, \\ (i = 1,\dots,s), & b_{s+1} = 1, & \alpha_{s+1} = 1, \\ A_{s+1} = n^{10s}, & B = 2n, & d = 1, \end{array}$$

we get the assertion of the Corollary 3.2. □

Taking into account Theorem 7.2, we get that to prove the Theorem 1.1 it suffices to get the upper bound of the variance of $\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])$ (see section 4), the lower bound of the variance of $\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])$ (see section 5), the four moment and Lévy's conditional variance of $\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])$ (see section 6).

## 4. Upper bound of the variance of the discrepancy $\ddot{\mathcal{D}}([\theta N])$

By (23), $W_0 = 50s^2 p_0 \log_2 n$ and $U = \{(r_1,\dots,r_s) \in [1,n]^s \mid \max_i r_i > W_0\}$. In order to apply the martingale CLT (see below Theorem 7.2) we divide the summation area $U$ into the essential area $U_1$ and into the negligible areas $U_2 - U_5$. Minor areas include the initial area $U_5$, the intermediate area $U_2$ and the last areas $U_3$ and $U_4$. We will apply Theorem 7.2 only to the domain $U_1$.

Let

$$\begin{aligned}
W_1 &= [n/W_2] - 2, \quad W_2 = [\log_2^{20s} n], \quad W_3 = [\log_2^{10s} n],\\
A_k &= n - (k+1)W_2, \quad B_k = A_k + W_2 - 2W_3,\\
h(\mathbf{r}) &= r_1 \log_2(p_1) + \dots + r_s \log_2(p_s),\\
U_1 &= \bigcup_{1\le k\le W_1} \mathbb{U}_k, \quad \mathbb{U}_k = \{\mathbf{r} \in U \mid h(\mathbf{r}) \in [A_k, B_k)\},\\
U_2 &= \bigcup_{1\le k\le W_1} \mathbb{U}'_k, \quad \mathbb{U}'_k = \{\mathbf{r} \in U \mid h(\mathbf{r}) \in [B_{k+1}, A_k)\},\\
U_3 &= \{\mathbf{r} \in U \mid h(\mathbf{r}) \in [B_1, n+W_2]\}, \quad U_4 = \{\mathbf{r} \in U \mid h(\mathbf{r}) > n + W_2\},\\
U_5 &= \{\mathbf{r} \in U \mid h(\mathbf{r}) < B_{W_1+1}\}, \quad \ddot{\mathcal{D}}_j(L) := \sum_{\mathbf{r}\in U_j} \ddot{\mathbb{D}}_{\mathbf{r},L}, \quad \mathfrak{D}_k(L) := \sum_{\mathbf{r}\in \mathbb{U}_k} \ddot{\mathbb{D}}_{\mathbf{r},L}. \qquad (38)
\end{aligned}$$

According to Lemma 2.1 and (31) we have

$$\mathfrak{D}_k(L) \ll n^{s+1}, \quad L \in [1,N]. \qquad (39)$$

We see that

$$U = \bigcup_{1\le j\le 5} U_j, \quad U_i \cap U_j = \varnothing, \quad i \ne j, \quad 1 \le i,j \le 5.$$

By (31) and (38), we get that

$$\ddot{\mathcal{D}}(L) = \sum_{1\le j\le 5} \ddot{\mathcal{D}}_j(L) \quad \text{and} \quad \ddot{\mathcal{D}}_1(L) = \sum_{1\le k\le W_1} \mathfrak{D}_k(L). \qquad (40)$$

Let

$$\mathfrak{U} \subset \mathbb{Z}^s, \quad \mathfrak{U} \dot{-} \boldsymbol{\alpha} = \{\mathbf{r} \in \mathbb{Z}^s \mid \mathbf{r} + \boldsymbol{\alpha} \in \mathfrak{U}\}, \quad \mathfrak{U}^{\cap,\boldsymbol{\alpha}} = \mathfrak{U} \cap (\mathfrak{U} \dot{-} \boldsymbol{\alpha}),$$
$$\mathfrak{U}^{\setminus,\boldsymbol{\alpha}} = \mathfrak{U} \setminus (\mathfrak{U} \dot{-} \boldsymbol{\alpha}), \qquad \mathfrak{U}^{\boldsymbol{\alpha},\setminus} = (\mathfrak{U} \dot{-} \boldsymbol{\alpha}) \setminus \mathfrak{U}. \qquad (41)$$

We need these notations to describe the expression

$$(U_1\dot{-}\boldsymbol{\alpha}_1)\cap(U_1\dot{-}\boldsymbol{\alpha}_2) \qquad \text{(see Lemma 5.1).}$$

It is easy to verify that

$$\mathfrak{U}=\mathfrak{U}^{\cap,\boldsymbol{\alpha}}\cup\mathfrak{U}^{\setminus,\boldsymbol{\alpha}},\quad \mathfrak{U}^{\cap,\boldsymbol{\alpha}}\cap\mathfrak{U}^{\setminus,\boldsymbol{\alpha}}=\varnothing,\quad \mathfrak{U}\dot{-}\boldsymbol{\alpha}=\mathfrak{U}^{\cap,\boldsymbol{\alpha}}\cup\mathfrak{U}^{\boldsymbol{\alpha},\setminus},\quad \mathfrak{U}^{\boldsymbol{\alpha}}\cap\mathfrak{U}^{\boldsymbol{\alpha},\setminus}=\varnothing$$

and

$$\begin{aligned}(\mathfrak{U}\dot{-}\boldsymbol{\alpha}_1)\cap(\mathfrak{U}\dot{-}\boldsymbol{\alpha}_2) &= \left(\mathfrak{U}^{\cap,\boldsymbol{\alpha}_1}\cup\mathfrak{U}^{\boldsymbol{\alpha}_1,\setminus}\right)\cap\left(\mathfrak{U}^{\cap,\boldsymbol{\alpha}_2}\cup\mathfrak{U}^{\boldsymbol{\alpha}_2,\setminus}\right)\\ &= \left((\mathfrak{U}\setminus\mathfrak{U}^{\setminus,\boldsymbol{\alpha}_1})\cup\mathfrak{U}^{\boldsymbol{\alpha}_1,\setminus}\right)\bigcap\left((\mathfrak{U}\setminus\mathfrak{U}^{\setminus,\boldsymbol{\alpha}_2})\cup\mathfrak{U}^{\boldsymbol{\alpha}_2,\setminus}\right)\\ &= (\mathfrak{U}\cup\dot{\mathfrak{U}})\setminus\ddot{\mathfrak{U}}\end{aligned}$$

with some

$$\dot{\mathfrak{U}}\subseteq\mathfrak{U}^{\setminus,\boldsymbol{\alpha}_1}\cup\mathfrak{U}^{\setminus,\boldsymbol{\alpha}_2},\quad \ddot{\mathfrak{U}}\subseteq\mathfrak{U}^{\boldsymbol{\alpha}_1,\setminus}\cup\mathfrak{U}^{\boldsymbol{\alpha}_2,\setminus}$$

and

$$\#\dot{\mathfrak{U}}\ +\ \#\ddot{\mathfrak{U}}\ \le\ 2\max_{i=1,2}(\#\mathfrak{U}^{\setminus,\boldsymbol{\alpha}_i}+\#\mathfrak{U}^{\boldsymbol{\alpha}_i,\setminus}). \tag{42}$$

In the following Lemma 4.1, we obtain estimates for the measures of the essential domain $U_1=\bigcup_{1\le k\le W_1}\mathbb{U}_k$ and measures of negligible domains. We need the above partition because the discrepancy functions on $(\mathbb{U}_k)_{1\le k\le W_1}$ (see $\mathfrak{D}_k(L)$ in (40)) form a sequence of martingale differences (see Lemma 7.1).

**Lemma 4.1.** *Let* $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_s)$, $\alpha_i\in[-10s\log_2 n,10s\log_2 n]$, $i=1,\dots,s$. *With the notations as in* (38), *we have*

$$\#\mathbb{U}_k\ll n^{s-1}\log_2^{20s}n,\quad \#U_2\le n^s\log_2^{-10s}n,\quad \#U_3\le n^{s-1}\log_2^{40s}n,\quad \#U_5\ll\log_2^{20s^2}n,$$

$$\#U_1^{\setminus,\boldsymbol{\alpha}}+\#U_1^{\boldsymbol{\alpha},\setminus}\ll\ n^s\log_2^{-19s}n,$$

$$U_1\cup U_2\cup[0,2W_2]^s\supset U\cap[1,n/(2p_0)]^s,\quad \#U_1\le n^s.$$

P r o o f. We put $\rho_r=r_1\log_2(p_1)+\cdots+r_{s-1}\log_2(p_{s-1})$.
**Let us consider** $U_3$ **and** $\mathbb{U}_k=\{\mathbf{r}\in U\mid h(\mathbf{r})\in[A_k,B_k)\}$
($A_k=n-(k+1)W_2$, $B_k=A_k+W_2-2W_3$, $h(\mathbf{r})=r_1\log_2(p_1)+\cdots+r_s\log_2(p_s)$, see (38)).

It is easy to see that

$$\#\mathbb{U}_k\le n^{s-1}\max_{1\le r_1,\dots,r_{s-1}\le n}\#\{1\le r_s\le n\mid A_k-\rho_r\le r_s\log_2(p_s)<A_k+W_2-2W_3-\rho_r\}.$$

Hence

$$\#\mathbb{U}_k\le n^{s-1}W_2\le n^{s-1}\log_2^{20s}n,\qquad \#U_3=\sum_{k=B_1}^{n+W_2-1}\#\mathbb{U}_k\ll n^{s-1}W_2^2\ll n^{s-1}\log_2^{40s}n,$$

where

$$W_2=[\log_2^{20s}n],\quad B_1=n-W_2-2W_3\quad\text{and}\quad W_3=[\log_2^{10s}n].$$

**Let us consider** $U_2$**.** By (38), we get

$$\mathbb{U}'_k = \{\mathbf{r} \in U \ : \ h(\mathbf{r}) \in [B_{k+1}, A_k)\} \quad \text{with} \quad A_k - B_{k+1} = 2W_3 = 2[\log_2^{10s} n],$$

$$U_2 = \bigcup_{1 \le k \le W_1} \mathbb{U}'_k \quad \text{and}$$

$$\begin{aligned}\#\mathbb{U}'_k &\le n^{s-1} \max_{1 \le r_1, \dots, r_{s-1} \le n} \#\{1 \le r_s \le n \mid B_{k+1} - \rho_r \le r_s \log_2(p_s) < A_k - \rho_r\} \\ &\le 2n^{s-1} \ \log_2^{10s} n.\end{aligned}$$

Hence $\#U_2 \le W_1 2n^{s-1} \ \log_2^{10s} n \ll n^s \ \log_2^{-10s} n$.

**Let us consider** $U_5 = \{\mathbf{r} \in U \ : \ h(\mathbf{r}) < B_{W_1+1}\}$**.** By (38), we get

$$\begin{aligned}B_{W_1+1} &= n - (W_1 + 2)W_2 + W_2 - 2W_3 \\ &= n - [n/W_2]W_2 + W_2 - 2W_3 \le 2W_2 - 2W_3. \qquad (43)\end{aligned}$$

Hence

$$\#\mathbb{U}_5 \le \left\{1 \le r_1, \dots, r_s \le n \ : \ \sum_{i=1}^{s} r_i \log_2(p_i) \le 2W_2 - 2W_3\right\} \le (2W_2)^s \ll \log_2^{20s^2} n.$$

**Let us consider** $U_1^{\searrow,\boldsymbol{\alpha}}$ **(see** (38) **and** (41)**).** It is easy to verify that

$$U_1^{\searrow,\boldsymbol{\alpha}} = \bigcup_{k=1}^{W_1} \mathbb{U}_k^{\searrow,\boldsymbol{\alpha}} \quad \text{and} \quad \mathbb{U}_k^{\cap,\boldsymbol{\alpha}} \cap \mathbb{U}_k^{\searrow,\boldsymbol{\alpha}} = \varnothing.$$

By (38), we get

$$\begin{aligned}\#U_1^{\searrow,\boldsymbol{\alpha}} &\le 2n^{s-1} W_1 \max_{1 \le r_1, \dots, r_{s-1} \le n} \ \max_{1 \le k \le W_1} \ \max_{0 \le \ell \le W_2 - 2W_3} \\ &\qquad \#\big\{1 \le r_s \le n \,\big|\, A_k + \ell - \rho_r - |\alpha_s| \le r_s \log_2(p_s) < A_k + \ell - \rho_r + |\alpha_s|\big\} \\ &\ll n^s \log_2^{-20s+1} n.\end{aligned}$$

The similar estimate is true for $U_1^{\boldsymbol{\alpha},\searrow}$ .

**Let us consider** $U_1$**.** The inequality $\#U_1 \le n^s$ is obvious. It is easy to see that if $\mathbf{r} \in [1, n/(2p_0)]^s$, then

$$\begin{aligned}h(\mathbf{r}) &= r_1 \log_2(p_1) + \dots + r_s \log_2(p_s) \le n \log_2 p_0/(2p_0) < B_1 \\ &= n - 2W_2 + W_2 - W_3.\end{aligned}$$

In view of (38) and (43), we get that

$$B_{W_1+1} \le 2W_2 \quad \text{and} \quad U_1 \cup U_2 \cup [0, 2W_2]^s \supset U \cap [1, n/(2p_0)]^s.$$

Hence Lemma 4.1 is proved. $\square$

Let

$$\chi_{1,\mathbf{m},\mathbf{r}} = \Delta(m_1 P_{\boldsymbol{\alpha}_1-\mathbf{r}_1} = -m_2 P_{\boldsymbol{\alpha}_2-\mathbf{r}_2}), \qquad \chi_{2,\mathbf{m},\mathbf{r}} = \Delta(m_1 P_{\boldsymbol{\alpha}_1-\mathbf{r}_1} = m_2 P_{\boldsymbol{\alpha}_2-\mathbf{r}_2}),$$
$$\chi_{3,\mathbf{m},\mathbf{r}} = 1 - \chi_{1,\mathbf{m},\mathbf{r}} - \chi_{2,\mathbf{m},\mathbf{r}}, \tag{44}$$

and let

$$\mathbf{D}_{\nu,\mu} = \sum_{\mathbf{r}_1,\mathbf{r}_2\in U_\nu} \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, j=1,2}} \sum_{\substack{0<|m_j|\le n^{10s}\\ (m_j,p_0)=1, j=1,2}} \varphi_{\mathbf{r}_1,[\theta N],m_1P_{\boldsymbol{\alpha}_1}} \varphi_{\mathbf{r}_2,[\theta N],m_2P_{\boldsymbol{\alpha}_2}}$$
$$\times \psi_{\mathbf{r}_1}(m_1P_{\boldsymbol{\alpha}_1},\mathbf{y})\psi_{\mathbf{r}_2}(m_2P_{\boldsymbol{\alpha}_2},\mathbf{y}) \tag{45}$$
$$\times e\left(\frac{m_1P_{\boldsymbol{\alpha}_1}}{P_{\mathbf{r}_1}}(\mathcal{X}_{\mathbf{r}_1,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_1,\mathbf{y}}) + \frac{m_2P_{\boldsymbol{\alpha}_2}}{P_{\mathbf{r}_2}}(\mathcal{X}_{\mathbf{r}_2,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_2,\mathbf{y}})\right)\chi_{\mu,\mathbf{m},\mathbf{r}}.$$

Using (31), we have

$$\ddot{\mathcal{D}}_\nu^2([\theta N]) = \mathbf{D}_{\nu,1} + \mathbf{D}_{\nu,2} + \mathbf{D}_{\nu,3}. \tag{46}$$

In the following Lemma 4.2 – Lemma 4.4, we estimate the expectation and variance of the discrepancy in essential and negligible areas. In all these lemmas, we use deep tools from Section 3 ("Diophantine inequalities").

