sciendo

uniform distribution theory

DOI: 10.2478/UDT-2025-0008
Unif. Distrib. Theory **20** (2025), no.1, 144-153

CONVERGENCE OF MULTIPLE SERIES AND $\mathcal{I}-$VARIATION

Vladimír Baláž,[1] — Alain Faisant[2] — Georges Grekos[2] — Juraj Kostra[3]

[1]Slovak Academy of Sciences, Bratislava, SLOVAKIA

[2]Institut Camille Jordan, University Jean Monnet, Saint-Étienne, FRANCE

[3]University of Pardubice, Pardubice, CZECH REPUBLIC

ABSTRACT. In the classical theory of multiple series the notion of variation of a sequence plays an important role. According to the concept of $\mathcal{I}-$convergence the notion of $\mathcal{I}-$variation of a sequence was recently introduced. In this paper we will examine applications of $\mathcal{I}-$variation of a sequence mainly in the theory of multiple series in particular in relation with the notion of $\mathcal{I}-$convergence.

Communicated by Friedrich Pillichshammer

1. Definitions and notations

We recall the basic definitions and connections that will be used throughout this paper. Let $\mathbb{N}$ be the set of all positive integers. Let $\mathbf{x} = (x_n)$ be a sequence of real (complex) numbers,

$$K \subseteq \mathbb{N}, \quad K = \{k_1 < k_2 < \cdots < k_n < \cdots\}$$

(an infinite set) and the restriction of the sequence $\mathbf{x}$ to the part K is

$$\mathbf{x}\,|_K = \{x_{k_1}, x_{k_2}, \ldots, x_{k_n}, \ldots\} = (x_{k_n}).$$

2010 Mathematics Subject Classification: 13C15, 40A35, 40B05.
Keywords: sequences, ideals, $\mathcal{I}$-convergence, $\mathcal{I}$-variation, multiple series.
[1] Supported by the The Slovak Research and Development Agency (research project VEGA No. 2/0119/23.)

A family $\mathcal{I}$, $\emptyset \neq \mathcal{I} \subseteq 2^{\mathbb{N}}$, is called an ideal, provided that $\mathcal{I}$ is additive ($A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$) and hereditary ($A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$).

The ideal is called non-trivial if $\mathcal{I} \neq 2^{\mathbb{N}}$. If $\mathcal{I}$ is a non-trivial ideal, then $\mathcal{I}$ is called admissible if it contains the singletons ($\{n\} \in \mathcal{I}$ for every $n \in \mathbb{N}$). In other words, for an admissible ideal $\mathcal{I}$ we have $\mathcal{I}_{\text{fin}} \subseteq \mathcal{I}$, where $\mathcal{I}_{\text{fin}}$ is the ideal of all finite subsets of $\mathbb{N}$. In the whole paper, we will only consider admissible ideals. The associated filter to an ideal $\mathcal{I}$ is defined by

$$\varphi(\mathcal{I}) = \{M \subseteq \mathbb{N} : \mathbb{N} \setminus M \in \mathcal{I}\}.$$

The notion of $\mathcal{I}-$variation of a sequence $\mathbf{x}$ was recently introduced in [5].

Definition 1.1. A sequence $\mathbf{x} = (x_n)$ is said to be of *bounded $\mathcal{I}-$variation* for an ideal $\mathcal{I}$ if there exists a set

$$K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \in \varphi(\mathcal{I})$$

such that

$$\operatorname{Var} \mathbf{x}|_K = \sum_{n=1}^{\infty} |x_{k_n} - x_{k_{n+1}}| < +\infty.$$

A sequence $\mathbf{x}$ is said to be of bounded variation if

$$\operatorname{Var} \mathbf{x}|_{\mathbb{N}} < +\infty.$$

It is clear that bounded variation of a sequence $\mathbf{x}$ implies bounded $\mathcal{I}-$variation of the sequence $\mathbf{x}$ for all admissible ideals $\mathcal{I} \subseteq 2^{\mathbb{N}}$.

Further properties of sequences having bounded $\mathcal{I}-$variation are investigated in [2], [4] and [5].

The notion of statistical convergence was introduced independently by Fast [6] and Schoenberg [12]. The notion of $\mathcal{I}-$convergence introduced by Kostyrko, Šalát and Wilczyński [10] corresponds to the natural generalization of the notion of statistical convergence.

