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RAMSEY-COLLATZ CORRELATION AND SOME EXTREMAL COMBINATORICS, ALONG WITH RAMSEY-MAHLER RARE CONCURRENCIES

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ABSTRACT. For a ternary tree T of depth n with 0-1 labeled edges, its weight $f(T)$ is the least number of path labels among binary subtrees. The maximum $f(n)$, over all labelings, of these weights starts with 1,2,3,4,8. We show $f(6) \geq 12$, but focus on depth 5 (with 2^{363} trees). We approximate the percentages for weights 1–8: 0, 1.04, 23.6, 55.0, 18.8, 1.54, 0, 0; our linked supplements include thousands of mined trees of rare weights 7-8. Our next products additionally relate to Mahler's $\frac{3}{2}$-problem and large stopping times showing a real number is not a Z-number. Our version is iterated multiplication of integers by $\frac{2}{3}$. For a certain sequence of integer intervals, we present a choice function. Our interval I_g has left endpoint $a(g) = \min\{k|\{g \cdot (\frac{2}{3})^k\} \geq \frac{1}{2} \vee k = \lceil \log_{\frac{3}{2}}(g) \rceil\}$, and right endpoint at the stopping time for "the 1st intermediate rounding error", but the interpolation is a "no sudden death" function. We present simultaneous peculiarity in base 2 and base 3: numbers g with large values (for the size of g) of $a(g)$ and length of I_g, which also have a less common Ramsey weight (when written in base 2 and used to edge-label a tree), or are prime. Then we cross the weight notion with the Collatz scaled total stopping time $\gamma_\infty(n)$. We construct sizable low-high sequences of 8-tuples of same-weight pairs of numbers below 2^{363} with certain monotonicity in values of γ_∞, and in six apartness levels $> 10^{-i}$ for $i = 1, \ldots, 6$. Apartness of γ_∞-values would be in the 'low' and 'high' halves as well as between the corresponding components of terms of the sequence. We get lower bounds for their lengths, and for Collatz-landing-apart just for weight 8. We statistically establish an unexpected correlation between Collatz and Ramsey.

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1. Introduction and preparation

1.1. Ternary trees, binary subtrees, and a Ramseyan property

A ternary tree T is complete if all the leaves (terminal nodes) are of the same distance, say n, to the root. Then T is of depth n. In [3, Downey et al.], the authors considered complete ternary trees of depths $n \geq 1$ where the edges are labeled either 0 or 1. For a complete ternary tree T of depth n with $\{0,1\}$--labeled edges, they considered the complete binary subtrees of the same depth n which use the least number, called $f(T)$, of path labels in $\{0,1\}^n$. Call this the weight of T. Let $f(n) = \max\{f(T) \mid T \text{ is a complete ternary tree of depth } n\}$. It is clear that $f(1) = 1$ and $f(2) = 2$. As in op. cit. (proofs appear in Milan's thesis [8, Milan]), $f(3) = 3$ and $f(4) = 4$. In [3, Downey et al.], where the main results were bounds on f and certain corollaries in computability theory, it was also announced in a note added in proof that $f(5) = 8$. Their proof, kindly communicated to the author by Carl Jockusch, is presented in the current paper, where we also show $f(6) \geq 12$.

In [9, West], West reported a question by Milans as to what the expected value of $f(T)$ is for a tree T of depth n with a random $\{0,1\}$-assignment on its edges. We deal with this for the first few values of the depth when $n \leq 5$.

For depth 2, a complete determination of the exact probability of weights 1 and 2 and so the expected value of the weight we obtain. It is $\frac{11}{8} = 1.375$ and we also verify it with Mathematica (this whole sample space is of size only 4096).

Note that for any fixed n, the number of ternary trees of depth n whose edges are $\{0,1\}$-labeled is $2^{\frac{3^{n+1}-3}{2}}$, and for each such tree the number of binary subtrees is 3^{2^n-1}. These are doubly exponential and that explains the limitations, not just for a similar comprehensive approach but even for simulations, beyond the first few values of n. E.g., after the first few values of n, a reasonable sample size for a simulation would be larger than what can now be computed by brute force with ordinary resources. Our computations and illustrations, thus restricted to small n, are via Mathematica. The codes are provided in our supplements.

For depth 3, we again obtain the precise expected value and produce computational evidence via random sample simulations, but, unlike the depth 2 case, not for the whole class of such trees. There are $2^{39} = 549,755,813,888$ edge-labeled trees T of depth 3, we determine the exact expected value of $f(T)$ to be $\frac{31033}{16384} \approx 1.8941$. Simulations with samples give means close to this. A Ramseyan property in depth 3 is used later: "three any two of which" does not reduce the maximum possible for total number of path labels for a binary subtree for each tree in the collection, triple or pair, if considered for just "two" (both turn out to be 4).

Starting from depth 4 (where there are $2^{120} > 1.3 \times 10^{36}$ trees), we follow the strategy to first remove the one-in-three terminal leaves keeping the majority (or a certain described pair if all three are the same). This does not increase the calculated weight of the tree. We use a Mathematica code to simulate the distribution of all possible values $f(T) = 1, 2, 3, 4$ via random samples of size 10^i for $i = 1, \dots, 6$. We present the outputs when we ran each size for three times. These random distributions, with their rather convincing consistencies especially for the three samples of size 1,000,000, are expected to come close to calculations that one can pursue based on heuristic arguments (we include the argument and precise calculation for one case). It is safe to say that the expected value is in $(2.5, 2.9)$. We give examples of the weights 1–4 for depth 4.

Our main contributions are in depth 5, where there are

$$2^{363} > 1.8 \times 10^{109}$$

trees. We present the weight distribution for 25 random samples of various sizes from 100 to 1500, and totaling 8200. All random trees we got were of weights 2-6. We show some corresponding confidence intervals for the mean, all included in $(3.6, 4.4)$, sample means were in $(3.86, 4.11)$. We also consider two classes of structured random trees, in certain senses: via either an Inductively Almost Balanced method, or the Log Super-additivity property

$$f(m+n) \geq f(m)f(n),$$

see [3, Downey et al.]. The first group trees are constructed by extending level-wise and at each new level, if a path label is used k times, of the $3k$ new edges, half of them ($+\frac{1}{2}$ if needed) are labeled 0 and the rest labeled 1. The trees in the other group are constructed to have weight at least 6, using the instance

$$f(2+3) \geq f(2)f(3)$$

of the Log SA property. For these two groups too, we present the weight distributions of 1535 trees coming from 14 samples of different sizes for each group, and observe some over-performance of the Inductive AB method compared to what it was perceived of. We are also able to construct and analyze sizable sets of rare trees of weights 7–8. In the supplements, we provide over 8400 trees-including 3000 trees of weight 8 (using the Ramseyan property in depth 3) plus trees of the Random, Inductively AB, and Log SA types.

For depth 6 (with $2^{1092} > 5.3 \times 10^{328}$ trees), using $f(m+n) \geq f(m)f(n)$, we have

$$f(6) \geq f(3)f(3) = 9.$$

We improve this to $f(6) \geq 12$. In depth 6, we give a tree where all binary subtrees that when restricted to any terminal depth-4 cluster are optimal for that cluster, would not be optimal.

1.2. Iterations of the left inverse $\lfloor\frac{2}{3}g\rfloor$ of Mahler's function $\lceil\frac{3}{2}g\rceil$

The function $\lfloor\frac{2}{3}\cdot\rfloor$ is onto but not 1-1, as

$$\left\lfloor\frac{2}{3}(3q+2)\right\rfloor = 2q+1 \quad \text{and} \quad \left\lfloor\frac{2}{3}(3q+1)\right\rfloor = \left\lfloor\frac{2}{3}(3q)\right\rfloor = 2q.$$

Its right inverse is Mahler's function $\lceil\frac{3}{2}g\rceil$ (see [7, Mahler]) as $\lfloor\frac{2}{3}\lceil\frac{3}{2}(2q)\rceil\rfloor = 2q$ and $\lfloor\frac{2}{3}\lceil\frac{3}{2}(2q+1)\rceil\rfloor = 2q+1$. Mahler's function itself is 1-1 but not onto, it does not take values congruent to 1 mod 3. For the reverse composition we have $\lceil\frac{3}{2}\lfloor\frac{2}{3}(3q+1)\rfloor\rceil \neq 3q+1$ (in contrast to a similar statement for the other two congruency classes mod 3) and so if we want the process of going backward in numbers by iterating $\lfloor\frac{2}{3}\cdot\rfloor$ starting from g to be recoverable (i.e., not to loose information), we should stop when we reach a number congruent to 1 mod 3. It would be a yellow light: at that point, Mahler's function applied to one more iteration of $\lfloor\frac{2}{3}\cdot\rfloor$ would not yield back the number before the last iteration. This type of stopping time and when it takes large values are of our interest here (see [4, Dubickas and Mossinghoff]), especially on inputs that are relevant for our ternary trees of depth 5 as above when the edges are labeled 0 or 1. Below we sandwich this function as an interpolation between two related stopping time functions.