**Lemma 4.2.** *With the notations as above*

$$\ddot{\mathcal{D}}_4([\theta N]) \ll 1, \qquad \ddot{\mathcal{D}}_5([\theta N]) \ll \log_2^{21s^2} n, \quad \mathbf{E}_{\boldsymbol{\theta}}\big(\ddot{\mathcal{D}}_1([\theta N]) + \ddot{\mathcal{D}}_2([\theta N])\big) \ll 1/n,$$
$$\mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}_1^2([\theta N]) \le 65 p_0^{2s+2} n^s, \quad \mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}_2^2([\theta N]) \ll n^s \log^{-10s} n, \qquad \mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,3} \ll 1.$$

P r o o f. **Let us consider $\ddot{\mathcal{D}}_4([\theta N])$.** Using (31), (38) and Lemma 2.1, we get

$$|\ddot{\mathcal{D}}_4([\theta N])| \le \sum_{\mathbf{r}\in U_4} \sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\le i\le s}} \sum_{\substack{0<|m|\le n^{10s}\\ (m,p_0)=1}} \frac{|\sin(2\pi m[\theta N]/P_{\mathbf{r}-\boldsymbol{\alpha}})|}{\bar{m}P_{\boldsymbol{\alpha}}} |\psi_{\mathbf{r}}(mP_{\boldsymbol{\alpha}},\mathbf{y})|.$$

By (38), we have $P_{\mathbf{r}} > 2^{n+W_2} \ge N 2^{\log_2^{20s} n-1}$ for $\mathbf{r} \in U_4$. Bearing in mind that $|\sin(x)| \le |x|$, we obtain from (40)

$$|\ddot{\mathcal{D}}_4([\theta N])| \le \sum_{\mathbf{r}\in U_4} \sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\le i\le s}} \sum_{\substack{0<|m|\le n^{10s}\\ (m,p_0)=1}} \frac{2\pi N}{P_{\mathbf{r}}} p_0 \Delta\big(P_{\mathbf{r}} > N 2^{\log_2^{20s} n-1}\big)$$
$$\ll \log_2^s n\, n^{10s} \sum_{\mathbf{r}\in U_4} \Delta\big(P_{\mathbf{r}} > N 2^{\log_2^{20s} n-1}\big) N/P_{\mathbf{r}}$$
$$\ll \log_2^s n\, n^{10s}\, n^s \times \max_{\mathbf{r}\in U_4} \Delta\big(P_{\mathbf{r}} > N 2^{\log_2^{20s} n-1}\big) N/P_{\mathbf{r}}$$
$$\ll n^s \log_2^s n\, n^{10s}\, n^{-300s} \ll 1. \tag{47}$$

**Let us consider $\ddot{\mathcal{D}}_5([\theta N])$.** From (31), (38), (40), Lemma 2.1 and Lemma 4.1, we derive

$$\begin{aligned}\ddot{\mathcal{D}}_5([\theta N]) &\leq p_0 \sum_{\mathbf{r}\in U_5} \sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\leq i\leq s}} \sum_{\substack{0<|m|\leq n^{10s}\\ (m,p_0)=1}} \frac{1}{\bar{m}P_{\boldsymbol{\alpha}}} \\ &\ll \sum_{\mathbf{r}\in U_5} \sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\leq i\leq s}} \log_2 n \times P_{\boldsymbol{\alpha}}^{-1} \\ &\ll \#U_5 \log_2 n \ll \log_2^{20s^2+1} n.\end{aligned}$$

**Let us consider $\ddot{\mathcal{D}}_1([\theta N]) + \ddot{\mathcal{D}}_2([\theta N])$.** Using (31), (38) and Lemma 2.1, we obtain

$$\begin{aligned}\ddot{\mathcal{D}}_1([\theta N]) + \ddot{\mathcal{D}}_2([\theta N]) \; &= \sum_{\mathbf{r}\in U_1\cup U_2} \sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\leq i\leq s}} \sum_{\substack{0<|m|\leq n^{10s}\\ (m,p_0)=1}} \varphi_{\mathbf{r},[\theta N],mP_{\boldsymbol{\alpha}}} \\ &\times\; \psi_{\mathbf{r}}(mP_{\boldsymbol{\alpha}},\mathbf{y}) e\left(\frac{mP_{\boldsymbol{\alpha}}}{P_{\mathbf{r}}}(\mathcal{X}_{\mathbf{r},\mathbf{x}} - \mathcal{X}_{\mathbf{r},\mathbf{y}})\right),\end{aligned}$$

with

$$\varphi_{\mathbf{r},L,m} = \frac{2i\sin(2\pi mL/P_{\mathbf{r}})}{P_{\mathbf{r}}(e(m/P_{\mathbf{r}})-1)}, \qquad |\varphi_{\mathbf{r},L,m}| \leq \frac{|\sin(2\pi mL/P_{\mathbf{r}})|}{\bar{m}} \leq \frac{1}{\bar{m}}.$$

In view of (88), we have

$$\begin{aligned}\left|\mathbf{E}_{\boldsymbol{\theta}}\big(\ddot{\mathcal{D}}_1([\theta N]) + \ddot{\mathcal{D}}_2([\theta N])\big)\right| &\leq \sum_{\mathbf{r}\in U_1\cup U_2} \sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\leq i\leq s}} \sum_{\substack{0<|m|\leq n^{10s}\\ (m,p_0)=1}} 1 \\ \times \left|\mathbf{E}_{\boldsymbol{\theta}} \sin\left(\frac{2\pi m[\theta N]}{P_{\mathbf{r}-\boldsymbol{\alpha}}}\right)\right| &\frac{1}{\bar{m}P_{\boldsymbol{\alpha}}} |\psi_{\mathbf{r}}(mP_{\boldsymbol{\alpha}},\mathbf{y})| \\ \leq p_0 \sum_{\mathbf{r}\in U_1} \sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\leq i\leq s}} &\sum_{\substack{0<|m|\leq n^{10s}\\ (m,p_0)=1}} \frac{1}{\bar{m}P_{\boldsymbol{\alpha}}} \left(1, \frac{1}{2N\langle\!\langle mP_{-\mathbf{r}+\boldsymbol{\alpha}}\rangle\!\rangle}\right).\end{aligned}$$

By (38), we get $U_1\cup U_2 \subset \{\mathbf{r}\in U \mid h(\mathbf{r}) \in [B_{W_1+1}, B_1)\}$ with

$$\begin{aligned}B_{W_1+1} &= n - (W_1+2)W_2 + W_2 - 2W_3 = n - ([n/W_2]-1)W_2 - 2W_3 \\ &\geq W_2 - 2W_3 > 10sp_0\log_2 n, \\ B_1 &= A_1 + W_2 - 2W_3 = n - 2W_2 + W_2 - 2W_3 = n - W_2 - 2W_3. \qquad (48)\end{aligned}$$

Hence

$$2^{\log_2^{20s} n} \leq P_{\mathbf{r}} \leq 2^{n-\log_2^{10s} n} \quad \text{for } \mathbf{r}\in U_1.$$

Taking into account that $|m| \leq n^{10s}$, we obtain

$$\left|\mathbf{E}_{\boldsymbol{\theta}}\left(\ddot{\mathcal{D}}_1([\theta N]) + \ddot{\mathcal{D}}_2([\theta N])\right)\right| \ll \sum_{\mathbf{r}\in U_1\cup U_2} \sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\leq i\leq s}} \sum_{\substack{0<|m|\leq n^{10s}\\ (m,p_0)=1}} \frac{1}{\bar{m}^2 P_{\boldsymbol{\alpha}}} \frac{P_{\mathbf{r}-\boldsymbol{\alpha}}}{N}$$

$$\ll n^s 2^{-\log_2^{10s} n} \ll 1/n.$$

**Let us consider $\mathbf{D}_{\nu,\mu}$ $(\nu,\mu = 1,2)$.** Suppose that $\chi_{\mu,\mathbf{m},\mathbf{r}} \geq 1$. Bearing in mind that $(m_j,p_0) = 1$, $j = 1,2$, we have from (44) that $|m_1| = |m_2|$ and $\mathbf{r}_2 = \mathbf{r}_1+\boldsymbol{\alpha}_2-\boldsymbol{\alpha}_1$. Applying (45) and Lemma 2.1, we obtain

$$|\mathbf{D}_{\nu,\mu}| \leq \sum_{\mathbf{r}_1,\mathbf{r}_2\in U_\nu} \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\leq i\leq s, j=1,2}}$$

$$\sum_{\substack{0<|m_j|\leq n^{10s}\\ (m_j,p_0)=1, j=1,2}} \frac{p_0^2}{\bar{m}_1\bar{m}_2 P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}} \Delta(|m_1| = |m_2|)\Delta(\mathbf{r}_2 = \mathbf{r}_1 + \boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_1)$$

$$\leq \#U_\nu \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\leq i\leq s, j=1,2}} \sum_{\substack{0<|m_1|\leq n^{10s}\\ (m_1,p_0)=1}} \frac{2p_0^2}{\bar{m}_1^2 P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}} \leq 32 p_0^{2s+2} \#U_\nu.$$

From Lemma 4.1, we get $\#U_1 \leq n^s$, $\#U_2 \leq n^s \log_2^{-10s} n$,

$$\mathbf{D}_{1,\mu} \leq 32 p_0^{2s+2} n^s \qquad \text{and} \qquad \mathbf{D}_{2,\mu} \ll n^s/\log_2^{10s} n, \qquad \mu = 1,2. \tag{49}$$

**Let us consider $\mathbf{D}_{\nu,3}$ $(\nu = 1,2)$.** Applying (45) and Lemma 2.1, we get

$$|\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{\nu,3}| \ll \sum_{\mathbf{r}_1,\mathbf{r}_2\in U_\nu} \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\leq i\leq s, j=1,2}}$$

$$\sum_{\substack{0<|m_j|\leq n^{10s}\\ (m_j,p_0)=1, j=1,2}} \left|\mathbf{E}_{\boldsymbol{\theta}}\left(\varphi_{\mathbf{r}_1,[\theta N],m_1 P_{\boldsymbol{\alpha}_1}} \times \varphi_{\mathbf{r}_2,[\theta N],m_2 P_{\boldsymbol{\alpha}_2}}\right)\right| \chi_{3,\mathbf{m},\mathbf{r}}$$

$$\ll \sum_{\mathbf{r}_1,\mathbf{r}_2\in U_\nu} \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\leq i\leq s, j=1,2}} \sum_{\substack{0<|m_j|\leq n^{10s}\\ (m_j,p_0)=1, j=1,2}} 1$$

$$\times \left|P_{\mathbf{r}_1}\left(e(m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}) - 1\right)\right|^{-1} \left|P_{\mathbf{r}_2}\left(e(m_2/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2}) - 1\right)\right|^{-1}$$

$$\times \left|\mathbf{E}_{\boldsymbol{\theta}}\left(\sin(2\pi m_1[\theta N]/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}) \sin(2\pi m_2[\theta N]/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2})\right)\right| \chi_{3,\mathbf{m},\mathbf{r}}.$$

Using (88) and Lemma 2.1, we derive

$$|\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{\nu,3}| \ll \sum_{\mathbf{r}_1,\mathbf{r}_2\in U_\nu} \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, j=1,2}} \sum_{\substack{0<|m_j|\le n^{10s}\\ (m_j,p_0)=1, j=1,2}} \frac{1}{\bar{m}_1\bar{m}_2 P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}} \times$$

$$\sum_{\ell=-1,1} \min\left(1, \frac{1}{2N\langle\!\langle m_1 P_{-\mathbf{r}_1+\boldsymbol{\alpha}_1} + \ell m_2 P_{-\mathbf{r}_2+\boldsymbol{\alpha}_2}\rangle\!\rangle}\right)\chi_{3,\mathbf{m},\mathbf{r}}. \quad (50)$$

By (38) and (48), we get $W_2 = [\log_2^{20s} n]$,

$$W_2-2W_3 < A_{W_1}-W_3 = B_{W_1+1} \le h_{\mathbf{r}_j} \le B_1 = n-W_2-2W_3 < n-W_2 \quad \text{for } \mathbf{r}_\nu \in U_\nu,$$

$$P_{\mathbf{r}_j} \ge 2^{W_2-2W_3}, \;\; N/P_{\mathbf{r}_j} \ge 2^{W_2} \ge 2^{\log^{20s} n-1} \text{ and } m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1} \ne \ell m_2/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2} \quad (51)$$

$$\text{for } \ell = -1,1, \quad \nu, j = 1,2 \quad \text{and} \quad \chi_{3,\mathbf{m},\mathbf{r}} > 0.$$

From (30), we obtain $0 < |m_j/P_{\mathbf{r}_j-\alpha_j}| < 1/32$ for $\mathbf{r}_\nu \in U_\nu$. Using Corollary 3.4 (the theorem on linear forms in the logarithm), we have

$$N\langle\!\langle m_1 P_{-\mathbf{r}_1+\boldsymbol{\alpha}_1} + (-1)^\ell m_2 P_{-\mathbf{r}_2+\boldsymbol{\alpha}_2}\rangle\!\rangle = N|m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1} + (-1)^\ell m_2/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2}|$$

$$\ge \frac{N}{2}\left(P_{-\mathbf{r}_2+\boldsymbol{\alpha}_2}|m_2|\left|\frac{|m_1|}{|m_2|}P_{\mathbf{r}_2-\mathbf{r}_1+\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2} - 1\right| + P_{-\mathbf{r}_1+\boldsymbol{\alpha}_1}|m_1|\,\left|\frac{|m_2|}{|m_1|}P_{\mathbf{r}_1-\mathbf{r}_2+\boldsymbol{\alpha}_2-\boldsymbol{\alpha}_1} - 1\right|\right)$$

$$\ge N/2\min\bigl(P_{-\mathbf{r}_2+\boldsymbol{\alpha}_2}|m_2|, P_{-\mathbf{r}_1+\boldsymbol{\alpha}_1}|m_1|\bigr)\exp\left(-C_1\ln^2 n\right)$$

$$\ge 2^{W_2-1}\exp\left(-C_1\ln^2 n\right) \ge 2^{\log_2^{20s} n-4}\exp\left(-C_1\ln^2 n\right) \gg n^{100s}.$$

We get from (50) that

$$|\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{\nu,3}| \ll \sum_{\mathbf{r}_1,\mathbf{r}_2\in U_\nu} \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, j=1,2}} \sum_{\substack{0<|m_j|\le n^{10s}\\ (m_j,p_0)=1, j=1,2}} \frac{1}{\bar{m}_1\bar{m}_2} n^{-100s} \ll 1, \quad \nu = 1,2. \quad (52)$$

By (46) $\ddot{\mathcal{D}}^2_\nu([\theta N]) = \mathbf{D}_{\nu,1} + \mathbf{D}_{\nu,2} + \mathbf{D}_{\nu,3}$ $(\nu = 1,2)$. Taking into account (49) and (52), we obtain that

$$\mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}^2_1([\theta N]) \le 65 p_0^{2s+2} n^s, \quad \mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}^2_2([\theta N]) \ll n^s \log^{-10s} n, \quad \mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{\nu,3} \ll 1.$$

Hence Lemma 4.2 is proved. □

In the following Lemma 4.3, we calculate the variance of $\ddot{\mathcal{D}}_3$ (see (31) and (38)). We can't obtain an estimate for everyone $\mathbf{x}$. We will get an estimate for almost all $\mathbf{x} \in [0,1)^s$. For the proof, we will use Theorem 3.1 (S-unit theorem).