Definition 1.2. We say that a sequence $\mathbf{x} = (x_n)$ of real (complex) numbers is *$\mathcal{I}-$convergent* to a number L and we write $\mathcal{I} - \lim\ x_n = L$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N}\ :\ |x_n - L| \geq \varepsilon\}$ belongs to the ideal $\mathcal{I}$.

If $\mathcal{I}$ is an admissible ideal, then for every sequence $\mathbf{x} = (x_n)$ we immediately have that $\lim_{n\to\infty} x_n = L$ (classic limit) implies that $\mathbf{x}$ also $\mathcal{I}-$converges to the number L but the converse needs not to be true.

Let $\mathcal{I}_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$, where $d(A)$ is the asymptotic density of the set $A \subseteq \mathbb{N}$ ($d(A) = \lim_{n\to\infty} \frac{\sharp\{a \leq n : a \in A\}}{n}$, where $\sharp M$ denotes the cardinality of the set M). In the case when $\mathcal{I} = \mathcal{I}_d$, then $\mathcal{I}_d-$convergence coincides with the statistical convergence. If $\mathcal{I} = \mathcal{I}_{\text{fin}}$, then $\mathcal{I}_{\text{fin}}-$convergence is the usual convergence.

$\mathcal{I}-$convergence satisfies usual axioms of convergence, i.e., the uniqueness of limit, arithmetical properties, etc. The class of all $\mathcal{I}-$convergent sequences is a linear space (see [10]). In the same paper there also was defined another type of convergence related to an ideal $\mathcal{I}$ so called $\mathcal{I}^*-$convergence.

Definition 1.3. A sequence $\mathbf{x} = (x_n)$ of real (complex) numbers is said to be $\mathcal{I}^*-$*convergent* to a number L and we write $\mathcal{I}^*-\lim\ x_n = L$, if there is a set $H \in \mathcal{I}$, such that for the set

$$M = \mathbb{N} \setminus H = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \varphi(\mathcal{I})$$

we have $\lim_{k\to\infty}\ x_{m_k} = L$, where the limit is in the usual sense.

It is easy to see that for an admissible ideal $\mathcal{I}$ we have that $\mathcal{I}^*-$convergence implies $\mathcal{I}-$convergence. The converse does not have to be true (see [10], where authors moreover give a characterization of ideals $\mathcal{I}$ for which $\mathcal{I}-$convergence and $\mathcal{I}^*-$convergence are equivalent by means of property (AP)).

Definition 1.4. An ideal $\mathcal{I}$ (not necessarily admissible) is said to satisfy *the condition* (AP) if for every countable family of mutually disjoint sets (A_n) belonging to $\mathcal{I}$ there exists countable family of sets (B_n) such that the symmetric difference

$$A_j \bigtriangleup B_j \quad \text{is finite for } j \in \mathbb{N} \quad \text{and} \quad B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}.$$

There exist many examples of ideals having or not having the property (AP) (see, e.g. [1], [10]).

Proposition 1.5 (see [10]). *From* $\mathcal{I}-\lim\ x_n = L$ *the statement* $\mathcal{I}^*-\lim\ x_n = L$ *follows if and only if* $\mathcal{I}$ *satisfies the property (AP).*

An ideal $\mathcal{I}$ (not necessarily admissible) is called $P-$ideal if for each sequence (A_n) of sets belonging to $\mathcal{I}$ there exists a set $A_\infty \in \mathcal{I}$ such that $A_n \setminus A_\infty$ is finite for all $n \in \mathbb{N}$. The notions of $P-$ideal and ideal having the property (AP) coincide (see [3]).

We will also use the following facts.

Proposition 1.6 (see [10]). *Let* $\mathcal{I}_1$, $\mathcal{I}_2$ *be admissible ideals such that* $\mathcal{I}_1 \subseteq \mathcal{I}_2$. *If* $\mathcal{I}_1 - \lim\ x_n = L$, *then* $\mathcal{I}_2 - \lim\ x_n = L$.

Proposition 1.7 (see [5]). *Let* $\mathbf{x} = (x_n)$ *be a sequence of real (complex) numbers having bounded* $\mathcal{I}-$*variation for an admissible ideal* $\mathcal{I}$. *Then* $\mathbf{x}$ *is* $\mathcal{I}^*-$*convergent and so it is also* $\mathcal{I}-$*convergent.*

Definition 1.8. An infinite series $\sum_{n=1}^{\infty} a_n$ for real (complex) numbers a_n, $n = 1, 2, \ldots$ is said to be $\mathcal{I}-$*convergent* if the sequence of the partial sums (s_n) is $\mathcal{I}-$convergent, where $s_n = \sum_{k=1}^{n} a_k$. The number $s = \mathcal{I} - \lim\ s_n$ is called $\mathcal{I}-$*sum of the series.*

$\mathcal{I}-$notions have been developed in several directions and used in various parts of mathematics, for example [1], [7], [11], [9] and [13].