For any integer $g \geq 1$, define the function f_g with

$$f_g(0) = g, \quad \text{and} \quad f_g(n) = \left\lfloor\frac{2}{3}f_g(n-1)\right\rfloor.$$

Consider the following functions (where as usual, bounded maximization or minimization would take the value of the bound when there are no witnesses to the matrix of the formula below the bound):

$a(g) = 1 + \max\ \{k \leq \lfloor\log_{\frac{3}{2}}(g)\rfloor \mid (\forall m \leq k)\{g(\frac{2}{3})^m\} < \frac{1}{2}\}$,

$b(g) = 1 + \max\ \{k \leq \lfloor\log_{\frac{3}{2}}(g)\rfloor \mid (\forall m \leq k)\lfloor g(\frac{2}{3})^m\rfloor \not\equiv 1 \pmod 3\}$,

$c(g) = 1 + \max\ \{k \leq \lfloor\log_{\frac{3}{2}}(g)\rfloor \mid (\forall m \leq k)f_g(m) \not\equiv 1 \pmod 3\}$,

$d(g) = 1 + \max\ \{k \leq \lfloor\log_{\frac{3}{2}}(g)\rfloor \mid (\forall m \leq k)\lfloor g(\frac{2}{3})^m\rfloor = f_g(m)\}$.

Note that:

$a(g) = \min\ \{k \leq 1 + \lfloor\log_{\frac{3}{2}}(g)\rfloor \mid \{g(\frac{2}{3})^k\} \geq \frac{1}{2}\}$,

$b(g) = \min\ \{k \leq 1 + \lfloor\log_{\frac{3}{2}}(g)\rfloor \mid \lfloor g(\frac{2}{3})^k\rfloor \equiv 1 \pmod 3\}$,

$c(g) = \min\ \{k \leq 1 + \lfloor\log_{\frac{3}{2}}(g)\rfloor \mid f_g(k) \equiv 1 \pmod 3\}$,

$d(g) = \min\ \{k \leq 1 + \lfloor\log_{\frac{3}{2}}(g)\rfloor \mid \lfloor g(\frac{2}{3})^k\rfloor \neq f_g(k)\}$.

Proposition 1.1. *For any $g \geq 1$, the difference $\lfloor g(\frac{2}{3})^k \rfloor - f_g(k)$ is 0 or 1.*

P r o o f. An easy induction on k. Note that as k varies starting from 0, the two values could be different only if $\lfloor g(\frac{2}{3})^k \rfloor$ is $1 \mod 3$ and $\{g(\frac{2}{3})^k\} \geq \frac{1}{2}$, and in that case the mentioned difference is 1. $\square$

Proposition 1.2. *For all g with $a(g), b(g), c(g) < 1 + \lfloor \log_{\frac{3}{2}}(g) \rfloor$ (this bound for a, b, c is indeed expected not to be achieved), we have:*

$$a(g) \leq 1 + b(g) = 1 + c(g) \leq d(g), \quad \text{and} \quad a(g) < d(g).$$

$$\limsup_{g \to \infty} a(g) = \infty \quad \text{and} \quad \liminf_{g \to \infty} d(g) = 2.$$

P r o o f. To prove $a(g) \leq 1 + b(g)$, note that even if $b(g)$ could happen to be as high as $\lfloor \log_{\frac{3}{2}}(g) \rfloor$ or $1 + \lfloor \log_{\frac{3}{2}}(g) \rfloor$, then certainly $a(g) \leq 1 + b(g)$ (by the bound on $a(g)$). In the other case (which appears to be the universal case), assume $b(g) \leq \lfloor \log_{\frac{3}{2}}(g) \rfloor - 1$. Suppose by way of contradiction that $a(g) > b(g) + 1$. Then $g(\frac{2}{3})^{b(g)}$ and $g(\frac{2}{3})^{b(g)+1}$ both have fractional part less than $\frac{1}{2}$. We have $\lfloor g(\frac{2}{3})^{b(g)} \rfloor \equiv 1 \pmod 3$, and write $g(\frac{2}{3})^{b(g)} = 3q + 1 + r$, with fractional part $r < \frac{1}{2}$. Then $g(\frac{2}{3})^{b(g)+1} = 2q + \frac{2}{3} + \frac{2r}{3}$. So $\frac{2r}{3} > \frac{1}{3}$ (otherwise the fractional part exceeds $\frac{2}{3} > \frac{1}{2}$), hence $r > \frac{1}{2}$, a contradiction.

We show $1 + b(g) \leq d(g)$ (under the condition $b(g) < 1 + \lfloor \log_{\frac{3}{2}}(g) \rfloor$) and $1 + c(g) \leq d(g)$ (under the condition $c(g) < 1 + \lfloor \log_{\frac{3}{2}}(g) \rfloor$) and because of the way $b(g)$, $c(g)$, and $d(g)$ are defined, these inequalities would then also imply that $b(g) = c(g)$.

To prove $1 + b(g) \leq d(g)$, consider ℓ with $1 \leq \ell \leq 1 + \lfloor \log_{\frac{3}{2}}(g) \rfloor$ and assume $b(g) \geq \ell$ and we show that $d(g) \geq \ell + 1$. We have

$$\lfloor g \rfloor, \quad \left\lfloor g\left(\frac{2}{3}\right) \right\rfloor, \ldots, \left\lfloor g\left(\frac{2}{3}\right)^{\ell-1} \right\rfloor \not\equiv 1 \pmod 3.$$

Let $j \leq \ell$, we show $\lfloor g(\frac{2}{3})^j \rfloor = f_g(j)$. This is clear for $j = 0$. Assume for some $k \leq \ell - 1$ that $\lfloor g(\frac{2}{3})^k \rfloor = f_g(k)$. Let $\{g(\frac{2}{3})^k\} = r$. For $f_g(k+1) = \lfloor \frac{2}{3} \lfloor g(\frac{2}{3})^k \rfloor \rfloor = \lfloor g(\frac{2}{3})^{k+1} - \frac{2r}{3} \rfloor$ to be equal to $\lfloor g(\frac{2}{3})^{k+1} \rfloor$, it suffices that $\{g(\frac{2}{3})^{k+1}\} \geq \frac{2r}{3}$. If $g(\frac{2}{3})^k = 3q + r$, then $g(\frac{2}{3})^{k+1} = 2q + \frac{2r}{3}$, and if $g(\frac{2}{3})^k = 3q + 2 + r$, then $g(\frac{2}{3})^{k+1} = (2q+1) + \frac{2r+1}{3}$.

To prove that $1 + c(g) \leq d(g)$, consider some ℓ with $1 \leq \ell \leq 1 + \lfloor \log_{\frac{3}{2}}(g) \rfloor$ and assume $c(g) \geq \ell$ and we show that $d(g) \geq \ell + 1$. So assume

$$f_g(0), \ldots, f_g(\ell - 1) \not\equiv 1 \pmod 3.$$

Let $j \leq \ell$, we show $\lfloor g(\frac{2}{3})^j \rfloor = f_g(j)$. This is clear for $j = 0$. Assume for some $k \leq \ell - 1$ that $\lfloor g(\frac{2}{3})^k \rfloor = f_g(k)$. Either $[g(\frac{2}{3})^k = 3q + r$ & $f_g(k) = 3q]$ or $[g(\frac{2}{3})^k = 3q + 2 + r$ & $f_g(k) = 3q + 2]$, for some $r \in [0, 1)$. So either $[g(\frac{2}{3})^{k+1} = 2q + \frac{2r}{3}$ & $f_g(k+1) = 2q]$ or $[g(\frac{2}{3})^{k+1} = (2q+1) + \frac{2r+1}{3}$ & $f_g(k+1) = 2q + 1]$. This shows $\lfloor g(\frac{2}{3})^{k+1} \rfloor = f_g(k+1)$.

To prove $a(g) < d(g)$, let g be such that $a(g) = 1 + b(g)$, therefore both are also equal to $1 + c(g)$; call their common values ℓ. Then $(\forall k \leq \ell - 1)\{g(\frac{2}{3})^k\} < \frac{1}{2}$. Note that $b(g) = \ell - 1 \leq \lfloor \log_{\frac{3}{2}}(g) \rfloor$ (since $a(g) \leq 1 + \lfloor \log_{\frac{3}{2}}(g) \rfloor$), so $\lfloor g(\frac{2}{3})^{\ell - 1} \rfloor = 3q + 1$, for some q. Write $g(\frac{2}{3})^{\ell-1} = 3q + 1 + r$, with $0 \leq r < \frac{1}{2}$. Then from $a(g) \leq d(g)$, we get $f_g(\ell - 1) = \lfloor g(\frac{2}{3})^{\ell - 1} \rfloor$, so $f_g(\ell - 1) = 3q + 1$. We have $f_g(\ell) = 2q$. Also from $g(\frac{2}{3})^\ell = 2q + \frac{2}{3} + \frac{2r}{3}$, we get $\lfloor g(\frac{2}{3})^\ell \rfloor = 2q$. Hence $f_g(\ell) = \lfloor g(\frac{2}{3})^\ell \rfloor$. This means that if $a(g) < 1 + \lfloor \log_{\frac{3}{2}}(g) \rfloor$, we have $d(g) \geq \ell + 1$.