**Lemma 4.3.** *With the notations as above*

$$\mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}_3^2([\theta N]) \ll n^{s-1/8} \qquad \textit{for a.s. } \mathbf{x}.$$

Proof. By (31) and (38), we get

$$\mathbf{E}_x\mathbf{E}_{\boldsymbol{\theta}}^2\ddot{\mathcal{D}}_3^2([\theta N]) = \sum_{\substack{\mathbf{r}_j\in U_3\\ j\in[1,4]}} \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, j\in[1,4]}} \sum_{\substack{0<|m_j|\le n^{10s}\\ (m_j,p_0)=1, j\in[1,4]}} \vartheta$$
$$\times \prod_{1\le j_1\le 2}\mathbf{E}_{\boldsymbol{\theta}}\Bigg(\prod_{1\le j_2\le 2}\varphi_{\mathbf{r}_{j_2+2(j_1-1)},[\theta N],m_{j_2+2(j_1-1)}P_{\boldsymbol{\alpha}_{j_2+2(j_1-1)}}}\Bigg)$$
$$\times \prod_{1\le j\le 4}\psi_{\mathbf{r}_j}(m_jP_{\boldsymbol{\alpha}_j},\mathbf{y})$$

with

$$\vartheta = \mathbf{E}_x e\Bigg(\sum_{1\le j\le 4}\frac{m_jP_{\boldsymbol{\alpha}_j}}{P_{\mathbf{r}_j}}(\mathcal{X}_{\mathbf{r}_j,\mathbf{x}}-\mathcal{X}_{\mathbf{r}_j,\mathbf{y}})\Bigg). \tag{53}$$

From Lemma 2.3, we obtain

$$\vartheta = \Delta\Bigg(\sum_{1\le j\le 4}m_jP_{-\mathbf{r}_j+\boldsymbol{\alpha}_j}=0\Bigg).$$

By Lemma 4.1, $\#U_3 \ll n^{s-1}\log^{40s} n$. Applying Corollary 3.2 (S-unit theorem), we obtain

$$\sum_{\mathbf{r}_j\in U_3,\,1\le j\le 4}\Delta\Bigg(\sum_{1\le j\le 4}m_jP_{-\mathbf{r}_j+\boldsymbol{\alpha}_j}=0\Bigg) \ll (\#U_3)^2 \ll n^{2s-2}\log_2^{80s} n. \tag{54}$$

From (53) – (54) and Lemma 2.1, we get

$$\mathbf{E}_x\mathbf{E}_{\boldsymbol{\theta}}^2\ddot{\mathcal{D}}_3^2([\theta N]) \ll \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n],\ 0<|m_j|\le n^{10s}\\ (m_j,p_0)=1,\ j\in[1,4],\ 1\le i\le s}} \frac{n^{2(s-1)}\log_2^{80s} n}{\bar{m}_1\cdots\bar{m}_4P_{\boldsymbol{\alpha}_1+\cdots+\boldsymbol{\alpha}_4}} \ll n^{2s-7/4}.$$

By Chebyshev's inequality, we have

$$\mathrm{Prob}\big(\mathbf{x}\in[0,1)^s \mid \mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}_3^2([\theta N]) > n^{s-1/8}\big) \le$$
$$\mathbf{E}_x\left(\mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}_3^2([\theta N])\right)^2/n^{2s-1/4} \ll n^{2s-7/4}/n^{2s-1/4} = n^{-3/2}. \tag{55}$$

Now, using the Borel-Cantelli lemma, we get the assertion of Lemma 4.3. □

The proof of the following Lemma 4.4 is similar to the proof of Lemma 4.3. The difference is that in Lemma 4.3 we consider only four moment of discrepancy, whereas in Lemma 4.4 we are forced to consider the eighth moment.

**Lemma 4.4.** *With the notations as above*

$$\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,2} \ll n^{s-1/16} \quad \text{for } a.s. \ \mathbf{x}, \ s \geq 1.$$

P r o o f. According to the Borel-Cantelli lemma, it is sufficient to prove that $\mathrm{Prob}(\mathbf{x} \in [0,1)^s \mid (\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,2})^4 > n^{4(s-1/16)}) \ll n^{-5/4}$. Using Chebyshev's inequality, we get that it is enough to verify that $\mathbf{E}_x(\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,2})^4 \ll n^{4(s-1/16)-5/4} = n^{4s-3/2}$. We will prove that

$$\mathbf{E}_x(\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,2})^4 \ll n^{2s} < n^{4s-3/2}.$$

By (44) and (45), we have that the expression of $(\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,2})^4$ includes the summation over

$$\mathbf{r}_1,\dots,\mathbf{r}_8,\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_8,m_1,\dots,m_8 \ \text{ with } \ m_j/P_{\mathbf{r}_j-\boldsymbol{\alpha}_j} = m_{j+4}/P_{\mathbf{r}_{j+4}-\boldsymbol{\alpha}_{j+4}}, \ \ 1 \leq j \leq 4.$$

Hence

$$m_j = m_{j+4}, \qquad \mathbf{r}_j - \boldsymbol{\alpha}_j = \mathbf{r}_{j+4} - \boldsymbol{\alpha}_{j+4}, \qquad 1 \leq j \leq 4.$$

From (45), we derive

$$\mathbf{E}_x(\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,2})^4 \leq \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\leq i\leq s,1\leq j\leq 8}} \sum_{\substack{\mathbf{r}_j\in U_1\\ 1\leq j\leq 8}} \sum_{\substack{0<|m_j|\leq n^{10s}\\ (m_j,p_0)=1,1\leq j\leq 8}} \prod_{j=1}^{4}\Bigg(\Big|\mathbf{E}_{\boldsymbol{\theta}}\big(\varphi_{\mathbf{r}_j,[\theta N],m_jP_{\boldsymbol{\alpha}_j}}$$
$$\times \varphi_{\mathbf{r}_{j+4},[\theta N],m_{j+4}P_{\boldsymbol{\alpha}_{j+4}}}\big)\Big|\Delta(m_j = m_{j+4})\Delta(\mathbf{r}_j - \boldsymbol{\alpha}_j = \mathbf{r}_{j+4} - \boldsymbol{\alpha}_{j+4})\Bigg)$$
$$\times \prod_{j=1}^{8}|\psi_{\mathbf{r}_j}(m_jP_{\boldsymbol{\alpha}_j},\mathbf{y})| \ \left|\mathbf{E}_x e\left(\sum_{1\leq j\leq 8}\frac{m_jP_{\boldsymbol{\alpha}_j}}{P_{\mathbf{r}_j}}(\mathcal{X}_{\mathbf{r}_j,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_j,\mathbf{y}})\right)\right|.$$

Hence

$$\mathbf{E}_x(\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,2})^4 \ll \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\leq i\leq s,1\leq j\leq 8}} \sum_{\substack{\mathbf{r}_j\in U_1\\ 1\leq j\leq 8}} \sum_{\substack{0<|m_j|\leq n^{10s}\\ (m_j,p_0)=1,1\leq j\leq 8}} \frac{1}{m_1^2\cdots m_4^2 P_{\boldsymbol{\alpha}_1+\cdots+\boldsymbol{\alpha}_8}}$$
$$\times \prod_{j=1}^{4}\Delta(m_j = m_{j+4})\Delta(\mathbf{r}_j - \boldsymbol{\alpha}_j = \mathbf{r}_{j+4} - \boldsymbol{\alpha}_{j+4})$$
$$\times \left|\mathbf{E}_x e\left(\sum_{1\leq j\leq 8}\frac{m_jP_{\boldsymbol{\alpha}_j}}{P_{\mathbf{r}_j}}(\mathcal{X}_{\mathbf{r}_j,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_j,\mathbf{y}})\right)\right|. \tag{56}$$

In view of Lemma 2.3, we have

$$\mathbf{E}_x e\left(\sum_{j=1}^{8}\frac{m_jP_{\boldsymbol{\alpha}_j}}{P_{\mathbf{r}_j}}(\mathcal{X}_{\mathbf{r}_j,\mathbf{x}}-\mathcal{X}_{\mathbf{r}_j,\mathbf{y}})\right) = \Delta\left(\sum_{j=1}^{8} m_jP_{-\mathbf{r}_j+\boldsymbol{\alpha}_j}=0\right) = \Delta\left(\sum_{j=1}^{4} 2m_jP_{-\mathbf{r}_j+\boldsymbol{\alpha}_j}=0\right).$$

By Lemma 4.1, we get $\#U_1 \le n^s$. Applying Corollary 3.2, we obtain

$$\sum_{\mathbf{r}_j \in U_1,\ 1 \le j \le 8} \Delta\left(\sum_{1 \le j \le 4} 2m_j P_{-\mathbf{r}_j+\boldsymbol{\alpha}_j} = 0\right) \prod_{j=1}^{4} \Delta(\mathbf{r}_j - \boldsymbol{\alpha}_j = \mathbf{r}_{j+4} - \boldsymbol{\alpha}_{j+4}) \ll \#U_1^2 \ll n^{2s}.$$

From (56), we obtain

$$\mathbf{E}_x(\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,2})^4 \le n^{2s} \sum_{\substack{\alpha_{j,i} \in [0, 10s \log_2 n] \\ 1 \le i \le s, 1 \le j \le 8}} \sum_{\substack{0 < |m_j| \le n^{10s} \\ (m_j, p_0) = 1, 1 \le j \le 4}} \frac{1}{m_1^2 \cdots m_4^2 P_{\boldsymbol{\alpha}_1 + \cdots + \boldsymbol{\alpha}_8}} \ll n^{2s}.$$

Hence Lemma 4.4 is proved. □

## 5. Lower bound of the variance of the discrepancy $\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])$

In Lemma 5.2, we will prove that $\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,1} \ge \kappa_3 n^s$, with some $\kappa_3 > 0$. This is the main result of the section. Lemma 5.1 is auxiliary. In Lemma 5.3, we obtain the lower and upper bounds of the variance of $\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])$.

In order to prove Lemma 5.2, in Lemma 5.1 we have summed over $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ at the end. Further, direct calculation allows us to obtain a lower bound of the variance.

**Lemma 5.1.** *Let*

$$G_{\mathbf{r}} := \sum_{\substack{0 < |m| \le n^{10s} \\ (m, p_0) = 1}} \frac{1}{2\pi^2 |m|^2} \left| \sum_{\substack{\alpha_i \in [0, 10s \log_2 n] \\ 1 \le i \le s}} \psi_{\mathbf{r}+\boldsymbol{\alpha}}(mP_{\boldsymbol{\alpha}}, \mathbf{y})/P_{\boldsymbol{\alpha}} \right|^2 . \tag{57}$$

*Then*

$$\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,1} = \sum_{\mathbf{r} \in U_1} G_{\mathbf{r}} + O(n^s \log^{-19s} n), \tag{58}$$

*where* $\mathbf{D}_{1,1}$ *is defined in* (45).

P r o o f. By (44), we have that if $\chi_{1,\mathbf{m},\mathbf{r}} = 1$, then $m_1 = -m_2$ and $\mathbf{r}_2 = \mathbf{r}_1 - \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$. Applying (45) and Lemma 2.1, we get

$$\frac{m_1}{P_{\mathbf{r}_1 - \boldsymbol{\alpha}_1}}(\mathcal{X}_{\mathbf{r}_1,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_1,\mathbf{y}}) \equiv -\frac{m_2}{P_{\mathbf{r}_2 - \boldsymbol{\alpha}_2}}(\mathcal{X}_{\mathbf{r}_2,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_2,\mathbf{y}}) \bmod 1$$

and

$$\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,1} = \sum_{\mathbf{r}_1,\mathbf{r}_2\in U_1} \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, j=1,2}} \sum_{\substack{0<|m_j|\le n^{10s}\\ (m_j,p_0)=1, j=1,2}} \mathbf{E}_{\boldsymbol{\theta}}(\varphi_{\mathbf{r}_1,[\theta N],m_1 P_{\boldsymbol{\alpha}_1}}$$

$$\times \varphi_{\mathbf{r}_2,[\theta N],m_2 P_{\boldsymbol{\alpha}_2}})e\left(\frac{m_1}{P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}}(\mathcal{X}_{\mathbf{r}_1,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_1,\mathbf{y}}) + \frac{m_2}{P_{\mathbf{r}_2-\boldsymbol{\alpha}_2}}(\mathcal{X}_{\mathbf{r}_2,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_2,\mathbf{y}})\right)$$

$$\times \psi_{\mathbf{r}_1}(m_1 P_{\boldsymbol{\alpha}_1},\mathbf{y})\psi_{\mathbf{r}_2}(m_2 P_{\boldsymbol{\alpha}_2},\mathbf{y})\ \chi_{1,\mathbf{m},\mathbf{r}}$$

$$= \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, j=1,2}} \sum_{\mathbf{r}_1,\mathbf{r}_2=\mathbf{r}_1-\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2\in U_1} \sum_{\substack{0<|m_1|\le n^{10s}\\ (m_1,p_0)=1}} \mathbf{E}_{\boldsymbol{\theta}}(|\varphi_{\mathbf{r}_1,[\theta N],m_1 P_{\boldsymbol{\alpha}_1}}|^2)$$

$$\times \psi_{\mathbf{r}_1}(m_1 P_{\boldsymbol{\alpha}_1},\mathbf{y})\psi_{\mathbf{r}_1-\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}(-m_1 P_{\boldsymbol{\alpha}_2},\mathbf{y})$$

$$= \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, j=1,2}} \sum_{\mathbf{r}_1,\mathbf{r}_2=\mathbf{r}_1-\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2\in U_1} \sum_{\substack{0<|m_1|\le n^{10s}\\ (m_1,p_0)=1}} \frac{4\mathbf{E}_{\boldsymbol{\theta}}\sin^2(2\pi m_1[\theta N]/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1})}{P_{\mathbf{r}_1+\mathbf{r}_2}|e(m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1})-1|^2}$$

$$\times \psi_{\mathbf{r}_1}(m_1 P_{\boldsymbol{\alpha}_1},\mathbf{y})\psi_{\mathbf{r}_1-\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}(-m_1 P_{\boldsymbol{\alpha}_2},\mathbf{y}). \tag{59}$$

Taking into account that

$$|m_1| \le n^{10s}, \qquad \max_{i,j}\alpha_{j,i} \le 10s\log_2 n$$

and

$$\min_j \max_i r_{j,i} \ge W_0 = 50s^2 p_0 \log_2 n,$$

we have

$$|m_1|^3 P_{-\mathbf{r}_1+2\boldsymbol{\alpha}_1+\alpha_2}| \ll n^{-15s}$$

and

$$e(m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}) - 1 = 2\pi i m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1} + O\big((m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1})^2\big),$$

$$P_{\mathbf{r}_1}|e(m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}) - 1| = 2\pi|m_1|P_{\boldsymbol{\alpha}_1} + O(m_1^2 P_{-\mathbf{r}_1+2\boldsymbol{\alpha}_1}),$$

$$\begin{aligned}
P_{\mathbf{r}_1+\mathbf{r}_2}|e(m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}) - 1|^2 &= P_{\boldsymbol{\alpha}_2-\boldsymbol{\alpha}_1}\big(P_{\mathbf{r}_1}|e(m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}) - 1|\big)^2\\
&= P_{\boldsymbol{\alpha}_2-\boldsymbol{\alpha}_1}\big(4\pi^2|m_1|^2 P_{2\boldsymbol{\alpha}_1} + O(|m_1|^3 P_{-\mathbf{r}_1+3\boldsymbol{\alpha}_1})\big)\\
&= 4\pi^2|m_1|^2 P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2} + O(|m_1|^3 P_{-\mathbf{r}_1+2\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2})\\
&= 4\pi^2|m_1|^2 P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2} + O(n^{-15s}),
\end{aligned}$$

where $\mathbf{r}_2 = \mathbf{r}_1 - \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$.