2. Main results

In this section we are going to study the application of the concept of $\mathcal{I}-$variation of a sequence in the theory of multiple series ($\sum_{n=1}^{\infty}\ a_n\beta_n$, where a_n and β_n are real (complex) numbers for $n = 1, 2, \ldots$) in particular in relation to the notion of $\mathcal{I}-$convergence. In connection with this we formulate two theorems and we are going to investigate what happens when the classical notions will be replaced by its $\mathcal{I}-$analogs in these theorems.

The following result is ascribed by Hardy [8] to Dedekind.

Theorem 2.1. *Let a series $\sum_{n=1}^{\infty} a_n$ be convergent and (β_n) be a sequence of bounded variation. Then the series $\sum_{n=1}^{\infty}\ a_n\beta_n$ is convergent, too.*

The previous theorem can be reversed in a certain sense. In [8] Hardy attributes this result to Hadamard.

Theorem 2.2. *Let (β_n) be a sequence such that for each convergent series $\sum_{n=1}^{\infty} a_n$ also the series $\sum_{n=1}^{\infty} a_n\beta_n$ is convergent. Then the sequence (β_n) is of bounded variation.*

The following theorem shows that an $\mathcal{I}-$analog of Theorem 2.1 is valid only in the case when $\mathcal{I} = \mathcal{I}_{\text{fin}}$. Moreover by means of an $\mathcal{I}-$analog of Theorem 2.1 is the ideal $\mathcal{I}_{\text{fin}}$ characterized.

Theorem 2.3. *The following assertions are equivalent:*

(i) *For every sequence (β_n) with bounded $\mathcal{I}-$variation the $\mathcal{I}-$convergence of a series $\sum_{n=1}^{\infty} a_n$ implies the $\mathcal{I}-$convergence of the series $\sum_{n=1}^{\infty} a_n\beta_n$.*

(ii) $\mathcal{I} = \mathcal{I}_{\text{fin}}$.

Proof.

(ii)$\Rightarrow$(i) is Theorem 2.1 (see [8]).

(i) $\Rightarrow$(ii) Let $\mathcal{I}$ be an admissible ideal such that $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$; there exists an infinite set

$$K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \in \mathcal{I}.$$

Then the set

$$M = \{k_2 < k_4 < \cdots < k_{2n} < \cdots\}$$

belongs also to $\mathcal{I}$. We define the sequence (s_n) of partial sums of a series $\sum_{n=1}^{\infty} a_n$ that unambiguously determines terms of the series $\sum_{n=1}^{\infty} a_n$. Put $s_n = 1$ if $n \in M$, and $s_n = 0$, otherwise. We obtain the terms of the series

$$a_1 = s_1 \quad \text{and} \quad a_n = s_n - s_{n-1} \text{ if } n \geq 2.$$

It follows that the series $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}^*-$convergent to zero since $s_n = 0$ for $n \in \mathbb{N} \setminus M \in \varphi(\mathcal{I})$. Thus the series $\sum_{n=1}^{\infty} a_n$ is also $\mathcal{I}-$convergent to zero.

Now we define the sequence (β_n) as follows:

$$\beta_n = 1 \text{ if } n \in M, \quad \text{and} \quad \beta_n = 0, \text{ otherwise.}$$

The sequence (β_n) has bounded $\mathcal{I}-$variation, since Var $\beta\ |_{\mathbb{N}\setminus M} = 0$. But the series $\sum_{n=1}^{\infty} a_n\beta_n$ is not $\mathcal{I}-$convergent since it has non-negative terms and contains infinitely many 1's. Put

$$S_n = \sum_{k=1}^{n} a_k\beta_k.$$

Since $a_n\beta_n = 1$ if $n \in M$, and $a_n\beta_n = 0$ otherwise, we have $S_n \geq 0$, and moreover, $a_n\beta_n \geq 0$, so the sequence (S_n) is nondecreasing. As we can see $S_{k_{2n}} = n$. Thus for every real (complex) number L and $\varepsilon = 1$ the set

$$\mathbb{N} \setminus A(1) = \{n \in \mathbb{N} : \ |S_n - L| < \varepsilon\}$$

is finite, so that $\mathbb{N} \setminus A(1) \in \mathcal{I}$, thus the set $A(1)$ cannot be in $\mathcal{I}$, because $\mathcal{I}$ is an admissible ideal. $\square$