To show that $\limsup_{g \to \infty} a(g) = \infty$, consider $g(u) = \frac{3^{2u-1} + 13}{8}$. We claim that for all $u \geq 1$ and $0 \leq \ell \leq 2u - 1$, we have $\{g(u)(\frac{2}{3})^\ell\} < \frac{1}{2}$. To see this for $\ell \geq 3$, note that $g(u)(\frac{2}{3})^\ell = 3^{2u-1-\ell} 2^{\ell - 3} + \frac{13}{8}(\frac{2}{3})^\ell$ and $\frac{13}{8}(\frac{2}{3})^\ell < \frac{1}{2}$ for $\ell \geq 3$. If $\ell = 0$, $g(u)(\frac{2}{3})^\ell$ is an integer. If $\ell = 1$, $g(u)(\frac{2}{3})^\ell = \frac{3^{2u-1} + 13}{12}$ has fractional part $\frac{4}{12} = \frac{1}{3} < \frac{1}{2}$. If $\ell = 2$, then $u \geq 2$, so $3^{2u-1} \equiv 9 \pmod{18}$ and $\{g(u) \cdot \frac{4}{9}\} = \frac{4}{18} = \frac{2}{9} < \frac{1}{2}$. Note that for any such number $g(u) = \frac{3^{2u-1}+13}{8}$, we have $a = 1 + b = 1 + c = 2u < d$.

To show that $\liminf_{g \to \infty} d(g) = 2$, consider $g(u) = 9u + 7$. We have $g = g$ and $\lfloor g \cdot (\frac{2}{3}) \rfloor = \lfloor g \cdot (\frac{2}{3}) \rfloor$ for any g (not even necessarily of this form), but $\lfloor \lfloor (9u+7)\frac{2}{3} \rfloor \frac{2}{3} \rfloor = 4u + 2$ while $\lfloor (9u+7)\frac{4}{9} \rfloor = 4u + 3$. Note that for these numbers $9u + 7$, we have $a = 1$, while $1 + b = 1 + c = d = 2$. □

1.3. Crossing Ramsey with Collatz

For $i = 1, \ldots, 6$, we define a LoH-iSeq to be a sequence of terms LoH-i (for LowHigh of apartness $> 10^{-i}$) each of which an ordered 8-tuple of ordered pairs of nonnegative integers $n < 2^{363}$:

$$\big((l_1, h_1),\ (l_2, h_2),\ (l_3, h_3),\ (l_4, h_4),\ (l_5, h_5),\ (l_6, h_6),\ (l_7, h_7),\ (l_8, h_8)\big),$$

neither of which a counterexample to Collatz Conjecture (if there are any) and which are presented in eight rows 1-8 and two columns l and h, with the following three properties:

Property R: The complete ternary tree of depth 5, when edge-labeled with any of the LoH-iSeq's 16 numbers in binary-padleft 0's as necessary to reach length 363, will have its weight equal to the row number of the pair in the 8-tuple: w(l_j)=w(h_j) $= j$, for $1 \leq j \leq 8$.

Property C: For the Collatz (the shortcut $\frac{3x+1}{2}$ version) scaled total stopping times $\gamma_\infty(n)$ rounded to i decimals, of the 16 numbers n in any LoH-i term of the LoH-iSeq, we have the following different monotonicity towards the row 4 low and high:

$$\begin{aligned}\gamma_\infty(l_1) > \gamma_\infty(l_2) > \gamma_\infty(l_3) > \gamma_\infty(l_4) \\ < \gamma_\infty(l_5) < \gamma_\infty(l_6) < \gamma_\infty(l_7) < \gamma_\infty(l_8),\end{aligned}$$

and

$$\begin{aligned}\gamma_\infty(h_1) < \gamma_\infty(h_2) < \gamma_\infty(h_3) < \gamma_\infty(h_4) \\ > \gamma_\infty(h_5) > \gamma_\infty(h_6) > \gamma_\infty(h_7) > \gamma_\infty(h_8).\end{aligned}$$

Recall that these values $\gamma_\infty(n)$ are the quotient of the total stopping time $\sigma_\infty(n)$ (number of iterations to reach 1 under the 3x+1 map $2m \mapsto m$, $2m+1 \mapsto 3m+2$) divided by ln of the initial number n. For any such LoH-i, all consecutive differences corresponding to the inequality chains above we require to be more than 10^{-i}. Different monotonicity requirements are motivated by the frequencies of each weight (and the fact that in weight 1, much bigger and much smaller $\gamma_\infty(n)$ are known, the smaller ones at powers of 2). See [1, Applegate and Lagarias], [6, Lagarias and Weiss], and [5, Lagarias] for background on the unsolved Collatz 3x+1 problem.

Properties S1-3: (S1) For the first LoH-i (i.e., the LoH-i with no predecessor in our LoH-iSeq), the numbers $\gamma_\infty(l_4) < \gamma_\infty(h_4)$ are on opposite sides of the expected value of $\gamma_\infty(n)$; indeed, as for this range of numbers the expected value empirically appears to be around ≈ 6.946, slightly less than 6.952 that some probabilistic models suggest, for the first LoH-i we require $\gamma_\infty(l_4) < 6.945$ and $\gamma_\infty(h_4) > 6.955$; (S2) Each LoH-i number $k+1$ in the LoH-iSeq is more peculiar than its predecessor LoH-i number k: in each row it has a lower low and a higher high value of γ_∞, each by more than 10^{-i} again. (S3) (a) For the first LoH-i (i.e., the LoH-i with no predecessor in the LoH-iSeq), the average of $\gamma_\infty(h_j) - \gamma_\infty(l_j)$, for $j = 1, \ldots, 8$, is greater than three standard deviations (with $\sigma \approx 0.634$ found experimentally for this range, $3\sigma \approx 1.90$) and (b) for the last LoH-i in the LoH-iSeq that average is more than $11\sigma \approx 6.97$. If in 3(a), we replace 3σ by $5\sigma \approx 3.17$, the sequence will be called an Outer LoH-iSeq.

We also present similar results on just Ramsey weight 8 for Collatz values present in small subintervals of its usual values but not its most condensed part.

1.4. Concurrencies, from the routine nearly linear to the rare

In the latter parts of the paper, we pursue concurrency between less frequent or rare weight on one hand, and either high stopping times for our variant of Mahler or the Collatz on the other. We present some mining results in these cases, like examples of weight-6 or weight-2 numbers below 2^{363} (when used in binary to edge-label a ternary tree) that have large stopping times for our variants of Mahler — some of which considerably apart for our numbers — and are prime too, and in another section establish some lower bounds for how long the LoH-iSEq's could be. We did not anticipate a clear relation between the weight and either of the Mahler or Collatz stopping times. However, we spotted a correlation between Collatz and Ramsey, see the final section of the paper. There are also situations when counting functions viewed or calculated differently, not only are related but actually output the same number. Here is an example:

Proposition 1.3. *Let the double sequence (a_n, b_n) be defined by*

$$a_1 = 1, \quad b_1 = 0, \quad a_{n+1} = 2a_n + b_n, \quad \text{and} \quad b_{n+1} = -a_n + 2b_n,$$

and the double sequence (c_n, d_n) by

$$c_1 = 1, \quad d_1 = 2, \quad c_{n+1} = 2c_n d_n, \qquad \text{and} \quad d_{n+1} = d_n^2 - c_n^2.$$

Then for all n we have $\#\{m \leq 2^n \mid b_m b_{m+1} < 0\}$ is odd if and only if $c_n d_n < 0$.

Proof. With a_m and b_m as in the theorem, an easy induction shows $a_m + b_m i = (2-i)^{m-1}$, for all $m \geq 1$. Let $\beta = \frac{\arctan(\frac{1}{2})}{\pi} \approx 0.147583617650433 26\ldots$ This number (called Plouffe's number) has been shown to be transcendental in some preprints, e.g., by B. Margolius. The mere irrationality of β can also be seen in a preprint by J. Jahnel. For completeness, we include the following irrationality argument. Assume $m\beta \in \mathbb{N}$ for some $m > 1$. We would then have $(\frac{2+i}{\sqrt{5}})^m = \cos(m\beta\pi) = \cos(-m\beta\pi) = (\frac{2-i}{\sqrt{5}})^m \in \{-1, 1\}$. (Note that the same conclusion would hold if we assumed $b_m = 0$.) Therefore $(\frac{2+i}{2-i})^m = (\frac{3+4i}{5})^m = 1$. Thus for some $k \in \{0, 1, \ldots, m-1\}$, we have $\frac{3}{5} = \cos\frac{2k\pi}{m}$. Now if T_m is the m-th Chebyshev polynomial of the first kind, then $T_m(\frac{3}{5}) - 1 = 0$, $T_m - 1$ has integer coefficients and leading coefficient 2^{m-1}, but the denominator 5 does not divide 2^{m-1}, contradiction. Knowing that for $n \geq 2$ the numbers b_n are never zero, their sign variations is of our interest. Note that $\lfloor m\beta \rfloor - \lfloor (m-1)\beta \rfloor = 1$ if $b_m b_{m+1} < 0$, and $\lfloor m\beta \rfloor - \lfloor (m-1)\beta \rfloor = 0$ if $b_m b_{m+1} > 0$, also when $m = 1$. Therefore $\lfloor n\beta \rfloor = \#\{m \leq n \mid b_m b_{m+1} < 0\}$. In particular, $\#\{m \leq 2^n \mid b_m b_{m+1} < 0\}$ equals $\lfloor 2^n\beta \rfloor$, and is therefore odd if and only if the n-th binary digit of β, which we call r_n, is 1. By another easy induction, we have $\frac{c_n}{d_n} = \tan(2^{n-1}\beta\pi)$. This will be negative if and only if the angle $2^{n-1}\beta\pi$ ends in quadrant either II or IV. The former case is equivalent to $(r_{n-1}, r_n) = (0, 1)$, and the latter to $(r_{n-1}, r_n) = (1, 1)$. □

Remark 1.

(a) Here is a very rough sketch for another proof. We can state the recurrence system as a matrix difference equation. The eigenvalues of the 2×2 transition matrix are distinct. As usual, diagonalization gives the solution involving (the real and imaginary parts of) powers of an eigenvalue.