Hence

$$\left|\frac{1}{P_{\mathbf{r}_1+\mathbf{r}_2}(e(m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1})-1)}\right|^2 - \left|\frac{1}{4\pi^2|m_1|^2 P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}}\right|^2 = \frac{P_{\mathbf{r}_1+\mathbf{r}_2}|e(m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1})-1|^2 - 4\pi^2|m_1|^2 P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}}{P_{\mathbf{r}_1+\mathbf{r}_2}|e(m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1})-1|^2 4\pi^2|m_1|^2 P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}} \ll n^{-15s}.$$

Using (30), we obtain $|m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}| \le 1/32$ and $\langle m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}\rangle = |m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}|$. By (51), we have

$$P_{\mathbf{r}_1}/N = O(n^{-30s}) \quad \text{for } \mathbf{r}_1 \in U_1.$$

In view of (88), we derive

$$\begin{aligned}\mathbf{E}_{\boldsymbol{\theta}}\sin^2(2\pi m_1[\theta N]/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}) &= 1/2 + O\left(\frac{1}{N\langle m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}\rangle}\right)\\ &= 1/2 + O(P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}/N) = 1/2 + O(n^{-30s})\end{aligned}$$

and

$$\begin{aligned}\frac{4\mathbf{E}_{\boldsymbol{\theta}}\sin^2(2\pi m_1[\theta N]/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1})}{P_{\mathbf{r}_1+\mathbf{r}_2}|e(m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1})-1|^2} &= \frac{1+O(n^{-30s})}{2\pi^2|m_1|^2 P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2} + O(n^{-15s})}\\ &= \frac{1}{2\pi^2|m_1|^2 P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}} + O(n^{-15s}).\end{aligned}$$

From (59), we get

$$\begin{aligned}\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,1} = &\sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, j=1,2}} \sum_{\mathbf{r}_1,\mathbf{r}_1-\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2\in U_1} \sum_{\substack{0<|m_1|\le n^{10s}\\ (m_1,p_0)=1}} \frac{1}{2\pi^2|m|^2 P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_1}}\\ &\times\ \psi_{\mathbf{r}_1}(m_1 P_{\boldsymbol{\alpha}_1},\mathbf{y})\psi_{\mathbf{r}_1-\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}(-m_1 P_{\boldsymbol{\alpha}_2},\mathbf{y}) + O(n^{-10s}).\end{aligned} \tag{60}$$

Let us consider the summation area

$$\mathbf{r}_1, \mathbf{r}_1 - \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 \in U_1.$$

Let $\mathbf{r} = \mathbf{r}_1 - \boldsymbol{\alpha}_1$. We have $\mathbf{r}+\boldsymbol{\alpha}_1, \mathbf{r}+\boldsymbol{\alpha}_2 \in U_1$, hence $\mathbf{r} \in (U_1 \dot{-} \boldsymbol{\alpha}_1) \cap (U_1 \dot{-} \boldsymbol{\alpha}_2)$.

Using (41), (42) and Lemma 4.1, we derive

$$(U_1\dot{-}\boldsymbol{\alpha}_1)\cap(U_1\dot{-}\boldsymbol{\alpha}_2) = (U_1 \cup \dot{U}_1)\setminus \ddot{U}_1 \quad \text{with} \quad \#\dot{U}_1 + \#\ddot{U}_1 \ll n^s \log_2^{-19s} n. \tag{61}$$

Bearing in mind that $|\psi_{\mathbf{r}_j+\boldsymbol{\alpha}_j}(m_j P_{\boldsymbol{\alpha}_j},\mathbf{y})| \le p_0$ $(j=1,2)$, we obtain

$$\sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, j=1,2}} \sum_{\mathbf{r}\in\mathfrak{U}} \sum_{\substack{0<|m|\le n^{10s}\\ (m,p_0)=1}} \frac{1}{2\pi^2|m|^2 P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_1}} \times$$

$$\psi_{\mathbf{r}+\boldsymbol{\alpha}_1}(m_1 P_{\boldsymbol{\alpha}_1},\mathbf{y})\,\psi_{\mathbf{r}+\boldsymbol{\alpha}_2}(-m_1 P_{\boldsymbol{\alpha}_2},\mathbf{y}) \ll \#\mathfrak{U}. \quad (62)$$

In view of (60)–(62), we have

$$\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,1} = \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, j=1,2}} \sum_{\mathbf{r}\in(U_1\cup\dot{U}_1)\setminus\ddot{U}_1} \sum_{\substack{0<|m|\le n^{10s}\\ (m,p_0)=1}} \frac{\psi_{\mathbf{r}+\boldsymbol{\alpha}_1}(mP_{\boldsymbol{\alpha}_1},\mathbf{y})}{2\pi^2|m|^2 P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_1}}$$

$$\times\,\psi_{\mathbf{r}+\boldsymbol{\alpha}_2}(-mP_{\boldsymbol{\alpha}_2},\mathbf{y}) + O(n^s\log^{-19s} n)$$

$$= \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, j=1,2}} \sum_{\mathbf{r}\in U_1} \sum_{\substack{0<|m|\le n^{10s}\\ (m,p_0)=1}} \frac{\psi_{\mathbf{r}+\boldsymbol{\alpha}_1}(mP_{\boldsymbol{\alpha}_1},\mathbf{y})\psi_{\mathbf{r}+\boldsymbol{\alpha}_2}(-mP_{\boldsymbol{\alpha}_2},\mathbf{y})}{2\pi^2|m|^2 P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_1}}$$

$$+\,O(n^s\log^{-19s} n).$$

Taking into account that

$$G_{\mathbf{r}} = \sum_{\substack{0<|m|\le n^{10s}\\ (m,p_0)=1}} \frac{1}{2\pi^2|m|^2}\left|\sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\le i\le s}} \psi_{\mathbf{r}+\boldsymbol{\alpha}}(mP_{\boldsymbol{\alpha}},\mathbf{y})/P_{\boldsymbol{\alpha}}\right|^2 \quad \text{(see (57))},$$

we get

$$\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,1} = \sum_{\mathbf{r}\in U_1} G_{\mathbf{r}} + O(n^s\log^{-19s} n).$$

Hence Lemma 5.1 is proved. □

Although the following Lemma 5.2 is the main part of the estimate of the lower bound for the variance of the discrepancy, we do not use any special number theory tools in the proof. We only use direct calculations. Almost the entire proof is devoted to calculating the lower bound of the following part of $G_{\mathbf{r}}$:

$$\sum_{\substack{1\le m<p_0\\ (m,p_0)=1}} \left|\sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\le i\le s}} \psi_{\mathbf{r}+\boldsymbol{\alpha}}(mP_{\boldsymbol{\alpha}},\mathbf{y})/P_{\boldsymbol{\alpha}}\right|^2 .$$

**Lemma 5.2.** *There exists $n_1>0$ such that*

$$\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,1} \ge \kappa_3 n^s \quad \text{with} \quad \kappa_3 = \pi^{-2}p_0^{-7-s}2^{-s}\kappa_1^{2s}\kappa_2^{s}, \quad \text{for} \quad n\ge n_1. \quad (63)$$

P r o o f. By Lemma 2.1, we get

$$\psi_{\mathbf{r}}(m,\mathbf{y}) = \prod_{i=1}^{s} \dot{\psi}(i,\{-mM_{i,\mathbf{r}}/p_i\}p_i, y_{i,r_i}), \quad \dot{\psi}(i,m,0) = 0,$$
$$\dot{\psi}(i,m,y_{i,r_i}) = \sum_{0\le b<y_{i,r_i}} e\big(m(b-y_{i,r_i})/p_i\big) \qquad \text{for } y_i > 0.$$

From (57), we obtain

$$20\pi^2 p_0^3 (2p_0)^{2s} \ge 2\pi^2 p_0^3 G_{\mathbf{r}} \ge \sum_{\substack{1\le m<p_0\\(m,p_0)=1}} \Bigg| \sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\1\le i\le s}} \psi_{\mathbf{r}+\boldsymbol{\alpha}}(mP_{\boldsymbol{\alpha}},\mathbf{y})/P_{\boldsymbol{\alpha}} \Bigg|^2$$
$$= \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\1\le i\le s, j=1,2}} \frac{1}{P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}}$$
$$\times \sum_{\substack{1\le m<p_0\\(m,p_0)=1}} \prod_{1\le i\le s} \prod_{1\le j\le 2} \dot{\psi}\Big(i,\{(-1)^j mP_{\boldsymbol{\alpha}_j} M_{i,\mathbf{r}+\boldsymbol{\alpha}_j}/p_i\}p_i, y_{i,r_i+\alpha_{j,i}}\Big). \tag{64}$$

By the Chinese Remainder Theorem, we get

$$\{0\le m<p_0\} = \Bigg\{ \sum_{1\le i\le s} m_i p_0/p_i \bmod p_0 \mid 0\le m_i<p_i,\ i\in[1,s] \Bigg\},$$

with $p_0 = p_1p_2\cdots p_s$. Hence

$$\{0\le m<p_0 \mid (m,p_0)=1\} = \Bigg\{ \sum_{1\le i\le s} m_i p_0/p_i \bmod p_0 \mid 1\le m_i<p_i,\ i\in[1,s] \Bigg\}.$$

Therefore,

$$2\pi^2 p_0^3 G_{\mathbf{r}} \ge \hbar_{\mathbf{r}} := \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\1\le i\le s, j=1,2}} \frac{1}{P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}} \prod_{1\le i\le s} \wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2},$$

where

$$\wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2} = \sum_{1\le m_i<p_i} \prod_{1\le j\le 2} \dot{\psi}\big(i,\{(-1)^j m_i P_{\boldsymbol{\alpha}_j} M_{i,\mathbf{r}+\boldsymbol{\alpha}_j} p_0/p_i^2\}p_i, y_{i,r_i+\alpha_{j,i}}\big)$$
$$= \sum_{1\le m_i<p_i} \prod_{1\le j\le 2} \sum_{b_j=0}^{y_{i,r_i+\alpha_{j,i}}-1} e\big((b_j - y_{i,r_i+\alpha_{j,i}})(-1)^j m_i P_{\boldsymbol{\alpha}_j} M_{i,\mathbf{r}+\boldsymbol{\alpha}_j} p_0/p_i^2\big). \tag{65}$$

We have

$$\hbar_{\mathbf{r}} = \prod_{1\le i\le s} \hbar_{\mathbf{r},i} \quad \text{with} \quad \hbar_{\mathbf{r},i} = \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\j=1,2}} \wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2}/p_i^{\boldsymbol{\alpha}_{1,i}+\boldsymbol{\alpha}_{2,i}}. \tag{66}$$

Let $c_j = y_{i,r_i+\alpha_{j,i}} - b_j$. In view of (8), we have

$$\wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2} = \sum_{c_1=1}^{y_{i,r_i+\alpha_{1,i}}} \sum_{c_2=1}^{y_{i,r_i+\alpha_{2,i}}} \left( \sum_{m_i=0}^{p_i-1} e\left( \sum_{j=1}^{2} c_j(-1)^{j+1} m_i P_{\boldsymbol{\alpha}_j} M_{i,\mathbf{r}+\boldsymbol{\alpha}_j} p_0/p_i^2 \right) - 1 \right)$$

$$= \sum_{c_1=1}^{y_{i,r_i+\alpha_{1,i}}} \sum_{c_2=1}^{y_{i,r_i+\alpha_{2,i}}} \left( p_i \delta_{p_i} \left( \sum_{j=1}^{2} c_j(-1)^{j+1} P_{\boldsymbol{\alpha}_j} M_{i,\mathbf{r}+\boldsymbol{\alpha}_j} p_0/p_i \right) - 1 \right).$$

According to (15), we obtain $M_{i,\mathbf{r}} \equiv \left(P_{\mathbf{r}}/p_i^{r_i}\right)^{-1} \bmod p_i^{r_i}$ and $M_{i,\mathbf{r}} \not\equiv 0 \bmod p_i$. Hence

$$M_{i,\mathbf{r}} \equiv \left(P_{\mathbf{r}}/p_i^{r_i}\right)^{-1} \bmod p_i,$$

$$(P_{\mathbf{r}+\boldsymbol{\alpha}_j} p_i^{-r_i-\alpha_{j,i}}) M_{i,\mathbf{r}+\boldsymbol{\alpha}_j} \equiv 1 \bmod p_i, \qquad (P_{\mathbf{r}} p_i^{-r_i})(P_{\boldsymbol{\alpha}_j} p_i^{-\alpha_{j,i}} M_{i,\mathbf{r}+\boldsymbol{\alpha}_j}) \equiv 1 \bmod p_i,$$

$$P_{\boldsymbol{\alpha}_j} p_i^{-\alpha_{j,i}} M_{i,\mathbf{r}+\boldsymbol{\alpha}_j} \equiv M_{i,\mathbf{r}} \bmod p_i,$$

and

$$P_{\boldsymbol{\alpha}_j} M_{i,\mathbf{r}+\boldsymbol{\alpha}_j} \equiv p_i^{\alpha_{j,i}} M_{i,\mathbf{r}} \bmod p_i \quad \text{with} \quad M_{i,\mathbf{r}} \not\equiv 0 \bmod p_i, \quad j = 1, 2.$$

Therefore

$$\delta_{p_i} \left( \sum_{j=1,2} c_j(-1)^{j+1} P_{\boldsymbol{\alpha}_j} M_{i,\mathbf{r}+\boldsymbol{\alpha}_j} p_0/p_i \right) = \delta_{p_i} \left( \sum_{j=1,2} c_j(-1)^{j+1} p_i^{\alpha_{j,i}} \right),$$

and

$$\wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2} = \sum_{c_1=1}^{y_{i,r_i+\alpha_{1,i}}} \sum_{c_2=1}^{y_{i,r_i+\alpha_{2,i}}} \left( p_i \delta_{p_i} \left( \sum_{j=1,2} c_j(-1)^{j+1} p_i^{\alpha_{j,i}} \right) - 1 \right).$$

Let

$$\hbar_{\mathbf{r},i}^{j_1,j_2} = \sum_{\substack{\alpha_{j,i} \in [0, 10s \log_2 n] \\ j=1,2}} \wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2}^{j_1,j_2} / p_i^{\boldsymbol{\alpha}_{1,i}+\boldsymbol{\alpha}_{2,i}}$$

with

$$\wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2}^{1,1} = \wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2} \Delta(\alpha_{1,i} > 0) \Delta(\alpha_{2,i} > 0),$$

$$\wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2}^{1,2} = \wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2} \Delta(\alpha_{1,i} > 0) \Delta(\alpha_{2,i} = 0),$$

$$\wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2}^{2,1} = \wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2} \Delta(\alpha_{1,i} = 0) \Delta(\alpha_{2,i} > 0),$$

$$\wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2}^{2,2} = \wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2} \Delta(\alpha_{1,i} = 0) \Delta(\alpha_{2,i} = 0).$$

By (66), we get

$$\hbar_{\mathbf{r},i} = \sum_{j_1,j_2=1,2} \hbar_{\mathbf{r},i}^{j_1,j_2}, \qquad \wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2} = \sum_{j_1,j_2=1,2} \wp_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2}^{j_1,j_2}.$$

It is easy to verify that

$$\wp^{1,1}_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2} = (p_i-1)y_{i,r_i+\alpha_{1,i}}y_{i,r_i+\alpha_{2,i}},$$

$$\wp^{1,2}_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2} = -y_{i,r_i+\alpha_{1,i}}y_{i,r_i},$$

$$\wp^{2,1}_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2} = -y_{i,r_i}y_{i,r_i+\alpha_{2,i}}$$

and

$$\begin{aligned}\wp^{2,2}_{i,\mathbf{r},\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2} &= \sum_{c_1,c_2=1}^{y_{i,r_i}}\big(p_i\delta_{p_i}(c_1-c_2)-1\big)\\ &= p_iy_{i,r_i}-y^2_{i,r_i}.\end{aligned}$$

Hence

$$\hbar^{1,1}_{\mathbf{r},i}=(p_i-1)\beta^2_{i,r_i}\quad\text{with}\quad \beta_{i,r_i}:=\sum_{1\le\alpha\le[0,10s\log_2 n]}y_{i,r_i+\alpha}/p_i^{\alpha},$$

$$\hbar^{1,2}_{\mathbf{r},i}=\hbar^{1,2}_{\mathbf{r},i}=-y_{i,r_i}\beta_{i,r_i}\quad\text{and}\quad \hbar^{2,2}_{\mathbf{r},i}=y_{i,r_i}(p_i-y_{i,r_i}).$$