The following examples show that Theorem 2.3 is valid only for the ideal $\mathcal{I}_{\text{fin}}$ even if not all assumptions of Theorem 2.3 in statement (i) are replaced by their $\mathcal{I}-$variants. Example 1 shows that for every admissible ideal $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$ there exist a convergent series $\sum_{n=1}^{\infty} a_n$ and a sequence (β_n) with bounded $\mathcal{I}-$variation such that the series $\sum_{n=1}^{\infty} a_n\beta_n$ does not have to be $\mathcal{I}-$convergent. Example 2 shows that for every admissible ideal $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$ there exist an $\mathcal{I}-$convergent series $\sum_{n=1}^{\infty} a_n$ and a sequence (β_n) having bounded variation such that the series $\sum_{n=1}^{\infty} a_n\beta_n$ does not have to be $\mathcal{I}-$convergent.

Example 1. Let the set K have the same meaning as in the proof of Theorem 2.3 in the implication (ii)⇒(i). Put the terms of series $\sum_{n=1}^{\infty} a_n$ as follows: $a_n = \frac{1}{n^2}$ if $n \in K$, and $a_n = 0$, otherwise. It is clear that the series $\sum_{n=1}^{\infty} a_n$ is convergent. Define the sequence (β_n), $\beta_n = n^2$ if $n \in K$, and $\beta_n = 0$ otherwise. Since $\mathbb{N} \setminus K \in \varphi(\mathcal{I})$ so the sequence (β_n) has bounded $\mathcal{I}-$variation. The series $\sum_{n=1}^{\infty} a_n\beta_n$ is the series having non-negative terms containing infinitely many 1's, so it is not $\mathcal{I}-$convergent.

Example 2. Let the set M have the same meaning as in the proof of Theorem 2.3 in the implication (ii)⇒(i). The terms a_n of series $\sum_{n=1}^{\infty} a_n$ are defined by means of the partial sums of this series. Put $S_n = n^2$ if $n \in M$, and $S_n = 0$, otherwise. Then $a_1 = S_1$ and $a_n = S_n - S_{n-1}$ if $n \geq 2$. It is clear that the series $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}-$convergent to zero (it even is $\mathcal{I}^{*}-$convergent to zero, since $\mathbb{N} \setminus M \in \varphi(\mathcal{I})$). Consider the sequence (β_n), $\beta_n = \frac{1}{n^2}$ if $n \in M$, and $\beta_n = 0$, otherwise. The sequence (β_n) has bounded variation. Again the series $\sum_{n=1}^{\infty} a_n\beta_n$ having non-negative terms and containing infinitely many 1's, is not $\mathcal{I}-$convergent.

Let us focus on the $\mathcal{I}-$analog of Theorem 2.2. For this we need the following Abel theorem, which we can prove in a simpler way than the one stated in the literature, in our opinion.

Lemma 2.4 ([14], p. 75, Theorem 2.4.13)**.** *Let (u_n) be a sequence with $u_n \geq 0$ for all $n \geq 2$ and $u_1 > 0$ such that*

$$\sum_{n=1}^{\infty} u_n = +\infty\,.$$

Denote

$$U_n = u_1 + u_2 + \cdots + u_n, \quad n = 1, 2, \ldots$$

Then the series

$$\sum_{n=1}^{\infty} \frac{u_n}{U_n} = +\infty\,.$$

Proof. We will use Cauchy-Bolzano criterion. For $k \geq 1$, $m \geq 1$ we have

$$\begin{aligned}\sum_{i=1}^{m} \frac{u_{k+i}}{U_{k+i}} &= \frac{u_{k+1}}{U_{k+1}} + \frac{u_{k+2}}{U_{k+2}} + \cdots + \frac{u_{k+m}}{U_{k+m}} \\ &\geq \frac{u_{k+1} + u_{k+2} + \cdots + u_{k+m}}{U_{k+m}} = \frac{U_{k+m} - U_k}{U_{k+m}} \\ &= 1 - \frac{U_k}{U_{k+m}}\,.\end{aligned}$$