(b) The fact that the n-th binary digit of β is 1 exactly when $\tan(2^{n-1}\beta\pi) < 0$, can also be seen in [2, Borwein and Girgensohn, Example 3.4]. We have $\beta \approx 0.0010010111001000000010\ldots$

(c) All components a, b, c, d in Proposition 1.3 grow exponentially, e.g.,

$$a_n = 5^{\frac{n-1}{2}} \cos\Big((n-1)\arctan\big(\tfrac{1}{2}\big)\Big)$$

and

$$b_n = -5^{\frac{n-1}{2}} \sin\Big((n-1)\arctan\big(\tfrac{1}{2}\big)\Big).$$

2. Depth-2 trees

For depth $n = 1$, the expected value of $f(T)$ is 1. We consider $n = 2$.

Lemma 2.1. *For a random T of depth 2, the probability that $f(T) = 1$ is $\frac{5}{8}$ and the complementary probability that $f(T) = 2$ is $\frac{3}{8}$.*

Proof. Let T be a tree of depth 2 with root r and $f(T) = 2$. Let v_1, v_2, v_3 be the children of r. There must be two edges, say rv_1 and rv_2, which have the same edge-label and rv_3 which has a different one. (otherwise $f(T)$ would be 1). This would have a probability of $\frac{6}{8} = \frac{3}{4}$. For the subtrees of T rooted at v_1 and v_2 to have their majority edge label different from each other the probability is $\frac{1}{2}$. A dozen depth-2 trees are shown top of Fig. 1. □

Corollary 2.2. *The expected weight value for trees T of depth 2 is $\frac{11}{8} = 1.375$.*

The Mathematica code C1 in SC: shorter codes, pdf outputs $\{2560, 1536\}$ as the exact number of trees of depth 2 with $f(T) = 1, 2$, respectively.

3. Depth 3: still exact mean weight found

Lemma 3.1. *For a random T of depth 3,* (i) $p[f(T)=1]=\frac{2275}{16384}\approx 0.138855$, (ii) $p[f(T)=2]=\frac{13569}{16384}\approx 0.828186$, and (iii) $p[f(T)=3]=\frac{135}{4096}\approx 0.032959$.

P r o o f. We show (i) and (iii), then (ii) would also follow.

To show (i), consider the three edges from the root to its children. Either two of these have the same label and the other has the different label or they are all the same. These happen with probability $\frac{6}{8}=\frac{3}{4}$, and $\frac{1}{4}$, respectively. In the first case, using Lemma 2.1, the two subtrees of T rooted at the two children of the root where both edges have the same label will have their weights equal to 1 with probabilities $\frac{5}{8}$ and $\frac{5}{8}$ and then for one to have the same path labels as the other, the probability is $\frac{1}{4}$. So we get $(\frac{5}{8})^2\cdot\frac{1}{4}=\frac{25}{256}$, and implementing the condition of the first case, we get $\frac{3}{4}(\frac{25}{256})$. In the second case, and including the condition of the case, we get the same number $\frac{1}{4}(3)(\frac{5}{8})^2\cdot\frac{1}{4}=\frac{3}{4}(\frac{25}{256})$. But we need to subtract twice the probability when all three are the same and the subtrees have the same 1 (i.e., same unique path label sufficiency). Note that it is not possible for exactly two pairs to have the same 1 (by transitivity of equality and the fact that 1 cannot be realized in more than one way in any tree). The probability is therefore is $\frac{75}{512}-(2)(\frac{1}{4})(\frac{5}{8})^3(\frac{1}{4})^2=\frac{125}{2^{14}}=\frac{2275}{16384}$.

To show (iii), we use the criterion established by Milans (namely, the first level edges must not have same labels and for the two which do, one is followed by a depth 2 subtree with weight 2 and the other with weight 1 whose path label is not covered by binary subtrees of the former, and the other depth 2 subtree following the opposite label has weight 2), see [8, M i l a n s, Proposition 4.2.8], and our Lemma 2.1. The probability equals $\frac{6}{8}\cdot(\frac{3}{8})^2\cdot\frac{5}{8}\cdot\frac{2}{4}=\frac{3^3\times 5}{2^{12}}$. □

Corollary 3.2. *The expected value of $f(T)$ for T of depth 3 is* $\frac{31033}{16384}\approx 1.8941$.

P r o o f. We have $\frac{2275}{16384}(1)+\frac{13569}{16384}(2)+\frac{135}{4096}(3)=\frac{31033}{16384}$. □

We used the code C2 in SC: shorter codes, pdf to simulate the distribution of $f(T)=1,2,3$ for depth 3 via a random sample of size 10,000. Our two runs gave $\{1369, 8298, 333\}$ and $\{1382, 8269, 349\}$. See lower half of Fig. 1 for some.

2RemPrk The following is an interesting property of depth 3, that our mining of maximum weight depth-5 trees will use. Define the property $p(n)$ as follows: There are triples of depth-n trees such that binary subtrees for any two of which together require $f(n+1)$ path labels. (We will see that depth $n=3$ is the smallest depth for which $p(n)$ holds, so far we do not have any other example).

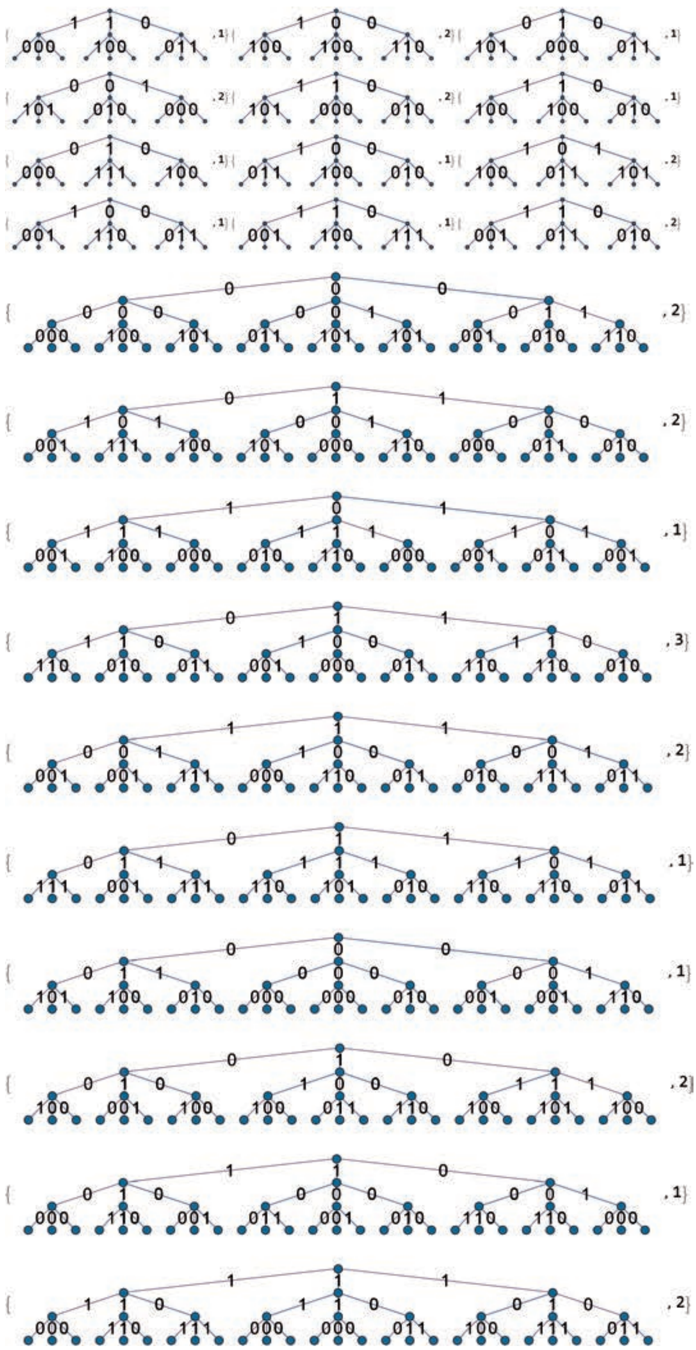


FIGURE 1. Some depth-2 and depth-3 trees with their weights to their right.

4. Depth 4: simulations for the mean

Lemma 4.1. *Take two trees T and T' both of which of depth and weight 3. If the majority edge-labels at the first level from the roots are different, then there are binary subtrees B and B' of, respectively, T and T' that together use exactly four path labels. Moreover, this is not possible with three path labels.*

Proof. For the first assertion, modify the proof by [8, Milans Lemma 4.2.9], case of different majority edge-labels, now for trees of weight 3. For the second assertion, note that for any choices of B and B', one would have either zero or two path labels starting with 0 while the other has either one or three. □

Therefore $f(4) = 4$. In particular, any two trees of depth 3 and weight 3 do not accommodate disjoint sets of path labels for any pair of their binary subtrees.