Therefore

$$\hbar_{\mathbf{r},i}=y_{i,r_i}(p_i-y_{i,r_i})-2y_{i,r_i}\beta_{i,r_i}+(p_i-1)\beta^2_{i,r_i}.$$

We consider the case $1\le y_{i,r_i}$ and $\{y_ip_i^{r_i}\}\le 1-\kappa_1$ (see (5)).
If $p_i=2$, then

$$\hbar_{\mathbf{r},i}=1-2\beta_{i,r_i}+\beta_i^2=(1-\beta_{i,r_i})^2\ge(1-\{y_{i,r_i}p_i^{r_i}\})^2\ge\kappa_1^2.$$

If $p_i\ge 3$ and $y_{i,r_i}\le p_i-2$, then

$$\hbar_{\mathbf{r},i}\ge 2y_{i,r_i}-2y_{i,r_i}\beta_{i,r_i}+2\beta^2_{i,r_i}\ge 1-\beta_{i,r_i}\ge 1-\{y_ip_i^{r_i}\}\ge\kappa_1.$$

If $p_i\ge 3$ and $y_{i,r_i}=p_i-1$, then

$$\hbar_{\mathbf{r},i}=p_i-1-2(p_i-1)\beta_{i,r_i}+(p_i-1)\beta^2_{i,r_i}=(p_i-1)(1-\beta_{i,r_i})^2\ge(1-\{y_ip_i^{r_i}\})^2\ge\kappa_1^2.$$

Therefore $\hbar_{\mathbf{r},i}\ge\kappa_1^2$ for all $p_i\ge 2$. Using (65) and (66), we derive

$$2\pi^2p_0^3G_{\mathbf{r}}\ge\hbar_{\mathbf{r}}\ge\kappa_1^{2s}\quad\text{for}\quad 1\le y_{i,r_i}\quad\text{and}\quad\{y_ip_i^{r_i}\}\le 1-\kappa_1,\quad 1\le i\le s.$$

Let $\tilde{U}=[1,n/(2p_0)]^s\cap\mathbb{Z}^s$. Applying (5), we have

$$2\pi^2p_0^3\sum_{\mathbf{r}\in\tilde{U}}G_{\mathbf{r}}\ \ge\ \kappa_1^{2s}\kappa_2^s\#\tilde{U}\ =\ \kappa_1^{2s}\kappa_2^s[n/(2p_0)]^s$$

and

$$\sum_{\mathbf{r}\in\tilde{U}}G_{\mathbf{r}}\ge\ \frac{1}{4\pi^2p_0^3}\kappa_1^{2s}\kappa_2^s\big(n/(2p_0)\big)^s.$$

From (64), we have $G_{\mathbf{r}}\le 10(2p_0)^{2s}$.

In view of (23), (38) and Lemma 4.1, we obtain

$$W_2 = [\log_2^{20s} n], \qquad \#U_2 \ll n^s \log_2^{-10s} n$$

and

$$\sum_{\mathbf{r} \in U_2 \cup [0,2W_2]^s} G_{\mathbf{r}} \ll n^s \log_2^{-10s} n,$$

$$U_1 \cup U_2 \cup [0, 2W_2]^s \supset U \cap [1, n/(2p_0)]^s = U \cap \tilde{U}$$

and

$$\#\big(U \cap [1, n/(2p_0)]^s\big) \geq [n/(2p_0)]^s - (50 p_0 s^2 \log_2 n)^s.$$

Thus

$$\sum_{\mathbf{r} \in U_1} G_{\mathbf{r}} \geq \sum_{\mathbf{r} \in \tilde{U}} G_{\mathbf{r}} - \sum_{\mathbf{r} \in U_2 \cup [0,2W_2]^s} G_{\mathbf{r}} - 10(2p_0)^{2s} (50 p_0 s^2 \log_2 n)^s$$
$$= \sum_{\mathbf{r} \in \tilde{U}} G_{\mathbf{r}} + O(n^s \log_2^{-10s} n) \geq (5\pi^2 p_0^3)^{-1} (2p_0)^{-s} \kappa_1^{2s} \kappa_2^s n^s$$

for $n \geq n_0$ with some $n_0 > 0$. By Lemma 5.1, we get

$$\mathbf{E}_{\boldsymbol{\theta}} \mathbf{D}_{1,1} = \sum_{\mathbf{r} \in U_1} G_{\mathbf{r}} + O(n^s \log^{-19s} n) \geq \kappa_3 n^s \quad \text{with} \quad \kappa_3 = \pi^{-2} p_0^{-7-s} 2^{-s} \kappa_1^{2s} \kappa_2^s$$

for $n \geq n_1$ with some $n_1 > 0$. Hence Lemma 5.2 is proved. $\square$

In the following Lemma 5.3, we collect the results of all the previous sections.

**Lemma 5.3.** *With the notations as above*

$$\mathbf{E}_{\boldsymbol{\theta}}\Big(\big(\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N]) - \ddot{\mathcal{D}}_1([\theta N])\big)^2\Big) \ll n^s \log^{-10s} n, \tag{67}$$

$$\mathbf{E}_{\boldsymbol{\theta}} \mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N]) \ll n^{s/2-1/16}, \tag{68}$$

$$\mathbf{E}_{\boldsymbol{\theta}}\big(\mathscr{D}^2_{\mathbf{y},\mathbf{x}}([\theta N]) - \ddot{\mathcal{D}}^2_1([\theta N])\big) \ll n^s \log^{-5s} n \tag{69}$$

$$\mathbf{E}_{\boldsymbol{\theta}} \mathscr{D}^2_{\mathbf{y},\mathbf{x}}([\theta N]) \leq 70 p_0^{2s+2} n^s, \tag{70}$$

$$\mathbf{E}_{\boldsymbol{\theta}} \mathscr{D}^2_{\mathbf{y},\mathbf{x}}([\theta N]) \geq 0.5 \kappa_3 n^s, \tag{71}$$

$n \geq n_2(\mathbf{x})$, *where* $\kappa_3 = \pi^{-2} p_0^{-7-s} 2^{-s} \kappa_1^{2s} \kappa_2^s$, *with some* $n_2(\mathbf{x}) > 0$ *for a.s.* $\mathbf{x}$.

P r o o f. From (40), Lemma 2.4, Lemma 4.2 – Lemma 5.2, we obtain

$$\mathbf{E}_{\boldsymbol{\theta}}\Big(\big(\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])\big) - \ddot{\mathcal{D}}([\theta N])\Big)^2 \leq \log_2^{2s+1} n,$$

$$\mathbf{E}_{\boldsymbol{\theta}} \ddot{\mathcal{D}}^2_1([\theta N]) \leq 65 p_0^{2s+2} n^s, \tag{72}$$

$$\ddot{\mathcal{D}}([\theta N]) = \sum_{1\le j\le 5}\ddot{\mathcal{D}}_j([\theta N]), \qquad \ddot{\mathcal{D}}_1^2([\theta N]) = \sum_{1\le j\le 3}\mathbf{D}_{1,j},$$

$$\mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}_2^2([\theta N]) \ll n^s \log^{-10s} n, \qquad \mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}_3^2([\theta N]) \ll n^{s-1/8},$$

$$|\ddot{\mathcal{D}}_4([\theta N])| + |\ddot{\mathcal{D}}_5([\theta N])| \ll \log^{21s^2} n, \tag{73}$$
$$\mathbf{E}_{\boldsymbol{\theta}}\big(\ddot{\mathcal{D}}_1([\theta N]) + \ddot{\mathcal{D}}_2([\theta N])\big) \ll 1/n,$$

$$\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,1} \ge \kappa_3 n^s, \quad \mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,2} \ll n^{s-1/16}, \quad \mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,3} \ll 1, \tag{74}$$

$n \ge n_3(\mathbf{x})$, with some $n_3(\mathbf{x}) > 0$ for a.s. $\mathbf{x}$.

**Let us consider** (67)**.** From (74), we get

$$\begin{aligned}
&\mathbf{E}_{\boldsymbol{\theta}}\Big(\big(\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])\big) - \ddot{\mathcal{D}}_1([\theta N])\Big)^2 \\
&\le 2\mathbf{E}_{\boldsymbol{\theta}}\big((\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])) - \ddot{\mathcal{D}}([\theta N])\big)^2 + 2\mathbf{E}_{\boldsymbol{\theta}}\big(\ddot{\mathcal{D}}([\theta N]) - \ddot{\mathcal{D}}_1([\theta N])\big)^2 \\
&\ll \log^{2s+1} n + \mathbf{E}_{\boldsymbol{\theta}}\left(\sum_{j=2}^{5}\ddot{\mathcal{D}}_j([\theta N])\right)^2 \\
&\ll \log^{2s+1} n + \sum_{j=2}^{5}\mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}_j^2([\theta N]) \\
&\ll n^s \log^{-10s} n.
\end{aligned}$$

**Let us consider** (68)**.** By (40), (67) and (74), we have

$$\mathbf{E}_{\boldsymbol{\theta}}\big(\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])\big) - \big(\ddot{\mathcal{D}}([\theta N])\big)^2 \le \log_2^{2s+1} n,$$

$$\ddot{\mathcal{D}}([\theta N]) = \ddot{\mathcal{D}}_1([\theta N]) + \ddot{\mathcal{D}}_2([\theta N]) + \ddot{\mathcal{D}}_3([\theta N]) + \ddot{\mathcal{D}}_4([\theta N]) + \ddot{\mathcal{D}}_5([\theta N]),$$

$$\Big|\mathbf{E}_{\boldsymbol{\theta}}\big(\ddot{\mathcal{D}}_3([\theta N])\big)\Big| \le \Big(\mathbf{E}_{\boldsymbol{\theta}}\big(\ddot{\mathcal{D}}_3([\theta N])\big)^2\Big)^{1/2} \ll n^{s/2-1/16},$$

$$\mathbf{E}_{\boldsymbol{\theta}}\big(\ddot{\mathcal{D}}_1([\theta N]) + \ddot{\mathcal{D}}_2([\theta N])\big) \ll 1/n, \quad \ddot{\mathcal{D}}_4([\theta N]) + \ddot{\mathcal{D}}_5([\theta N]) \ll \log_2^{21s^2} n,$$

and

$$\begin{aligned}
|\mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}([\theta N])| &\le \Big|\mathbf{E}_{\boldsymbol{\theta}}\big(\ddot{\mathcal{D}}_1([\theta N]) + \ddot{\mathcal{D}}_2([\theta N])\big)\Big| + \Big|\mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}_3([\theta N])\Big| \\
&\quad + \Big|\mathbf{E}_{\boldsymbol{\theta}}\big(\ddot{\mathcal{D}}_4([\theta N]) + \ddot{\mathcal{D}}_5([\theta N])\big)\Big| \\
&\ll n^{s/2-1/16}.
\end{aligned}$$

Hence

$$|\mathbf{E}_{\boldsymbol{\theta}}\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])| \le |\mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}([\theta N])| + \Big(\mathbf{E}_{\boldsymbol{\theta}}\big(\ddot{\mathcal{D}}([\theta N]) - \mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])\big)^2\Big)^{1/2} \ll n^{s/2-1/16},$$

and (68) follows.

**Let us consider** (69)**.** From (67) and (74), we derive

$$\begin{aligned}
&\left|\mathbf{E}_{\boldsymbol{\theta}}\big(\mathscr{D}^2_{\mathbf{y},\mathbf{x}}([\theta N]) - \ddot{\mathcal{D}}^2_1([\theta N])\big)\right| \\
&= \mathbf{E}_{\boldsymbol{\theta}}\big|\big(\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N]) - \ddot{\mathcal{D}}_1([\theta N])\big) \times \big(\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N]) + \ddot{\mathcal{D}}_1([\theta N])\big)\big| \\
&\le \big(\mathbf{E}_{\boldsymbol{\theta}}(\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N]) - \ddot{\mathcal{D}}_1([\theta N]))^2\big)^{1/2} \times \big(\mathbf{E}_{\boldsymbol{\theta}}(\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N]) + \ddot{\mathcal{D}}_1([\theta N]))^2\big)^{1/2} \\
&\ll n^{s/2}\log^{-5s} n\Big(\mathbf{E}_{\boldsymbol{\theta}}\big((\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N]) - \ddot{\mathcal{D}}_1([\theta N])) + 2\ddot{\mathcal{D}}_1([\theta N])\big)^2\Big)^{1/2} \\
&\ll n^{s/2}\log^{-5s} n\Big(2\mathbf{E}_{\boldsymbol{\theta}}\big(\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N]) - \ddot{\mathcal{D}}_1([\theta N])\big)^2 + 8\mathbf{E}_{\boldsymbol{\theta}}\big(\ddot{\mathcal{D}}_1([\theta N])\big)^2\Big)^{1/2} \\
&\ll n^{s/2}\log^{-5s} n(n^s\log^{-10s} + n^s)^{1/2} \ll n^s\log^{-5s} n.
\end{aligned}$$

**Let us consider** (70)**.** By (69) and (74), we have

$$\mathbf{E}_{\boldsymbol{\theta}}\mathscr{D}^2_{\mathbf{y},\mathbf{x}}([\theta N]) = \Big(\mathbf{E}_{\boldsymbol{\theta}}\big(\mathscr{D}^2_{\mathbf{y},\mathbf{x}}([\theta N]) - \ddot{\mathcal{D}}^2_1([\theta N])\big)\Big) + \mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}^2_1([\theta N]) \le 70p_0^{2s+2}n^s,$$

$n \ge n_4(\mathbf{x})$, with some $n_4(\mathbf{x}) > 0$ for a.s. $\mathbf{x}$.
**Let us consider** (71)**.** Bearing in mind that $\ddot{\mathcal{D}}^2_1([\theta N]) = \sum_{1\le j\le 3}\mathbf{D}_{1,j}([\theta N])$, we obtain from (74)

$$\kappa_3 n^s \le \mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,1} \le \mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}^2_1([\theta N]) + |\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,2}| + |\mathbf{E}_{\boldsymbol{\theta}}\mathbf{D}_{1,3}| = \mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}^2_1([\theta N]) + O(n^{s-1/16}).$$

Using (69), we get

$$\mathbf{E}_{\boldsymbol{\theta}}\mathscr{D}^2_{\mathbf{y},\mathbf{x}}([\theta N]) \ge \mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}^2_1([\theta N]) - n^s\log_2^{-4s} n \ge 0.5\kappa_3 n^s,$$

$n \ge n_5(\mathbf{x})$, with some $n_5(\mathbf{x}) > 0$ for a.s. $\mathbf{x}$. Hence, (71) and Lemma 5.3 are proved. ☐

## 6. Four moments and Lévy conditional variance estimates

**Lemma 6.1.** *With the notations as in* (31) *and* (38)*, we have*

$$\hat{\varpi} := \sum_{1\le k\le W_1}\mathbf{E}_{\boldsymbol{\theta}}\mathfrak{D}^4_k([\theta N]) \ll n^{2s-1/8} \qquad \text{for } a.s.\ \mathbf{x}. \tag{75}$$

P r o o f. The proof is similar to that of Lemma 4.3. Let

$$\Bbbk_{j,k_1,k_2} = k_1 \quad \text{for } j \in [1,4] \quad \text{and} \quad \Bbbk_{j,k_1,k_2} = k_2 \quad \text{for } j \in [5,8].$$