For k being fix we have $\lim_{m\to\infty} \frac{U_k}{U_{k+m}} = 0$. Then there exists $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$ we obtain

$$\frac{U_k}{U_{k+m}} \leq \frac{1}{2} \quad \text{and so} \quad \sum_{i=1}^{m} \frac{u_{k+i}}{U_{k+i}} \geq \frac{1}{2}\,.$$

Thus Cauchy-Bolzano criterion is not satisfied, so the series $\sum_{n=1}^{\infty} \frac{u_n}{U_n}$ is divergent. □

Before we state the $\mathcal{I}-$analogue of Theorem 2.2, we prove Theorem 2.2 in order to gain a slight generalization. Again the proof is simpler and more precise than the one given in the literature, of course in our opinion. In the assumption of the theorem an arbitrary convergent series is not necessary but it is sufficient to limit ourselves to series that are convergent to zero. Theorem 2.2 is equivalent to the following statement that we are going to prove.

Theorem 2.5. *If a sequence (β_n) has unbounded variation, then there exists a convergent series $\sum_{n=1}^{\infty} a_n$ such that the series $\sum_{n=1}^{\infty} a_n\beta_n$ is divergent.*

Proof. For brevity, we put $b_n = \beta_n - \beta_{n+1}$. Then by assumption we have

$$\sum_{n=1}^{\infty} |b_n| = +\infty.$$

Without loss of generality, we can assume $b_1 \neq 0$ thus

$$B_n = |b_1| + |b_2| + \cdots + |b_n| > 0 \quad \text{for} \ \ n = 1, 2, \ldots$$

Define the terms of the series $\sum_{n=1}^{\infty} a_n$ by means of its partial sums

$$s_n = \sum_{k=1}^{n} a_k = \frac{\operatorname{sign} b_n}{B_n}$$

and subsequently,

$$a_1 = s_1 \quad \text{and} \quad a_n = s_n - s_{n-1} \quad \text{for} \ \ n \geq 2.$$

Using Abel process we obtain for the series $\sum_{n=1}^{\infty} a_n\beta_n$ the following equation:

$$\begin{aligned}
\sum_{k=1}^{n} a_k\beta_k &= a_1\beta_1 + (s_2 - s_1)\beta_2 + \cdots + (s_n - s_{n-1})\beta_n \\
&= s_1(\beta_1 - \beta_2) + s_2(\beta_2 - \beta_3) + \cdots + s_{n-1}(\beta_{n-1} - \beta_n) + s_n\beta_n \\
&= \sum_{k=1}^{n-1} s_k b_k + s_n\beta_n\,. \qquad (1)
\end{aligned}$$

We have $s_k b_k = \frac{|b_k|}{B_k} \geq 0$. Using the Lemma 2.4, where we put

$$u_n = |b_n| \quad \text{and} \quad U_n = B_n, \ \ n \geq 1,$$

we obtain that

$$\sum_{n=1}^{\infty} s_n b_n = +\infty.$$

We show that the sequence $(s_n \beta_n)$ is bounded. Indeed

$$\beta_n = \beta_1 - \sum_{k=1}^{n-1} b_k,$$

so

$$|s_n \beta_n| = \frac{|\beta_n|}{B_n} \leq \frac{|\beta_1|}{B_n} + \frac{\sum_{k=1}^{n-1} |b_k|}{\sum_{k=1}^{n} |b_k|} \leq \frac{|\beta_1|}{B_n} + 1.$$

Since $\lim_{n\to\infty} B_n = +\infty$, it follows that the sequence $(s_n \beta_n)$ is bounded. By (1) we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k \beta_k = +\infty.$$

Moreover,

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{\pm 1}{B_n} = 0.$$ □

By this way the proof of Theorem 2.5 is finished and its analysis shows that we have proved a stronger claim as Theorem 2.2.

Theorem 2.6. *Let (β_n) be a sequence such that for each convergent series $\sum_{n=1}^{\infty} a_n$ with zero sum also the series $\sum_{n=1}^{\infty} a_n \beta_n$ is convergent then the sequence (β_n) is of bounded variation.*

Now we can move on the $\mathcal{I}-$analogue of Theorem 2.6, resp. Theorem 2.5.