The Mathematica code C3 in SC: shorter codes, pdf was used to simulate the distribution of $f(T) = 1, 2, 3, 4$ for depth 4, here are the outputs of three runs for several sample sizes.

$$\begin{array}{ccc}
\{0,3,6,1\} & \{0,4,4,2\} & \{0,3,7,0\}; \\
\{0,35,55,10\} & \{0,41,55,4\} & \{1,34,63,2\}; \\
\{2,376,538,84\} & \{3,366,551,80\} & \{4,355,562,79\}; \\
\{40,3541,5687,732\} & \{33,3600,5623,744\} & \{26,3542,5705,727\}; \\
\{395,36187,56246,7172\} & \{393,35991,56471,7145\} & \{362,35729,56692,7217\}; \\
\{3597,360207,564528,71668\} & \{3589,361906,563045,71460\} & \{3513,361603,563360,71524\}.
\end{array}$$

Remark 3. These, with their almost consistencies for the latter and larger samples, are expected to come close to calculations one can try to undertake based on heuristic arguments. E.g., we find the probability for the weight to be 1 as follows. Either the first level labels (on edges from the root) are not all the same ($p = \frac{6}{8}$), or they are. In the former case, we use the probability we found earlier for the two subtrees of depth 3 following the same outer labels to have weight 1 ($p = \frac{2275}{16384}$) and indeed the same path label of length 3 ($p = \frac{1}{8}$). In the latter case we calculate similarly, but have to subtract twice the probability that all the three siblings of depth 3 have weight 1 and that is witnessed by the same path label of length 3.

$$\left(\frac{6}{8}\right)\left(\frac{2275}{16384}\right)^2\left(\frac{1}{8}\right)+\left(\frac{2}{8}\right)(3)\left(\frac{2275}{16384}\right)^2\left(\frac{1}{8}\right)-(2)\left(\frac{2}{8}\right)\left(\frac{2275}{16384}\right)^3\left(\frac{1}{8}\right)^2\approx 0.00359422.$$

With the least likely event (weight 1) having this close simulation and theoretical probabilities, we achieve more confidence in our approximations for the other events (weights 2–4). We would therefore say that weight 1–4 trees constitute $\approx 0.4^{-}\%$, $36\ \%$, $56^{+}\%$, and $7\ \%$, respectively, of the $2^{120} > 1.3\times 10^{36}$ depth-4 trees.

Remark 4. These distributions indicate that the expected value of the weight in depth 4 is highly probable to belong to the interval $(2.5, 2.9)$.

Consider the doubling set D, the set of positive integers n such that $f(n+1) = 2f(n)$, the maximum possible increase going from depth n to depth $n+1$.

Remark 5. We have $n \in D$ if and only if there are two trees T_1 and T_2 of depths n with $f(T_1) = f(T_2) = f(n)$ such that any pair of a binary subtree for T_1 and another for T_2 use disjoint path labels.

Only if: Consider a full weight tree of depth $n+1$ and two same label edges emanating from its root, and take the two trees of depth n that follow those edges. If: Consider a tree of depth 1 and edge labels 0,0, and 1 and attach such two trees of depth n to the first two edges and an arbitrary full weight tree of depth n to the third.

Of course, $n = 1$ is in D. The number $n = 2$ is not in D, indeed in depth 2 any full weight tree has a set of three path labels any pair of which realized in some binary subtree, see [8, Milans, Proposition 4.2.3]. Therefore any two depth 2 trees of weight 2 will have overlapping path labels. In Lemma 4.1, we observed that $n = 3$ also fails the criterion. The next such depth is $n = 4$: $f(4+1) = 2f(4)$, as we see in the next section, based on Fig. 2. It is not known whether any other number, besides 1 and 4, belongs to D.

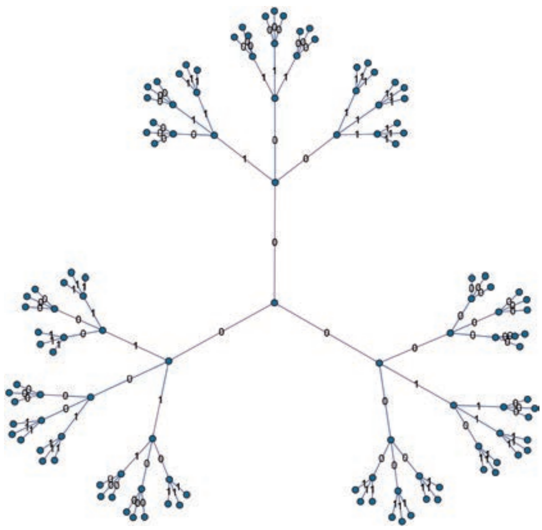


Figure 2. Example of T of depth 4 and weight 4 with the additional property that the first three edges are labelled 0, see code 15 in SC: shorter codes, pdf. This tree is used to construct two trees of depth 5, one of weight 7 and the other 8.

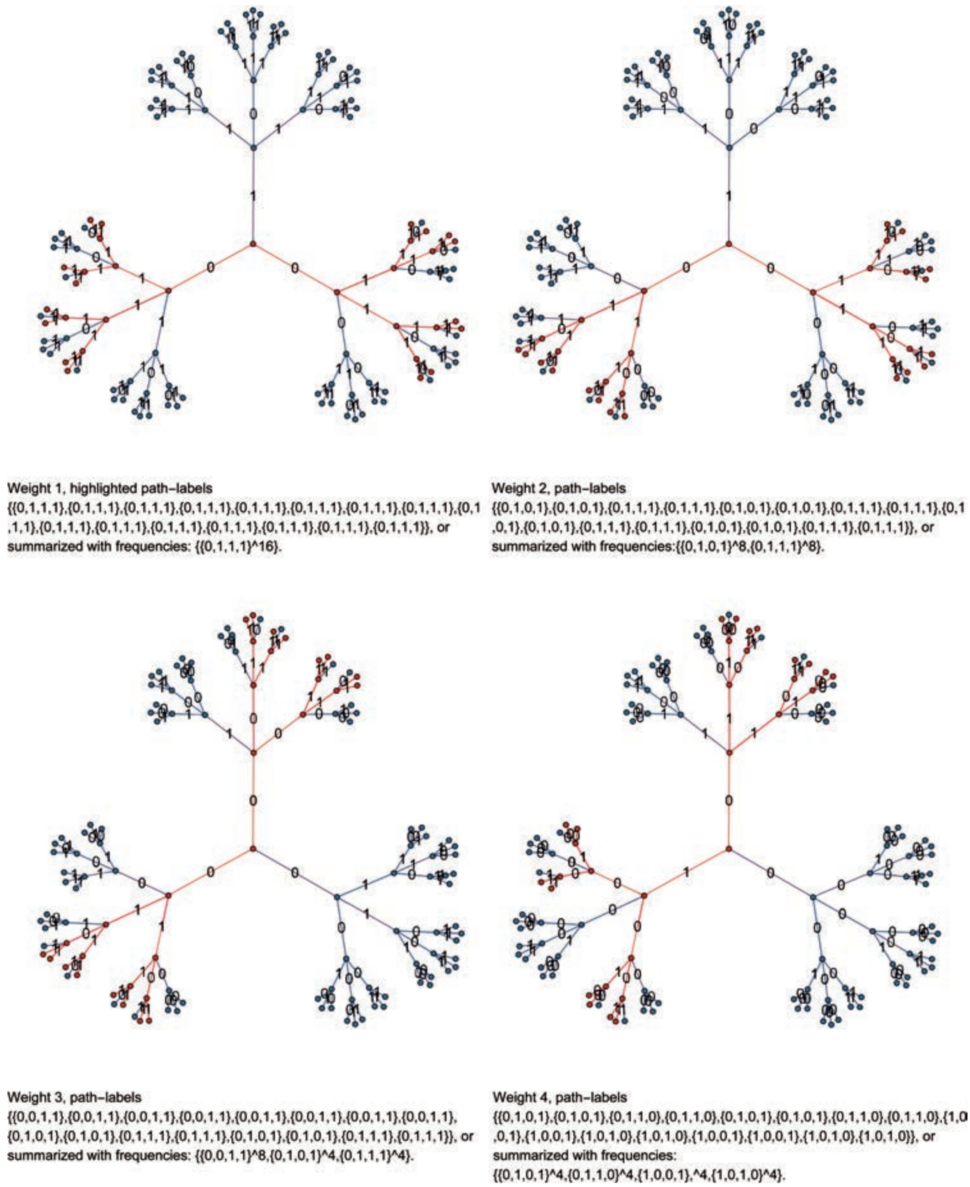


FIGURE 3. Four trees of depth 4 of weights 1–4.

5. Depth 5: three randomness types; means, and mining

Here computations get more involved, still we constructed sizable samples of trees of all weights.[1] Weights 1–2 are easy to construct and we have plenty in the section on Collatz. Some weight 2–6 trees also appear in the next section with further properties. For weight 6, we use the inequality $f(m+n) \geq f(m)f(n)$. A tree built up from trees of maximum weights in depths 2 and 3 would have weight at least 6 (some stats appear later on percentages of w=6,7).

The authors of [3, Downey et al.] announced in a Note Added in Proof that $f(5) = 8$. The proof was kindly communicated to us by C. Jockusch. It is based on the special depth-4 trees of weight 4 like above.

Lemma 5.1 (DGJM). *There is a tree T of depth 4 and weight 4 such that all three edges emanating from the root are labeled 0.*

See Figure 2 (labeling by DGJM), and code C4 in SC: shorter codes, pdf.