By Lemma 2.1, (31) and (38), we get

$$
\begin{aligned}
&\mathbf{E}_x \hat{\varpi}^2 \\
&= \sum_{1\le k_1,k_2\le W_1} \mathbf{E}_x \left( \mathbf{E}_{\boldsymbol{\theta}} \left( \sum_{\mathbf{r}\in\mathbb{U}_{k_1}} \ddot{\mathbb{D}}_{\mathbf{r},L} \right)^4 \mathbf{E}_{\boldsymbol{\theta}} \left( \sum_{\mathbf{r}\in\mathbb{U}_{k_2}} \ddot{\mathbb{D}}_{\mathbf{r},L} \right)^4 \right) \\
&= \sum_{1\le k_1,k_2\le W_1} \sum_{\substack{\mathbf{r}_j\in\mathbb{U}_{\Bbbk_{j,k_1,k_2}} \\ 1\le j\le 8}} \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n] \\ 1\le i\le s, 1\le j\le 8}} \sum_{\substack{0<|m_j|\le n^{10s} \\ (m_j,p_0)=1, 1\le j\le 8}} \mathbf{E}_{\boldsymbol{\theta}} \Big( \prod_{j=1}^4 \varphi_{\mathbf{r}_j,[\theta N], m_j P_{\boldsymbol{\alpha}_j}} \Big) \\
&\quad \times \mathbf{E}_{\boldsymbol{\theta}} \left( \prod_{j=5}^8 \varphi_{\mathbf{r}_j,[\theta N], m_j P_{\boldsymbol{\alpha}_j}} \right) \prod_{j=1}^8 \psi_{\mathbf{r}_j}(m_j P_{\boldsymbol{\alpha}_j}, \mathbf{y})\, \mathbf{E}_x e \left( \sum_{j=1}^8 \frac{m_j}{P_{\mathbf{r}_j - \boldsymbol{\alpha}_j}} (\mathcal{X}_{\mathbf{r}_j,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_j,\mathbf{y}}) \right) \\
&\ll \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n] \\ 1\le i\le s, 1\le j\le 8}} \sum_{\substack{0<|m_j|\le n^{10s} \\ (m_j,p_0)=1, 1\le j\le 8}} \prod_{1\le j\le 8} \frac{1}{|m_j P_{\alpha_j}|} \sum_{1\le k_1,k_2\le W_1} \\
&\quad \times \sum_{\substack{\mathbf{r}_j\in\mathbb{U}_{\Bbbk_{j,k_1,k_2}} \\ 1\le j\le 8}} \mathbf{E}_x e \left( \sum_{1\le j\le 8} \frac{m_j P_{\boldsymbol{\alpha}_j}}{P_{\mathbf{r}_j}} (\mathcal{X}_{\mathbf{r}_j,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_j,\mathbf{y}}) \right). \qquad (76)
\end{aligned}
$$

From Lemma 2.3, we obtain

$$
\mathbf{E}_x e \left( \sum_{1\le j\le 8} \frac{m_j P_{\boldsymbol{\alpha}_j}}{P_{\mathbf{r}_j}} (\mathcal{X}_{\mathbf{r}_j,\mathbf{x}} - \mathcal{X}_{\mathbf{r}_j,\mathbf{y}}) \right) = \Delta \left( \sum_{1\le j\le 8} m_j P_{-\mathbf{r}_j+\boldsymbol{\alpha}_j} = 0 \right).
$$

Applying Corollary 3.2 (S-unit theorem) and Lemma 4.1, we have

$$
\sum_{\substack{\mathbf{r}_j\in\mathbb{U}_{\Bbbk_{j,k_1,k_2}} \\ 1\le j\le 8}} \Delta \left( \sum_{1\le j\le 8} m_j P_{-\mathbf{r}_j+\boldsymbol{\alpha}_j} = 0 \right) \ll \max_k (\#\mathbb{U}_k)^4 \ll n^{4(s-1)} \log_2^{80s} n. \qquad (77)
$$

In view of (76) – (77) and (38), we get

$$
\begin{aligned}
\mathbf{E}_x \hat{\varpi}^2 &\ll \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n] \\ 1\le i\le s, 1\le j\le 8}} \sum_{\substack{0<|m_j|\le n^{10s} \\ (m_j,p_0)=1, 1\le j\le 8}} \prod_{1\le j\le 8} \frac{1}{|m_j P_{\alpha_j}|} \sum_{1\le k_1,k_2\le W_1} n^{4(s-1)} \log_2^{80s} n \\
&\ll \log_2^8 n\, W_1^2 n^{4(s-1+0.05)} \ll \log_2^8 n (n/\log_2^{20s} n)^2 n^{4s-4+0.2} \ll n^{4s-3/2}.
\end{aligned}
$$

By Chebyshev's inequality, we have

$$
P\big(\hat{\varpi} > n^{2s-1/8}\big) \le \mathbf{E}_x \hat{\varpi}^2 / n^{4s-1/4} \ll n^{4s-3/2-4s+1/4} = n^{-5/4}.
$$

Now, using the Borel-Cantelli lemma, we get the assertion of Lemma 6.1. □

Denote by $\dot{\mathcal{F}}(l)$ the sigma field on $[0,1)$ generated by

$$\Big\{\Big[\frac{j}{2^l},\frac{j+1}{2^l}\Big) \mid j=0,\dots,2^l-1\Big\}.$$

Let $l(0)=0, l_k=(k+1)W_2+W_3,\ \mathcal{F}_k=\dot{\mathcal{F}}(l_k)$. We will consider the probability space $\big(([0,1),B([0,1))),\lambda_1\big)$ and the conditional expectation

$$\mathbf{E}_{\boldsymbol{\theta}}(f(\theta)\mid \mathcal{F}_k)=2^{l_k}\int_{\mathfrak{n}/2^{l_k}}^{(\mathfrak{n}+1)/2^{l_k}} f(x)dx \qquad \text{for } \theta\in\Big[\frac{\mathfrak{n}}{2^{l_k}},\frac{\mathfrak{n}+1}{2^{l_k}}\Big). \tag{78}$$

The following Lemma 6.2 is the main part of the estimate of the Levy conditional variance.

**Lemma 6.2.** *With the notations as above*

$$\mathbf{E}_{\boldsymbol{\theta}}\big(\mathfrak{D}_k([\theta N])\mid \mathcal{F}_{k-1}\big)=O(n^{-30s}), \tag{79}$$

$$\mathbf{E}_{\boldsymbol{\theta}}\big(\mathfrak{D}_k([\theta N])\mid \mathcal{F}_{k}\big)=\mathfrak{D}_k([\theta N])+O(n^{-30s}),$$

$$\mathbf{E}_{\boldsymbol{\theta}}\big(\mathfrak{D}_k^2([\theta N])\mid \mathcal{F}_{k-1}\big)=\mathbf{E}_{\boldsymbol{\theta}}\big(\mathfrak{D}_k^2([\theta N])\big)+O(n^{-30s}). \tag{80}$$

P r o o f. By Lemma 2.1, (31), (38) and (93), we get

$$\mathbf{E}_{\boldsymbol{\theta}}\big(\mathfrak{D}_k([\theta N])\mid \mathcal{F}_{k-1}\big)=\sum_{\mathbf{r}\in\mathbb{U}_k}\ \sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\le i\le s}}\ \sum_{\substack{0<|m|\le n^{10s}\\ (m,p_0)=1}}\frac{\sqrt{-1}}{P_{\mathbf{r}}(e(m/P_{\mathbf{r}-\boldsymbol{\alpha}})-1)}\times$$

$$2\mathbf{E}_{\boldsymbol{\theta}}\big(\sin(2\pi m[\theta N]/P_{\mathbf{r}-\boldsymbol{\alpha}})\mid\mathcal{F}_{k-1}\big)\psi_{\mathbf{r}}(mP_{\boldsymbol{\alpha}},\mathbf{y})\, e\Big(\frac{mP_{\boldsymbol{\alpha}}}{P_{\mathbf{r}}}(\mathcal{X}_{\mathbf{r},\mathbf{x}}-\mathcal{X}_{\mathbf{r},\mathbf{y}})\Big)\ll$$

$$n^{-40s}\sum_{\mathbf{r}\in\mathbb{U}_k}\ \sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\le i\le s}}\ \sum_{\substack{0<|m|\le n^{10s}\\ (m,p_0)=1}}\frac{1}{\bar{m}P_{\boldsymbol{\alpha}}}\ll n^{-30s}.$$

Similarly, using (92), we have

$$\mathbf{E}_{\boldsymbol{\theta}}\big(\mathfrak{D}_k([\theta N])\mid \mathcal{F}_{k}\big)-\mathfrak{D}_k([\theta N])=\sum_{\mathbf{r}\in\mathbb{U}_k}\ \sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\le i\le s}}\ \sum_{\substack{0<|m|\le n^{10s}\\ (m,p_0)=1}}\frac{P_{\mathbf{r}}^{-1}\sqrt{-1}}{e(m/P_{\mathbf{r}-\boldsymbol{\alpha}})-1}$$

$$\times 2\Big(\mathbf{E}_{\boldsymbol{\theta}}\big(\sin(2\pi m[\theta N]/P_{\mathbf{r}-\boldsymbol{\alpha}})\mid\mathcal{F}_{k}\big)-\sin(2\pi m[\theta N]/P_{\mathbf{r}-\boldsymbol{\alpha}})\Big)\psi_{\mathbf{r}}(mP_{\boldsymbol{\alpha}},\mathbf{y})$$

$$\times\ e\Big(\frac{mP_{\boldsymbol{\alpha}}}{P_{\mathbf{r}}}(\mathcal{X}_{\mathbf{r},\mathbf{x}}-\mathcal{X}_{\mathbf{r},\mathbf{y}})\Big)\ll n^{-40s}\sum_{\mathbf{r}\in\mathbb{U}_k}\ \sum_{\substack{\alpha_i\in[0,10s\log_2 n]\\ 1\le i\le s}}\ \sum_{\substack{0<|m|\le n^{10s}\\ (m,p_0)=1}}\frac{1}{\bar{m}P_{\boldsymbol{\alpha}}}\ll n^{-30s}.$$

Therefore, the first part of Lemma 6.2 is proved. □

**Let us consider** (80)**.** By (31) and (38), we obtain

$$\mathfrak{D}_k^2([\theta N]) = -\sum_{\mathbf{r}_1,\mathbf{r}_2\in\mathbb{U}_k}\ \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, j=1,2}}\ \sum_{\substack{0<|m_j|\le n^{10s}\\ (m_j,p_0)=1, j=1,2}} 4$$

$$\times \frac{\sin(2\pi m_1[\theta N]/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1})\sin(2\pi m_2[\theta N]/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2})}{P_{\mathbf{r}_1}(e(m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1})-1)P_{\mathbf{r}_2}(e(m_2/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2})-1)}\psi_{\mathbf{r}_1}(m_1P_{\boldsymbol{\alpha}_1},\mathbf{y})$$

$$\times\,\psi_{\mathbf{r}_2}(m_2P_{\boldsymbol{\alpha}_2},\mathbf{y})\ \ e\Big(\frac{m_1P_{\boldsymbol{\alpha}_1}}{P_{\mathbf{r}_1}}(\mathcal{X}_{\mathbf{r}_1,\mathbf{x}}-\mathcal{X}_{\mathbf{r}_1,\mathbf{y}})+\frac{m_2P_{\boldsymbol{\alpha}_2}}{P_{\mathbf{r}_2}}(\mathcal{X}_{\mathbf{r}_2,\mathbf{x}}-\mathcal{X}_{\mathbf{r}_2,\mathbf{y}})\Big).$$

Now using (94), we get

$$\Big|\mathbf{E}_{\boldsymbol{\theta}}(\mathfrak{D}_k^2([\theta N])|\mathcal{F}_{k-1})-\mathbf{E}_{\boldsymbol{\theta}}\mathfrak{D}_k^2([\theta N])\Big|$$

$$\ll \sum_{\mathbf{r}_1,\mathbf{r}_2\in\mathbb{U}_k}\ \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, j=1,2}}\ \sum_{\substack{0<|m_j|\le n^{10s}\\ (m_j,p_0)=1, j=1,2}} 1$$

$$\times\Big|\mathbf{E}_{\boldsymbol{\theta}}\Big(\sin(2\pi m_1[\theta N]/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1})\sin(2\pi m_2[\theta N]/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2})\mid\mathcal{F}_{k-1}\Big)$$

$$-\,\mathbf{E}_{\boldsymbol{\theta}}\Big(\sin(2\pi m_1[\theta N]/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1})\sin(2\pi m_2[\theta N]/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2})\Big)\Big|\frac{1}{|m_1m_2|P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}}$$

$$\ll n^{-40s}\sum_{\mathbf{r}_1,\mathbf{r}_2\in\mathbb{U}_k}\ \sum_{\substack{\alpha_{j,i}\in[0,10s\log_2 n]\\ 1\le i\le s, j=1,2}}\ \sum_{\substack{0<|m_j|\le n^{10s}\\ (m_j,p_0)=1, j=1,2}}\frac{1}{|m_1m_2|P_{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}}$$

$$\ll n^{-30s}.$$

Hence Lemma 6.2 is proved. □

In what follows, we essentially repeat [19, Sec. 2.8 and Sec. 2.9].

## 7. Martingale approximation

Let

$$\xi_k = \mathbf{E}\big(\mathfrak{D}_k([\theta N])\mid\mathcal{F}_k\big)-\mathbf{E}\big(\mathfrak{D}_k([\theta N])\mid\mathcal{F}_{k-1}\big),\ k=1,2,\dots \tag{81}$$

Then $(\xi_k)_{k\ge1}$ is the martingale difference array satisfying $\mathbf{E}(\xi_k|\mathcal{F}_{k-1})=0$, $k=1,2,\dots$ Bearing in mind (38) and (40), we define

$$\dot{\mathbb{S}}_n := \sum_{k=1}^{W_1}\xi_k,\quad \ddot{\mathbb{S}}_n := \sum_{k=1}^{W_1}\mathfrak{D}_k([\theta N]) = \ddot{\mathcal{D}}_1([\theta N]),\quad \dot{\varrho}_n^2 := \mathbf{E}_{\boldsymbol{\theta}}(\dot{\mathbb{S}}_n^2),\quad \ddot{\varrho}_n^2 := \mathbf{E}_{\boldsymbol{\theta}}(\ddot{\mathbb{S}}_n^2). \tag{82}$$

**Lemma 7.1.** *With the notations as above*

$$\mathfrak{D}_k([\theta N]) - \xi_k = O(n^{-30s}), \quad \ddot{\mathbb{S}}_n - \dot{\mathbb{S}}_n = O(n^{-29s}), \quad \mathfrak{D}_k([\theta N])^2 - \xi_k^2 = O(n^{-28s}),$$

$$\ddot{\varrho}_n^2 - \dot{\varrho}_n^2 = O(n^{-11s}) \qquad \textit{and} \qquad |\xi_k|^4 \le 8|\mathfrak{D}_k([\theta N])|^4 + O(n^{-30s}).$$

P r o o f. By (38), (81) and Lemma 6.2, we get $W_1 < n$ and

$$\mathfrak{D}_k([\theta N]) - \mathbf{E}_\theta\big(\mathfrak{D}_k([\theta N]) \mid \mathcal{F}_k\big) \ll n^{-30s},$$
$$\mathbf{E}_\theta\big(\mathfrak{D}_k([\theta N]) \mid \mathcal{F}_{k-1}\big) \ll n^{-30s},$$

$$\mathfrak{D}_k([\theta N]) - \xi_k = \mathfrak{D}_k([\theta N]) - \mathbf{E}_{\boldsymbol{\theta}}\big(\mathfrak{D}_k([\theta N]) \mid \mathcal{F}_k\big) + \mathbf{E}_{\boldsymbol{\theta}}\big(\mathfrak{D}_k([\theta N]) \mid \mathcal{F}_{k-1}\big)$$
$$\ll n^{-30s}, \qquad \ddot{\mathbb{S}}_n - \dot{\mathbb{S}}_n = O(n^{-29s}).$$