Theorem 2.7. *Let $\mathcal{I}$ be an admissible ideal and (β_n) be a sequence such that for each $\mathcal{I}-$convergent series $\sum_{n=1}^{\infty} a_n$ having zero sum also the series $\sum_{n=1}^{\infty} a_n \beta_n$ is $\mathcal{I}-$convergent. Then the sequence (β_n) is of bounded $\mathcal{I}-$variation.*

Proof. By contradiction, suppose that the sequence (β_n) is of infinite $\mathcal{I}-$variation. Then for every set $K \in \varphi(\mathcal{I})$ we have

$$\operatorname{Var} \boldsymbol{\beta}|_K = +\infty.$$

Especially, for the set K we can take $\mathbb{N}$. Therefore

$$\sum_{k=1}^{\infty} |\beta_n - \beta_{k+1}| = +\infty.$$

Using Theorem 2.5, we find a sequence (a_n) satisfying the following properties:

$$\lim_{n\to\infty}\sum_{k=1}^{n} a_k = 0, \tag{2}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} a_k\beta_k = +\infty. \tag{3}$$

The first property (2) implies that $\sum_{n=1}^{\infty} a_n$ is $\mathcal{I}-$convergent to zero. By (3) we have that $\sum_{n=1}^{\infty} a_n\beta_n$ is not $\mathcal{I}-$convergent since the series has non-negative terms and contains infinitely many 1's. □

Acknowledgement. The authors would like to thank the anonymous reporter for having corrected errors in the first submitted version.

The author Vladimír Baláž wishes to thank The Slovak Research and Development Agency (research project VEGA No. 2/0119/23) for financial support.

REFERENCES

[1] BALÁŽ, V.—VISNIAY, T.: *$\mathcal{I}-$convergence of arithmetical functions*. In: Number Theory and Its Applications, chapter 8, IntechOpen, London, 2020, pp. 125–145,

[2] BALÁŽ, V.—FAISANT, A.—GREKOS, G.: *On Real Algebras Associated with Ideal Convergence*, Real Anal. Exchange **47** (2022), 281–296.

[3] BALCERZAK, M.—DEMS, K.—KOMISARSKI, A.: *Statistical convergence and ideal convergence for sequences of functions*, J. Math. Anal. Appl. **328** (2007), no. 1, 715–729.

[4] BARTOSZEWICZ, A.—GŁĄB, S.—WACHOWICZ, A.: *Remarks on ideal boundedness, convergence and variation of sequences*, J. Math. Anal. Appl. **375** (2011), 431–435.

[5] FAISANT, A.—GREKOS, G.—TOMA, V.: *On the statistical variation of sequences*, J. Math. Anal. Appl. **306** (2005), 432–439.

[6] FAST, H.: *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[7] FURSTENBERG, H.: *Recurrence in Ergodic Theory and Combinatorial Number Theory*, M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1981.

[8] HARDY, G. H.: *On convergence of certain multiple series*, Proc. Cambridge Phil. Soc. **19** (1917), 86–95.

[9] JIN, R.: *Applications of nonstandard analysis in additive number theory*, Bull. Symbolic Logic **6** (2000), no. 3, 331–341.

[10] KOSTYRKO, P.—ŠALÁT, T.—WILCZYŃSKI, W.: *I-convergence*, Real Anal. Exchange **26** (2000/2001), no. 2, 669–686.

[11] PAŠTÉKA, M.: *Density and Related Topics*. VEDA Bratislava, AC Praha, 2017.

[12] SCHOENBERG, I. J.: *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66**, 1959, 361–375.

[13] STRAUCH, O.—Š. PORUBSKÝ, V.: *Distribution of Sequences: A Sampler*. Schr. Slowak. Akad. Wiss. Vol. 1 [Series of the Slovak Academy of Sciences] Peter Lang, Frankfurt am Main, 2005.

[14] ŠALÁT, T.: *Infinite Series*. Academia, Praha, 1974. (In Slovak)

Received November 29, 2024
Accepted January 7, 2025

Vladimír Baláž
Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
814 73 Bratislava
SLOVAKIA
ORCID: 0000-0003-0574-1708
E-mail: vladimir.balaz@mat.savba.sk

Alain Faisant
Georges Grekos
Université Jean-Monnet -Saint-Étienne
CNRS, ICJ UMR 5208
11 rue du Dr. Rémy Annino
42000 Saint-Étienne
FRANCE
E-mail: faisant@univ-st-etienne.fr
grekos@univ-st-etienne.fr

Juraj Kostra
Faculty of Chemical Technology
Institute of Applied Physics and Mathematics
University of Pardubice
Studenská 95
Pardubice, 53210
CZECH REPUBLIC
E-mail: juraj.kostra@seznam.cz