Corollary 5.2 (DGJM). *There is a a tree of depth 5 and weight 8.*

Proof. Here is a tree U constructed by the authors of [3, Downey et al.]. Let the edges emanating from the root have labels 0, 0, and 1. Let T^* be the same as T that was constructed in Lemma 5.1, but with the three edges emanating from the root all labeled 1 instead of 0. Note that the path labels in T are disjoint from those in T^*. The subtrees of U below the edges emanating from the root are T, T^*, Y where T, T^* are glued to the two edges labeled 0, and Y is any tree of depth 4 and weight 4. Let the children of the root be a, b, and c, where the edges from the root to a and b have label 0. If a binary subtree S contains a and b, it has weight 8 by the disjointness. Otherwise it contains a and c, or b and c, and again we have disjointness of the paths as the top labels are different. $\square$

To get an example of depth 5 and weight 7, we put T in the upper left and T^* in the lower branch each following an edge labeled 0, and a tree of depth 4 and weight 3 in the upper right following an edge labeled 1. See Fig. 7.

Again, to get an example of depth 5 and weight 8, put two copies of T in the upper left and right where the first follows an edge labeled 0 and the other an edge labeled 1, and T^* below those following an edge labeled 0 (the upper right copy of T can be replaced by any tree of depth 4 and weight 4). See Fig. 8.

The codes C5 and C6 in SC: shorter codes, pdf convert a depth-5 tree to its edge-labels, or vice versa.

[1]Here are links to 8400 trees of weight 1–8:
250 weight 4 trees, 250 weight 3 trees, 250 weight 2 trees, 250 weight 1 trees,
500 weight 5 trees, 1000 weight 6 trees, 2900 weight 7 trees, 3000 weight 8 trees.

Each illustraed tree T in this section is followed by edge-labels, path labels and how many of each, the weights of the three children of the root of depth 4 ("west", "south", "east"), the minimum number of path labels needed pairwise $(\mathrm{WS}, \mathrm{WE}, \mathrm{SE})$ to path label binary subtrees for both, the weight of T, highlighted binary subtree using such a minimal number of path labels.

Before those, here are weights of trees in our 25 samples of various sizes, see Fig. 4. The overall distribution was $\{0, 85, 1936, 4509, 1544, 126, 0, 0\}$ of 8200.

Size	Sample's Weight 1-8 Freq.	Mean	Confid. Int., 99%
100	{0, 2, 24, 54, 20, 0, 0, 0}	3.92	{3.73, 4.11}
	{0, 1, 23, 57, 18, 1, 0, 0}	3.95	{3.77, 4.13}
	{0, 0, 28, 55, 17, 0, 0, 0}	3.89	{3.72, 4.06}
	{0, 1, 23, 46, 24, 6, 0, 0}	4.11	{3.88, 4.34}
	{0, 0, 25, 56, 19, 0, 0, 0}	3.94	{3.77, 4.11}
150	{0, 1, 37, 85, 26, 1, 0, 0}	3.93	{3.78, 4.07}
	{0, 1, 39, 70, 36, 4, 0, 0}	4.02	{3.85, 4.19}
	{0, 3, 32, 89, 25, 1, 0, 0}	3.93	{3.78, 4.07}
	{0, 2, 39, 84, 23, 2, 0, 0}	3.89	{3.74, 4.05}
	{0, 1, 35, 79, 34, 1, 0, 0}	3.99	{3.84, 4.15}
200	{0, 2, 55, 94, 44, 5, 0, 0}	3.98	{3.83, 4.12}
	{0, 3, 49, 98, 45, 5, 0, 0}	4.00	{3.85, 4.15}
	{0, 2, 50, 103, 43, 2, 0, 0}	3.97	{3.83, 4.1}
	{0, 3, 39, 109, 43, 6, 0, 0}	4.05	{3.91, 4.19}
	{0, 2, 45, 120, 33, 0, 0, 0}	3.92	{3.8, 4.04}
300	{0, 1, 71, 149, 71, 8, 0, 0}	4.05	{3.93, 4.16}
	{0, 1, 82, 178, 35, 4, 0, 0}	3.86	{3.76, 3.96}
	{0, 3, 73, 175, 46, 3, 0, 0}	3.91	{3.81, 4.01}
	{0, 0, 68, 165, 61, 6, 0, 0}	4.02	{3.91, 4.12}
	{0, 5, 71, 170, 51, 3, 0, 0}	3.92	{3.81, 4.03}
450	{0, 5, 90, 244, 105, 6, 0, 0}	4.04	{3.95, 4.13}
600	{0, 6, 139, 337, 107, 11, 0, 0}	3.96	{3.89, 4.04}
800	{0, 8, 198, 429, 146, 19, 0, 0}	3.96	{3.89, 4.03}
1100	{0, 14, 259, 628, 188, 11, 0, 0}	3.93	{3.88, 3.98}
1500	{0, 18, 342, 835, 284, 21, 0, 0}	3.97	{3.92, 4.01}

CI (99%) Plot

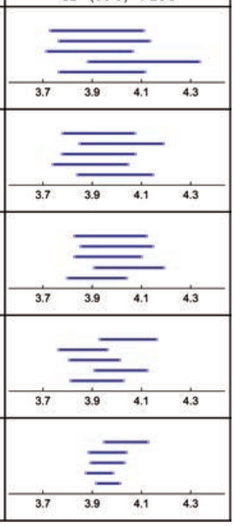


FIGURE 4. Confidence Intervals for the mean weight in depth 5.

Consider the sequence defined by $a_0 = 1$ and $a_m = \lceil \frac{3a_{m-1}}{2} \rceil$ for $m \geq 1$. As in [3, Downey et al., Lemma 4.1], for each m, there exist depth-m trees using each path label at most a_m times. We used this inductive level-wise argument in an algorithm for constructing sets of trees of higher average weight than random.

Our file Generating W1, Rand, IndAB&LogSA includes codes to generate trees of these classes: Random; [3, Downey et al., Lemma 4.1]-method for w4Plus or simply w4p, call it Inductively Almost Balanced; and op. cit. [Prop. 2.1 (2)] — method for w6Plus or simply w6p, using Log Super-Additivity instance $f(2+3) \geq f(2)f(3)$. (Note: The IndAB method often over-performs its perceived lower bound $\lceil \frac{2^5}{a_5} \rceil = 3$, it is not clear if it can give 3. Also, IndAB or LogSA trials did not give $w = 8$.) Fig. 13 shows the bar charts for our random, respectively IndAB, respectively LogSA trees.

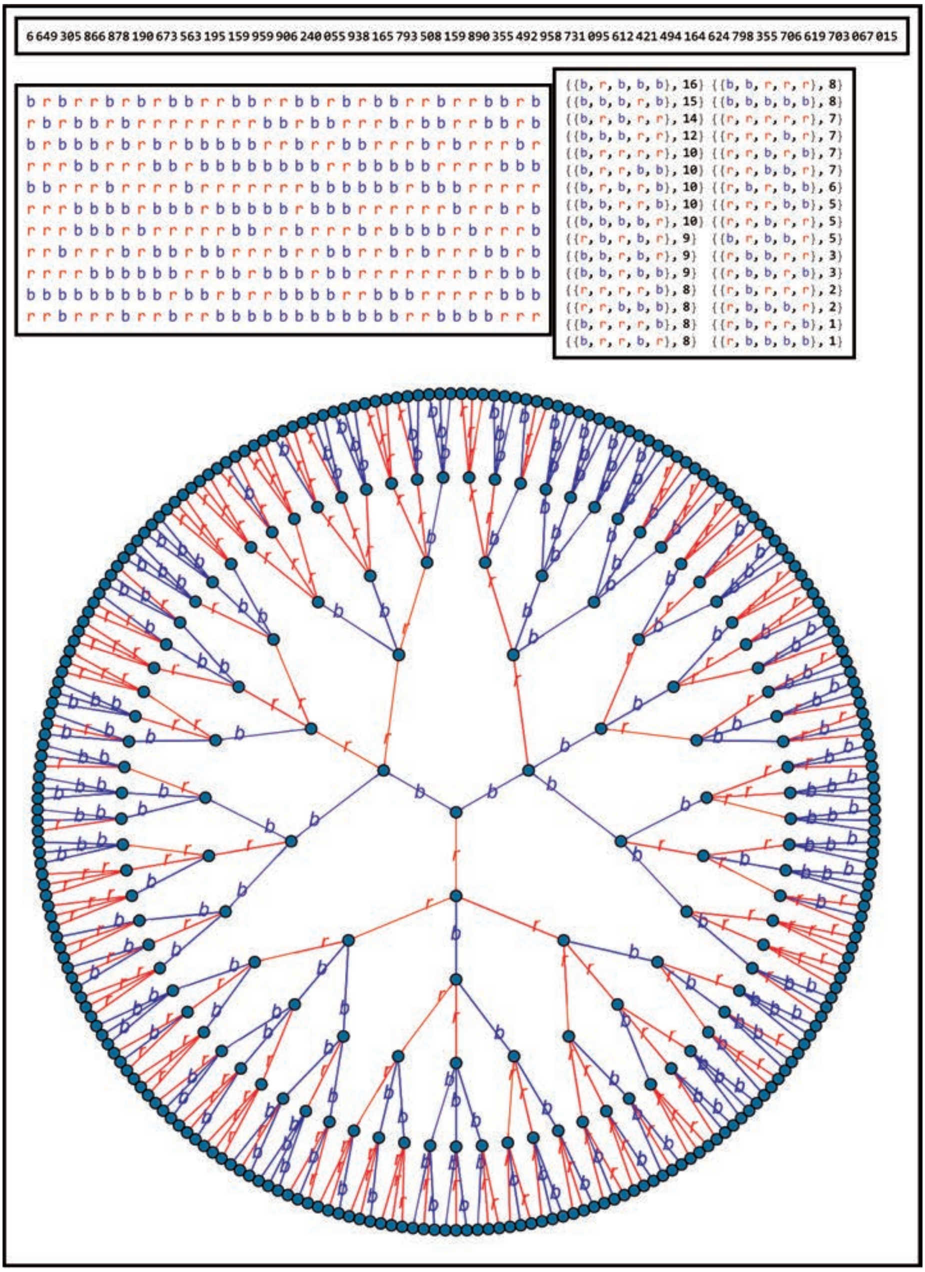
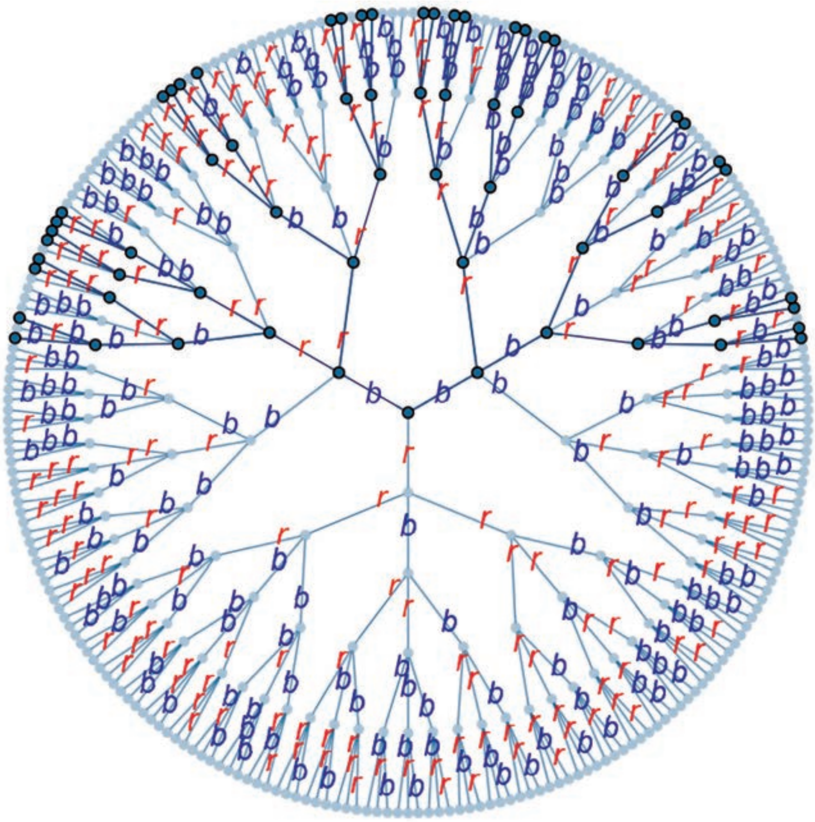