From (39) and (82), we have $\mathfrak{D}_k([\theta N]) \ll n^{s+1}$, and $|\dot{\mathbb{S}}_n| + |\ddot{\mathbb{S}}_n| \ll n^{s+2}$. Hence

$$|\mathfrak{D}_k^2([\theta N]) - \xi_k^2| \le |\mathfrak{D}_k([\theta N]) - \xi_k| \left(2|\mathfrak{D}_k([\theta N])| + |\mathfrak{D}_k([\theta N]) - \xi_k|\right)$$
$$\ll n^{-29s+1}$$

and

$$|\ddot{\varrho}_n^2 - \dot{\varrho}_n^2| = |\mathbf{E}_\theta(\ddot{\mathbb{S}}_n^2) - \mathbf{E}_\theta(\dot{\mathbb{S}}_n^2)| = \left|\mathbf{E}_\theta\big((\ddot{\mathbb{S}}_n - \dot{\mathbb{S}}_n)(\ddot{\mathbb{S}}_n + \dot{\mathbb{S}}_n)\big)\right|$$
$$\le \Big(\mathbf{E}_\theta\big((\ddot{\mathbb{S}}_n - \dot{\mathbb{S}}_n)^2\big)\mathbf{E}_\theta\big((\ddot{\mathbb{S}}_n + \dot{\mathbb{S}}_n)^2\big)\Big)^{1/2}$$
$$\ll n^{(-29s+2s+4)/2} \ll n^{-11s}.$$

By Hölder's inequality, we have $(a+b)^4 \le 8(a^4+b^4)$. Therefore,

$$|\xi_k|^4 = |\mathfrak{D}_k([\theta N]) + \xi_k - \mathfrak{D}_k([\theta N])|^4 \le 8|\mathfrak{D}_k([\theta N])|^4 + 8|\mathfrak{D}_k([\theta N]) - \xi_i|^4$$
$$= 8|\mathfrak{D}_k([\theta N])|^4 + O(n^{-40s}).$$

Hence Lemma 7.1 is proved. □

We shall use the following variant of the *martingale central limit theorem* (see [17, Corollary 2.1, p. 58]):

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{(\zeta_{n,k}, F_{n,k}) \mid k = 1, \dots, \ell_n\}$ be a martingale difference array with $\mathbf{E}(\zeta_{n,k} \mid F_{n,k-1}) = 0$ a.s. ($F_{n,0}$ is the trivial field). Let

$$\mathbb{S}_n = \sum_{k=1}^{\ell_n} \zeta_{n,k}, \quad L(n,\epsilon) = \sum_{k=1}^{\ell_n} \mathbf{E}\big(\zeta_{n,k}^2 \Delta(|\zeta_{n,k}| > \epsilon)\big), \quad \mathbb{V}_n^2 = \sum_{k=1}^{\ell_n} \mathbf{E}(\zeta_{n,k}^2 \mid F_{n,k-1}). \quad (83)$$

**Theorem 7.2.** *Let*

$$\sum_{1\le k\le \ell_n} \mathbf{E}(\zeta_{n,k}^2) = 1, \quad L(n,\epsilon) \xrightarrow{P} 0 \quad \forall \epsilon > 0 \quad \textit{and} \quad \mathbb{V}_n^2 \xrightarrow{P} 1, \quad \textit{then} \;\; \mathbb{S}_n \xrightarrow{w} \mathcal{N}(0,1).$$

Now, we apply Theorem 7.2 to the martingale difference array $\{(\zeta_{n,k}, F_{n,k}) : k = 1, \dots, \ell_n\}$ with $F_{n,k} = \mathcal{F}_k$, $\zeta_{n,k} = \xi_k/\dot{\varrho}_n$ and $\ell_n = W_1$.

**Lemma 7.3.** *With the notations as above* $\dot{\mathbb{S}}_n/\dot{\varrho}_n \xrightarrow{w} \mathcal{N}(0,1)$.

Proof. By (81), $(\xi_i)_{i\ge 1}$ is the martingale difference sequence (and consequently orthogonal). From (69) – (71), (82) and Lemma 7.1, we obtain

$$\begin{aligned}
\dot{\varrho}_n^2 &= \mathbf{E}_{\boldsymbol{\theta}}(\dot{\mathbb{S}}_n^2) = \mathbf{E}_{\boldsymbol{\theta}}\Bigg(\sum_{k\in[1,W_1]} \xi_k\Bigg)^2 \\
&= \sum_{k\in[1,W_1]} \mathbf{E}_{\boldsymbol{\theta}}\xi_k^2 = \ddot{\varrho}_n^2 + O(n^{-11s}) \\
&= \mathbf{E}_{\boldsymbol{\theta}}(\ddot{\mathbb{S}}_n^2) + O(n^{-11s}) \\
&= \mathbf{E}_{\boldsymbol{\theta}}\Bigg(\sum_{k\in[1,W_1]} \mathfrak{D}_k([\theta N])\Bigg)^2 + O(n^{-11s}) \\
&= \mathbf{E}_{\boldsymbol{\theta}}\ddot{\mathcal{D}}_1^2([\theta N]) + O(n^{-11s}) \\
&= \mathbf{E}_{\boldsymbol{\theta}}\big(\mathscr{D}_{\mathbf{y},\mathbf{x}}^2([\theta N])\big) + O(n^s \log_2^{-5s}) \ge 0.4\kappa_3 n^s. \qquad (84)
\end{aligned}$$

We derive from (83), (84), Lemma 6.1 and Lemma 7.1 that $\dot{\varrho}_n^2 \ge 0.4\kappa_3 n^s$ and

$$\begin{aligned}
L(n,\epsilon) &= \sum_{1\le k\le W_1} \int_0^1 |\xi_k/\dot{\varrho}_n|^2 \Delta(\xi_k/\dot{\varrho}_n| > \epsilon) d\theta \le \sum_{1\le k\le W_1} \epsilon^{-2} \int_0^1 |\xi_k/\dot{\varrho}_n|^4 d\theta \\
&\ll \sum_{1\le k\le W_1} \epsilon^{-2} n^{-2s}\big(\mathbf{E}_{\boldsymbol{\theta}}\mathfrak{D}_{k,[\theta N]}^4 + O(n^{-30s})\big) \ll \epsilon^{-2} n^{-2s} n^{2s-1/8} \ll n^{-1/8}.
\end{aligned}$$

**Let us consider** $\mathbb{V}_n^2 - 1$**.** By (84), Lemma 6.2 and Lemma 7.1, we derive

$$\begin{aligned}
\dot{\varrho}_n^2 &= \sum_{k\in[1,W_1]} \mathbf{E}_{\boldsymbol{\theta}}\xi_k^2, \quad \mathbf{E}_{\boldsymbol{\theta}}\big(\mathfrak{D}_k^2([\theta N]) \mid \mathcal{F}_{k-1}\big) \\
&= \mathbf{E}_{\boldsymbol{\theta}}\big(\mathfrak{D}_k^2([\theta N])\big) + O(n^{-30s}), \qquad \mathfrak{D}_k^2([\theta N]) - \xi_k^2 = O(n^{-28s}).
\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{V}_n^2 - 1 &= \dot{\varrho}_n^{-2} \sum_{1\le k\le W_1} \Big(\mathbf{E}_{\boldsymbol{\theta}}\big(\xi_k^2|\mathcal{F}_{k-1}\big) - \mathbf{E}_{\boldsymbol{\theta}}\xi_k^2\Big)\\ &= \dot{\varrho}_n^{-2} \sum_{1\le k\le W_1} \Big(\mathbf{E}_{\boldsymbol{\theta}}\big(\mathfrak{D}_k^2([\theta N])|\mathcal{F}_{k-1}\big) - \mathbf{E}_{\boldsymbol{\theta}}\mathfrak{D}_k^2([\theta N])\Big) + O(n^{-27s}) = O(n^{-27s}).\end{aligned}$$

Applying Theorem 7.2, we obtain the assertion of Lemma 7.3. □

We need the following "Converging Together Lemma" [15, p. 105, ex. 3.2.13]:

**Lemma 7.4.** *If $X_n \xrightarrow{w} X$ and $Z_n - X_n \xrightarrow{w} 0$, then $Z_n \xrightarrow{w} X$.*

## 8. The end of the proof of the Theorem 1.1

Let

$$\dddot{\mathbb{S}}_n = \mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N]) \qquad \text{and} \qquad \dddot{\varrho}_n^2 = \mathbf{E}_{\boldsymbol{\theta}}(\dddot{\mathbb{S}}_n^2). \tag{85}$$

By (82), we have

$$\ddot{\mathbb{S}}_n = \ddot{\mathcal{D}}_1([\theta N]), \qquad \dot{\varrho}_n^2 = \mathbf{E}_{\boldsymbol{\theta}}(\dot{\mathbb{S}}_n^2), \qquad \ddot{\varrho}_n^2 = \mathbf{E}_{\boldsymbol{\theta}}(\ddot{\mathbb{S}}_n^2).$$

From Lemma 7.1, we get

$$\dot{\mathbb{S}}_n - \ddot{\mathbb{S}}_n \ll n^{-29s}, \qquad \dot{\varrho}_n^2 - \ddot{\varrho}_n^2 \ll n^{-11s}.$$

In view of Lemma 5.3, we obtain

$$\begin{aligned}\mathbf{E}_{\boldsymbol{\theta}}(\dddot{\mathbb{S}}_n - \ddot{\mathbb{S}}_n)^2 &= \mathbf{E}_{\boldsymbol{\theta}}\Big(\big(\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N])\big) - \ddot{\mathcal{D}}_1([\theta N])\Big)^2 \ll n^s \log_2^{-10s},\\ \dddot{\varrho}_n^2 - \ddot{\varrho}_n^2 &= \mathbf{E}_{\boldsymbol{\theta}}\Big(\mathscr{D}_{\mathbf{y},\mathbf{x}}^2([\theta N]) - \ddot{\mathcal{D}}_1^2([\theta N])\Big) \ll n^s \log_2^{-5s}\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}_{\boldsymbol{\theta}}(\dddot{\mathbb{S}}_n^2) &= \dddot{\varrho}_n^2 \in [0.5\kappa_3, 70p_0^{2s+2}]n^s,\\ \mathbf{E}_{\boldsymbol{\theta}}(\dddot{\mathbb{S}}_n) &= \mathbf{E}_{\boldsymbol{\theta}}\mathscr{D}_{\mathbf{y},\mathbf{x}}([\theta N]) \ll n^{s/2-1/16},\end{aligned} \tag{86}$$

with $\kappa_3 = \pi^{-2}p_0^{-7-s}2^{-s}\kappa_1^{2s}\kappa_2^s$. According to (84), we get $\dot{\varrho}_n^2 \ge 0.4\kappa_3 n^s$. Hence

$$\dot{\varrho}_n^2 - \dddot{\varrho}_n^2 = (\dot{\varrho}_n^2 - \ddot{\varrho}_n^2) + (\ddot{\varrho}_n^2 - \dddot{\varrho}_n^2) \ll n^s \log_2^{-5s} n,$$

and

$$\mathbf{E}_{\boldsymbol{\theta}}(\dot{\mathbb{S}}_n - \dddot{\mathbb{S}}_n)^2 \le 2\mathbf{E}_{\boldsymbol{\theta}}\big((\dot{\mathbb{S}}_n - \ddot{\mathbb{S}}_n)^2\big) + 2\mathbf{E}_{\boldsymbol{\theta}}\big((\ddot{\mathbb{S}}_n - \dddot{\mathbb{S}}_n)^2\big) \ll n^s \log_2^{-5s} n.$$

Therefore,

$$\begin{aligned}\left|\frac{1}{\dot{\varrho}_n} - \frac{1}{\dddot{\varrho}_n}\right| &= \frac{|\dot{\varrho}_n - \dddot{\varrho}_n|}{\dot{\varrho}_n\, \dddot{\varrho}_n} = \frac{|\dot{\varrho}_n^2 - \dddot{\varrho}_n^2|}{\dot{\varrho}_n\, \dddot{\varrho}_n(\dot{\varrho}_n + \dddot{\varrho}_n)}\\ &\ll n^{s-3s/2}\log_2^{-5s} n = n^{-s/2}\log_2^{-5s} n.\end{aligned}$$

We derive

$$\begin{aligned}\mathbf{E}_{\boldsymbol{\theta}}\left(\frac{\dot{\mathbb{S}}_n}{\dot{\varrho}_n} - \frac{\dddot{\mathbb{S}}_n}{\dddot{\varrho}_n}\right)^2 &= \mathbf{E}_{\boldsymbol{\theta}}\left(\frac{\dot{\mathbb{S}}_n - \dddot{\mathbb{S}}_n}{\dot{\varrho}_n} + \dddot{\mathbb{S}}_n\left(\frac{1}{\dot{\varrho}_n} - \frac{1}{\dddot{\varrho}_n}\right)\right)^2 \\ &\le 2\dot{\varrho}_n^{-2}\mathbf{E}_{\boldsymbol{\theta}}(\dot{\mathbb{S}}_n - \dddot{\mathbb{S}}_n)^2 + 2(1/\dot{\varrho}_n - 1/\dddot{\varrho}_n)^2\mathbf{E}_{\boldsymbol{\theta}}\,\dddot{\mathbb{S}}_n^2 \\ &\ll n^{-s+s}\log_2^{-5s} n + (n^{-s/2}\log_2^{-5s} n)^2 n^s \ll \log_2^{-5s} n.\end{aligned}$$

Hence

$$\dot{\mathbb{S}}_n/\dot{\varrho}_n - \dddot{\mathbb{S}}_n/\ddot{\varrho}_n \overset{w}{\to} 0.$$

Bearing in mind that $\dot{\mathbb{S}}_n/\dot{\varrho}_n \overset{w}{\to} \mathcal{N}(0,1)$ and Lemma A, we get that

$$\dddot{\mathbb{S}}_n/\dddot{\varrho}_n \overset{w}{\to} \mathcal{N}(0,1).$$

By (85) and (86), we have the assertion of the Theorem 1.1. □

## 9. Appendix

The results of this section were used in Lemma 6.2. By (29), we get

$$\mathbf{E}_{\boldsymbol{\theta}} f([\theta N]) = \frac{1}{N}\sum_{k=0}^{N-1} f(k). \tag{87}$$

The calculation in Lemma 9.1 based on the following well-known expressions:

$$\begin{aligned}\sin(z) &= \big(\exp(iz) - \exp(-iz)\big)/2i, \\ \cos(z) &= \sin(z + \pi/2), \\ 2\sin^2(z) &= 1 - \cos(2z) = 1 - \sin(2z + \pi/2)\end{aligned}$$

and

$$\begin{aligned}2\sin(z_1)\sin(z_2) &= \cos(z_1 - z_2) - \cos(z_1 + z_2) \\ &= \sin(z_1 - z_2 + \pi/2) - \sin(z_1 + z_2 + \pi/2).\end{aligned}$$

We obtain from (9) and (87), that

$$\begin{aligned}&|\mathbf{E}_{\boldsymbol{\theta}}\sin(2\pi[\theta N]\alpha + \beta)| = N^{-1}\left|\sum_{k=0}^{N-1}\sin(2\pi k\alpha + \beta))\right| \le \min\left(1, \frac{1}{2N\langle\!\langle\alpha\rangle\!\rangle}\right), \\ &2|\mathbf{E}_{\boldsymbol{\theta}}\sin^2(2\pi[\theta N]\alpha) - 1/2| \le \min\left(1, \frac{1}{2N\langle\!\langle 2\alpha\rangle\!\rangle}\right), \qquad (88) \\ &2|\mathbf{E}_{\boldsymbol{\theta}}\sin(2\pi[\theta N]\alpha)\sin(2\pi[\theta N]\alpha)| \le \min\left(1, \frac{1}{2N\langle\!\langle\alpha - \beta\rangle\!\rangle}\right) + \min\left(1, \frac{1}{2N\langle\!\langle\alpha + \beta\rangle\!\rangle}\right).\end{aligned}$$

The main tools of the proof in Lemma 9.1 are the inequalities in (88). We use the deep tools of Theorem 3.3 (the theorem on linear forms in the logarithm) only in the estimate (94).

**Lemma 9.1.** *Let* $\alpha_i \in [0, 10s\log_2 n]$, $1 \le i \le s$, $1 \le |m| \le n^{10s}$, $\mathbf{r} \in \mathbb{U}_k$, $l_k = (k+1)W_2 + W_3$, $k \in [1, W_1]$. *Then*

$$\left|\mathbf{E}_{\boldsymbol{\theta}}\big(\sin(2\pi\beta[\theta N] + \eta) \mid \mathcal{F}_k\big)\right| \le \min\left(2, \frac{2^{l_k - 1}}{N\langle\!\langle \beta \rangle\!\rangle}\right), \tag{89}$$

$$\left|\mathbf{E}_{\boldsymbol{\theta}}\big(\sin^2(2\pi\beta[\theta N]) \mid \mathcal{F}_k\big) - 1/2\right| \le \min\left(1, \frac{2^{l_k - 2}}{N\langle\!\langle 2\beta \rangle\!\rangle}\right), \tag{90}$$

$$\left|\mathbf{E}_{\boldsymbol{\theta}}\big(\sin(2\pi\beta[\theta N]\beta)\sin(2\pi\gamma[\theta N]) \mid \mathcal{F}_{k-1}\big)\right| \le \sum_{i=1,2} \min\left(1, \frac{2^{l_{k-1}-2}}{N\langle\!\langle \beta + (-1)^i\gamma \rangle\!\rangle}\right), \tag{91}$$

$$\left|\mathbf{E}_{\boldsymbol{\theta}}\big(\sin(2\pi m[\theta N]/P_{\mathbf{r}-\boldsymbol{\alpha}} + \eta) \mid \mathcal{F}_k\big) - \sin(2\pi m[\theta N]/P_{\mathbf{r}-\boldsymbol{\alpha}} + \eta)\right| \le n^{-40s}, \tag{92}$$

$$\left|\mathbf{E}_{\boldsymbol{\theta}}\big(\sin(2\pi m[\theta N]/P_{\mathbf{r}-\boldsymbol{\alpha}} + \eta) \mid \mathcal{F}_{k-1}\big)\right| \le n^{-40s},$$

$$\left|\mathbf{E}_{\boldsymbol{\theta}}\big(\sin(2\pi m[\theta N]/P_{\mathbf{r}-\boldsymbol{\alpha}} + \eta)\big)\right| \le n^{-40s}. \tag{93}$$

$$\begin{aligned}\Bigg|\mathbf{E}_{\boldsymbol{\theta}}\Bigg(&\sin\left(\frac{2\pi m_1[\theta N]}{P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}}\right)\sin\left(\frac{2\pi m_2[\theta N]}{P_{\mathbf{r}_2-\boldsymbol{\alpha}_2}}\right) \mid \mathcal{F}_{k-1}\Bigg) - \\ &\mathbf{E}_{\boldsymbol{\theta}}\Bigg(\sin\left(\frac{2\pi m_1[\theta N]}{P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}}\right)\sin\left(\frac{2\pi m_2[\theta N]}{P_{\mathbf{r}_2-\boldsymbol{\alpha}_2}}\right)\Bigg)\Bigg| \ll n^{-40s}.\end{aligned} \tag{94}$$

P r o o f. **Let us consider** (89). Let $\theta \in [\frac{\mathfrak{n}}{2^{\ell_k}}, \frac{\mathfrak{n}+1}{2^{\ell_k}}]$. From (78), we see

$$\begin{aligned}\mathbf{E}_{\boldsymbol{\theta}}\big(e(\beta[\theta N] + \eta) \mid \mathcal{F}_k\big) &= 2^{l_k}\int_{\mathfrak{n}/2^{l_k}}^{(\mathfrak{n}+1)/2^{l_k}} e(\beta[xN] + \eta)dx \\ &= \int_0^1 e\big(\beta[(N\mathfrak{n} + zN)/2^{l_k}] + \eta\big)dz \\ &= \frac{1}{N}\sum_{\ell=0}^{N-1} e\big(\beta[(N\mathfrak{n} + \ell)/2^{l_k}] + \eta\big).\end{aligned}$$

Let

$$N = N_1 2^{l_k} + N_2, \quad \ell = \ell_1 2^{l_k} + \ell_2 \quad \text{with} \quad N_2, \ell_2 \in [0, 2^{l_k}).$$

Hence

$$0 \le \ell_1 2^{l_k} \le N - 1 - \ell_2, \quad 0 \le \ell_1 \le [(N_1 2^{l_k} + N_2 - 1 - \ell_2)/2^{l_k}], \quad 0 \le \ell_1 \le N_1 + N_3,$$

with

$$N_3 = [(N_2 - 1 - \ell_2)/2^{l_k}] \in \{-1, 0\}.$$

Using (9), we derive

$$\begin{aligned}\left|\mathbf{E}_{\boldsymbol{\theta}}\Big(e(\beta[\theta N]+\eta)\mid\mathcal{F}_k\Big)\right| &= \frac{1}{N}\left|\sum_{0\le \ell_2\le 2^{l_k}-1}\ \sum_{0\le \ell_1\le N_1+N_3} e\Big(\beta\ell_1+\beta[(N\mathfrak{n}+\ell_2)/2^{l_k}]\Big)\right| \\ &\le \frac{2^{l_k}}{N}\max_{0\le \ell_2\le 2^{l_k}-1}\left|\sum_{0\le \ell_1\le N_1+N_3} e(\beta\ell_1)\right| \\ &\le \frac{2^{l_k}}{N}\min\left(N_1+1,\frac{1}{2\langle\!\langle\beta\rangle\!\rangle}\right)\le \min\left(2,\frac{2^{l_k}}{2N\langle\!\langle\beta\rangle\!\rangle}\right),\end{aligned}$$

and (89) follows.

**Let us consider** (90) **and** (91). Bearing in mind (88) and (89), we obtain

$$\begin{aligned}2\left|\mathbf{E}_{\boldsymbol{\theta}}\Big(\sin^2(2\pi\beta[\theta N])\mid\mathcal{F}_k\Big)-1/2\right| &= \left|\mathbf{E}_{\boldsymbol{\theta}}\Big(-\cos(4\pi\beta[\theta N])\mid\mathcal{F}_k\Big)\right| \\ &= \left|\mathbf{E}_{\boldsymbol{\theta}}\Big(\sin(4\pi\beta[\theta N]+\pi/2)\mid\mathcal{F}_k\Big)\right| \\ &\le \min\left(2,\frac{2^{l_k}}{2N\langle\!\langle 2\beta\rangle\!\rangle}\right),\end{aligned}$$

and

$$\begin{aligned}&2\left|\mathbf{E}_{\boldsymbol{\theta}}\Big(\sin(2\pi\beta[\theta N])\sin(2\pi\gamma[\theta N])\mid\mathcal{F}_{k-1}\Big)\right| \\ &= \left|\mathbf{E}_{\boldsymbol{\theta}}\Big(\sin(2\pi[\theta N]\beta-\gamma+\pi/2)-\sin(2\pi[\theta N]\beta+\gamma+\pi/2)\mid\mathcal{F}_{k-1}\Big)\right| \\ &\le \sum_{i=1,2}\min\left(2,\frac{2^{l_{k-1}}}{2N\langle\!\langle\beta+(-1)^i\gamma\rangle\!\rangle}\right).\end{aligned}$$

**Let us consider** (92). By (38), we have $W_2=[\log_2^{20s} n]$, $W_3=[\log_2^{10s} n]$, $A_k=n-(k+1)W_2$, $B_k=A_k+W_2-2W_3$, $h(\mathbf{r})=r_1\log_2(p_1)+\cdots+r_s\log_2(p_s)$, $\mathbb{U}_k=\{\mathbf{r}\in U\mid h(\mathbf{r})\in[A_k,B_k)\}$. Taking into account that $l_k=(k+1)W_2+W_3$, we have for $\mathbf{r}\in\mathbb{U}_k$:

$$\begin{aligned}\frac{P_{\mathbf{r}-\boldsymbol{\alpha}}2^{l_{k-1}}}{N} &\ll 2^{n-(k+1)W_2+W_2-2W_3-n+kW_2+W_3-1}=2^{-W_3-1}\le 2^{-\log_2^{10s} n}\le n^{-40s}, \\ \frac{mN}{P_{\mathbf{r}-\boldsymbol{\alpha}}} &\ll 2^{10s\log_2 n+n+1-(n-(k+1)W_2)+10sp_0\log_2 n}\ll 2^{(k+1)W_2+20sp_0\log_2 n} \\ \frac{mN}{P_{\mathbf{r}-\boldsymbol{\alpha}}2^{l_k}} &\ll 2^{-W_3+20sp_0\log_2 n}\ll 2^{-W_3/2}\ll 2^{-0.5\log_2^{10s} n}\ll n^{-40s}.\end{aligned}\tag{95}$$

Let $\theta \in [\frac{\mathfrak{n}}{2^{l_k}}, \frac{\mathfrak{n}+1}{2^{l_k}}]$, $\theta = (\mathfrak{n}+\theta_1)/2^{l_k}$ with $\theta_1 \in [0,1]$. We see

$$\frac{m[\theta N]}{P_{\mathbf{r}-\boldsymbol{\alpha}}} = \frac{m}{P_{\mathbf{r}-\boldsymbol{\alpha}}}\left(\left[\frac{\mathfrak{n}N}{2^{l_k}}\right] + \left[\frac{\theta_1 N}{2^{l_k}}\right] + \epsilon\right) = \frac{m}{P_{\mathbf{r}-\boldsymbol{\alpha}}}\left[\frac{\mathfrak{n}N}{2^{l_k}}\right] + O\left(\frac{mN}{P_{\mathbf{r}-\boldsymbol{\alpha}}2^{l_k}}\right)$$
$$= \frac{m}{P_{\mathbf{r}-\boldsymbol{\alpha}}}\left[\frac{\mathfrak{n}N}{2^{l_k}}\right] + O(n^{-40s}), \quad \text{with} \quad \epsilon \in [-2,2].$$

Bearing in mind that $|\sin(x_1)-\sin(x_2)| = |x_1-x_2||\cos(x_1+z(x_2-x_1))| \le |x_1-x_2|$ $(z\in[0,1])$, we get

$$\sin\left(\frac{2\pi m[\theta N]}{P_{\mathbf{r}-\boldsymbol{\alpha}}}+\eta\right) - \sin\left(\frac{2\pi m}{P_{\mathbf{r}-\boldsymbol{\alpha}}}\left[\frac{\mathfrak{n}N}{2^{l_k}}\right]+\eta\right) = O(n^{-40s})$$

and

$$\mathbf{E}_{\boldsymbol{\theta}}\left(\sin\left(\frac{2\pi m[\theta N]}{P_{\mathbf{r}-\boldsymbol{\alpha}}}+\eta\right) \mid \mathcal{F}_k\right) - \sin\left(\frac{2\pi m}{P_{\mathbf{r}-\boldsymbol{\alpha}}}\left[\frac{\mathfrak{n}N}{2^{l_k}}\right]+\eta\right) = O(n^{-40s}).$$

Therefore

$$\sin\left(\frac{2\pi m[\theta N]}{P_{\mathbf{r}-\boldsymbol{\alpha}}}+\eta\right) - \mathbf{E}_{\boldsymbol{\theta}}\left(\sin\left(\frac{2\pi m[\theta N]}{P_{\mathbf{r}-\boldsymbol{\alpha}}}+\eta\right) \mid \mathcal{F}_k\right) = O(n^{-40s}).$$

**Let us consider** (93). By (30), we get $|m|/P_{\mathbf{r}-\boldsymbol{\alpha}} \le 1/32$. In view of (89) and (95), we obtain

$$\left|\mathbf{E}_{\boldsymbol{\theta}}\big(\sin(2\pi m[\theta N]/P_{\mathbf{r}-\boldsymbol{\alpha}}+\eta) \mid \mathcal{F}_{k-1}\big)\right| \le \min\left(2, \frac{2^{l_{k-1}-1}}{N\langle\!\langle m/P_{\mathbf{r}-\boldsymbol{\alpha}}\rangle\!\rangle}\right) =$$
$$\min\left(2, \frac{P_{\mathbf{r}-\boldsymbol{\alpha}}2^{l_{k-1}-1}}{|m|N}\right) \le \min\left(2, \frac{P_{\mathbf{r}-\boldsymbol{\alpha}}2^{l_{k-1}-1}}{N}\right) \ll n^{-40s},$$
$$\left|\mathbf{E}_{\boldsymbol{\theta}}\big(\sin(2\pi m[\theta N]/P_{\mathbf{r}-\boldsymbol{\alpha}}+\eta)\big)\right| \le \min\left(2, \frac{1}{N\langle\!\langle m/P_{\mathbf{r}-\boldsymbol{\alpha}}\rangle\!\rangle}\right) \ll n^{-40s}.$$

**Let us consider** (94). Let $|m_1|/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1} \neq |m_2|/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2}$. By (30), we have

$$|m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1} + (-1)^j m_2/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2}| \le 1/16.$$

Applying Corollary 3.4, we get

$$\langle\!\langle m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1} + (-1)^j m_2/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2}\rangle\!\rangle = |m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1} + (-1)^j m_2/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2}| >$$
$$\min(|m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}|, |m_2/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2}|)\exp(-C_1 \ln^2 n).$$

Using (88), (91) and (95), we have

$$2\left|\mathbf{E}_{\boldsymbol{\theta}}\left(\sin\left(\frac{2\pi m_1[\theta N]}{P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}}\right)\sin\left(\frac{2\pi m_2[\theta N]}{P_{\mathbf{r}_2-\boldsymbol{\alpha}_2}}\right)\;\Big|\;\mathcal{F}_{k-1}\right)-\right.$$
$$\left.\mathbf{E}_{\boldsymbol{\theta}}\left(\sin\left(\frac{2\pi m_1[\theta N]}{P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}}\right)\sin\left(\frac{2\pi m_2[\theta N]}{P_{\mathbf{r}_2-\boldsymbol{\alpha}_2}}\right)\right)\right| \leq$$
$$2\sum_{j=1,2}\min\left(2,\frac{2^{l_{k-1}}}{2N\langle\!\langle m_1/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}+(-1)^j m_2/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2}\rangle\!\rangle}\right) \leq$$
$$2^{l_{k-1}}\max_j P_{\mathbf{r}_j-\boldsymbol{\alpha}_j}N^{-1}\exp(C_1\ln^2 n) \ll 2^{-\log_2^{10s} n}\exp(C_1\ln^2 n) \ll n^{-40s}. \quad (96)$$

Now let $|m_1|/P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}=|m_2|/P_{\mathbf{r}_2-\boldsymbol{\alpha}_2}$. From (88), (93) and (95), we obtain

$$2\left|\mathbf{E}_{\boldsymbol{\theta}}\left(\sin^2\left(\frac{2\pi m_1[\theta N]}{P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}}\right)\;\Big|\;\mathcal{F}_{k-1}\right)-\mathbf{E}_{\boldsymbol{\theta}}\left(\sin^2\left(\frac{2\pi m_1[\theta N]}{P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}}\right)\right)\right|$$
$$=\left|\mathbf{E}_{\boldsymbol{\theta}}\left(\sin\left(\frac{4\pi m_1[\theta N]}{P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}}+\pi/2\right)\;\Big|\;\mathcal{F}_{k-1}\right)-\mathbf{E}_{\boldsymbol{\theta}}\left(\sin\left(\frac{4\pi m_1[\theta N]}{P_{\mathbf{r}_1-\boldsymbol{\alpha}_1}}+\pi/2\right)\right)\right| \ll n^{-40s}. \quad (97)$$

By (96) and (97), (94) is proved and Lemma 9.1 follows. □

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**Mordechay B. Levin**
*Department of Mathematics*
*Bar-Ilan University*
*5290002, Ramat-Gan*
*ISRAEL*
*E-mail*: mlevin@math.biu.ac.il