FIGURE 5. A tree of weight 7 of 'type 744', see next page.



{{4, 4, 4}, {4, 7, 4}, $\boxed{7}$}

{{b, r, r, r, r}, 6} {{b, r, b, b, b}, 6} {{b, b, r, b, r}, 4} {{b, r, r, b, r}, 2}
{{b, r, b, r, r}, 6} {{b, r, r, r, b}, 4} {{b, b, r, b, b}, 4}

FIGURE 6. Binary subtree, weights, and pathlabels of the special w7-tree. The lower depth-4 cluster of original tree has two binary subtrees using the minimal number 4 of pathlabels that have a single common path--label. The upper clusters are 'distant' but are 'close' to the lower one. (There are depth-4 trees of weight 4 that allow only three binary subtrees with 4 pathlabels and those sets of path-labels have pairwise intersections of size 3: example on Rarible.).

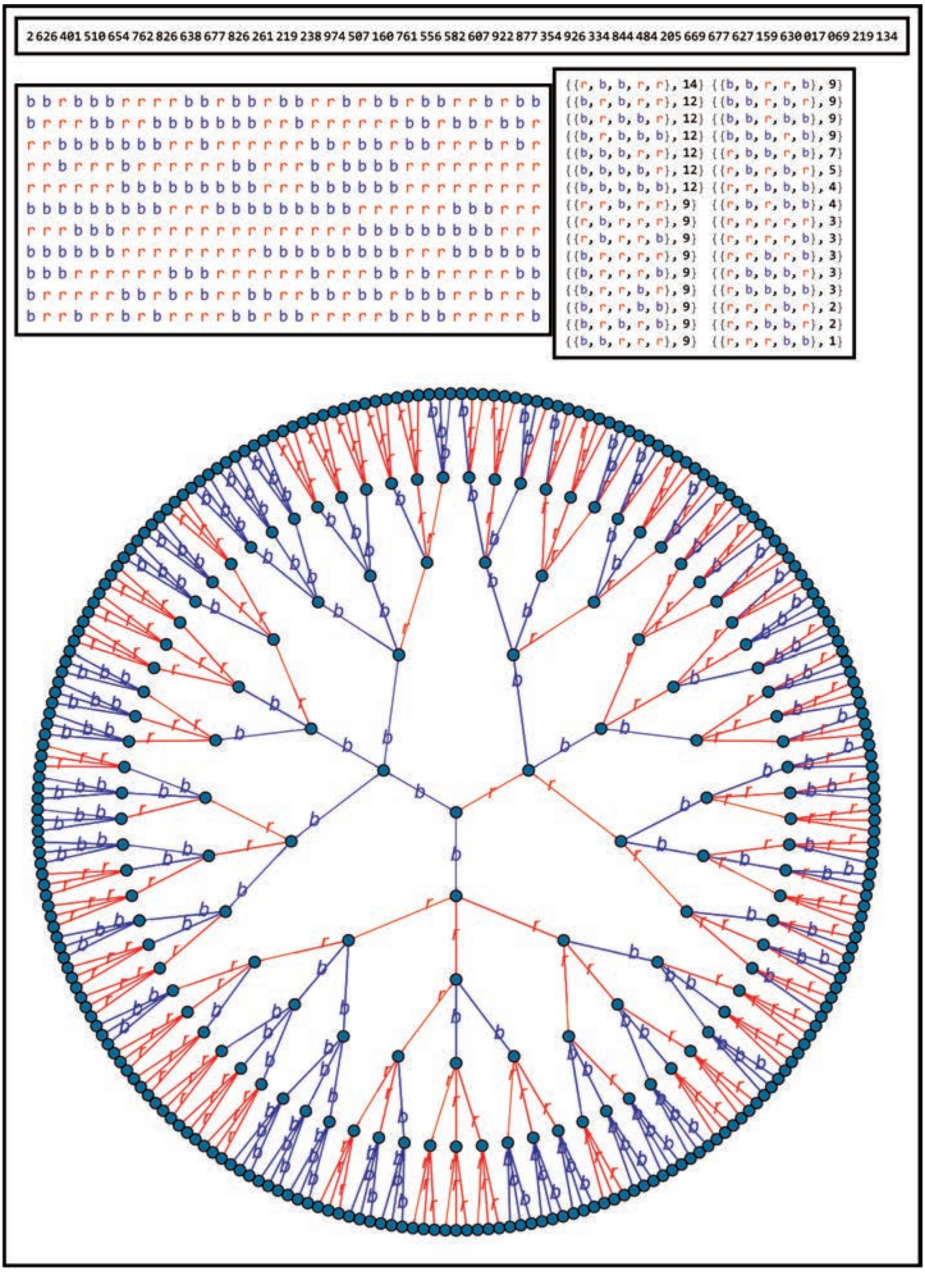


FIGURE 7. Another example of w(T)=7.

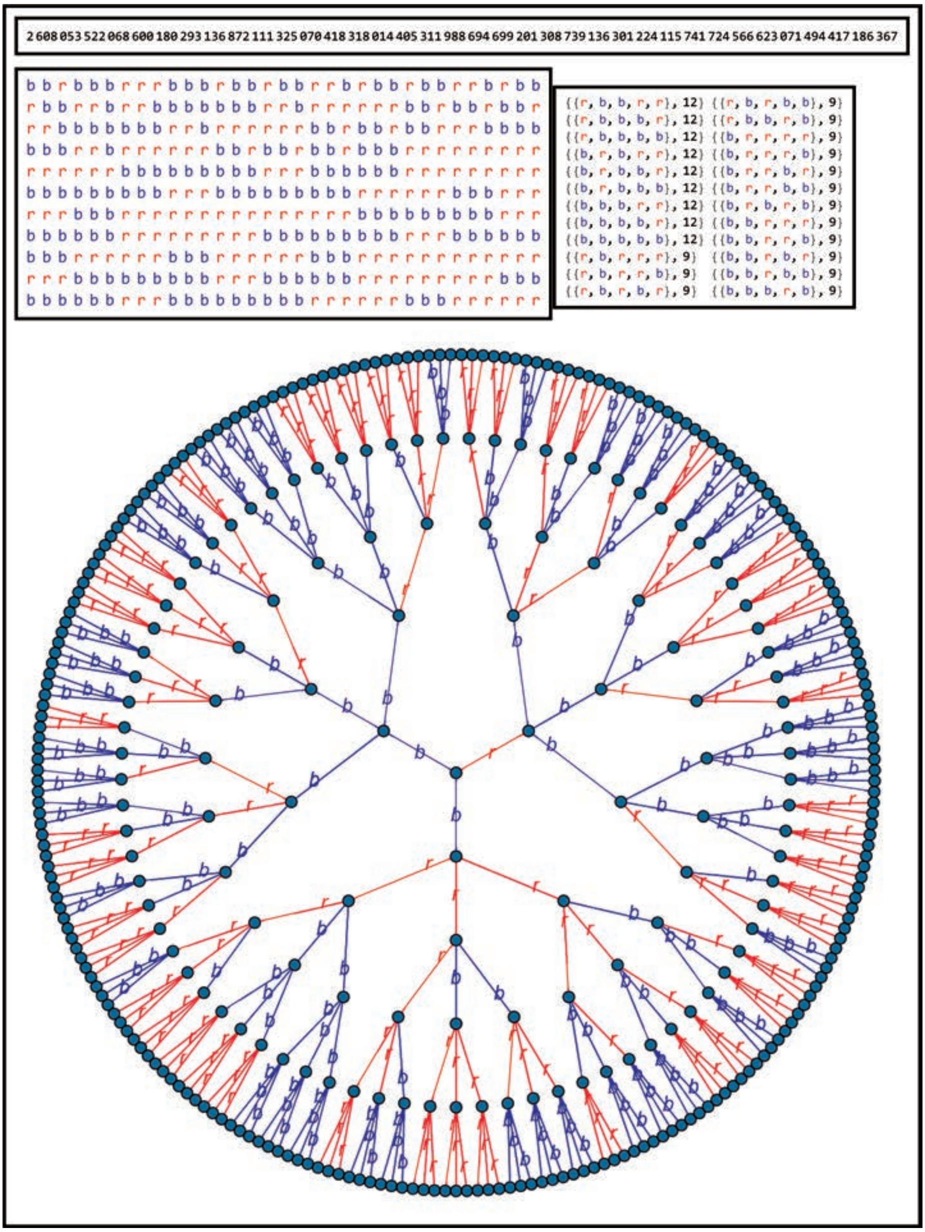


FIGURE 8. Example of w(T)=8.

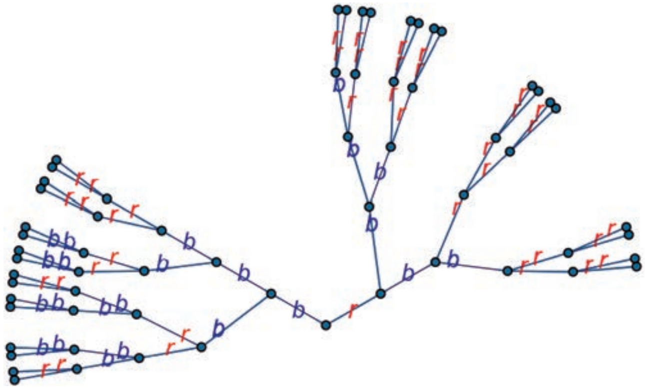


FIGURE 9. Binary subtree extracted from our 2nd weight-7 tree.

```
{{4, 4, 3}, {8, 4, 6}, 7}

{{r, b, b, r, r}, 10} {{b, b, r, b, r}, 4} {{b, b, b, r, r}, 4} {{r, b, b, b, r}, 2}
{{r, b, r, r, r}, 4}  {{b, b, r, b, b}, 4} {{b, b, b, r, b}, 4}
```

FIGURE 10. The weights, followed by pathlabels used.

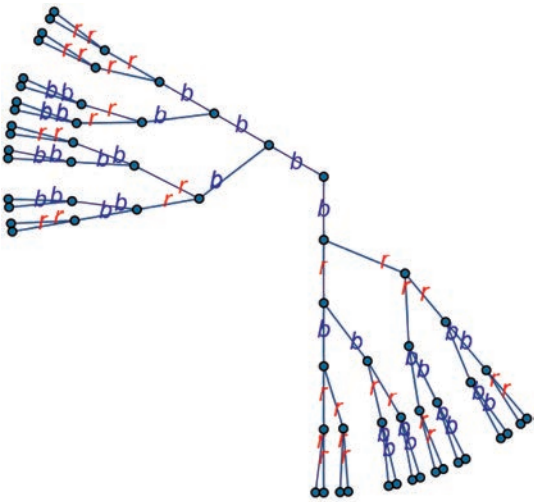


FIGURE 11. Binary subtree extracted from our weight-8 tree.

```
{{4, 4, 4}, {8, 4, 8}, 8}

{{b, r, r, b, r}, 4} {{b, r, b, r, r}, 4} {{b, b, r, b, r}, 4} {{b, b, b, r, r}, 4}
{{b, r, r, b, b}, 4} {{b, r, b, r, b}, 4} {{b, b, r, b, b}, 4} {{b, b, b, r, b}, 4}
```

FIGURE 12. The weights, followed by pathlabels used.

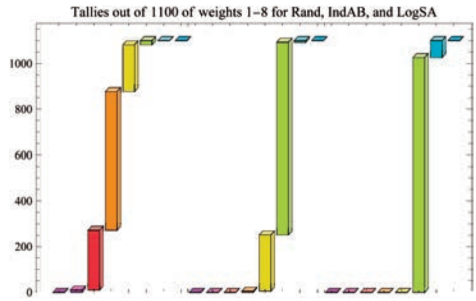


FIGURE 13. Bar charts for 1100 trees via Rand, IndAB, and LogSA. Links to three smaller samples: **Random**, **IndAB**, **LogSA**.

6. On trees of depth 6: a new lower bound

Computing optimal binary subtrees for edge-labeled ternary trees of higher depths is much more computationally expensive. It also appears an impossible task if a heuristic approach is undertaken to find the probabilities. We know $f(6) = f(3{+}3) \geq f(3)\times f(3) = 9$ (we also know $f(6) \leq 2f(5) = 16$). We improve this lower bound to $f(6) \geq 12$.

PROPOSITION 6.1. *We have $f(6) \geq 12$.*

Proof. We build a depth 6 tree by first taking a depth 2 tree R as shown in the first two layers in Fig. 14, and using the special depth 4 trees T and T^* of weight 4 as in the previous section (T and T^* require a total of 8 path labels for both to be covered).

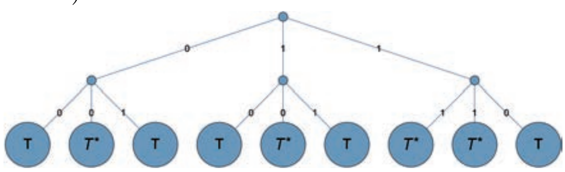


FIGURE 14. A tree of depth 6 and weight 12; here the clusters after the second level are abbreviated. In Fig. 15 we show this as an expanded tree.

Each of the first two clusters requires 8 path labels (these clusters after the root are the same as the depth 5 weight 8 tree constructed earlier, and so this is either due to path labels above them being different and circled children being of weight 4, or when the path labels above them are the same we have used the special *fully disjoint* trees). These first two clusters have disjoint path labels. The last cluster requires four path labels as a minimum. The only path it has in common with the second cluster is $10\,T$. If they are used in the binary subtree, then we still need to use from the middle cluster either $\{1, 0, T^*\}$ or $\{1, 1, T\}$.

Either of these together with $\{1, 1, T^*\}$ from the third cluster requires 8 more path labels, for a total of 12. (If the binary subtree intersects the third cluster, at least one of the paths with initial labels $\{1, 1\}$ must be used. This follows with T^*, while the $\{1, 1\}$ in the middle cluster is followed by T.) $\square$

Remark 6. We showed $f(6) \geq 12$. If $f(6) \geq^{(?)} 14$, that would slightly increase a lower bound in [3, Downey et al.] for $\lim_{n\to\infty} f(n)^{\frac{1}{n}}$ (as a byproduct of their lower bound $2^{\frac{n-3}{\log_2 3}}$ on $f(n)$), from $2^{\frac{1}{\log_2 3}} \approx 1.548$ to $\sqrt[6]{14} \approx 1.552$. Note that by Fekete's Lemma, due to log super additivity, the limit and the sup agree.

Remark 7. One can now get depth-6 trees of lower weights also. For weight 11, replace the T in the branch 11T by a depth 4 tree of weight 3 which has disjoint path labels from T^* (e.g., take a depth 3 tree of weight 3 and attach three copies of it to a common root with the new edges labeled 0). To get weight 10 (respectively 9), replace that by a depth 3 tree of weight 2 (respectively 1) with the first three edges labeled 0. (Weights 1–8 of depth 6 are easy to get examples, using same weight trees of depth 5.)

For the depth 6 tree of weight 12 that we presented, it was possible, as highlighted, to get a binary subtree using 12 path labels, in a way that its restriction to each terminal depth 4 cluster uses no more than the least possible number of path labels for a binary subtree there (which is $f(4) = 4$). Any two clusters in this tree have either disjoint sets of path labels, or (only for 10 T in clusters 2 and 3, or 11 T^* in cluster 3) had the same such sets and required four path labels.

That is not always the case, as we show below there are trees S_0, S_1, S_2 of depth 4 and weights 4,4,3 such that for any two of which any pair of individually optimal binary subtrees, one for each of the two, would together require at least seven path labels with one pair needing no more; while for any two of S_0, S_1, S_2 any pair of (not necessarily individually optimal) binary subtrees, one for each of the two, would together require at least five path labels with one pair needing no more. For our triple, the code C7 in SC: shorter codes, pdf returns $\{8, 7, 7\}$, (pairwise minimum needed path labels: 6,5,5). Individual trees' weights: 4, 4, and 3; the number of path labels that can be used in a binary subtree: 5, 5, and 3, and these three sets are disjoint. Putting these trees together, we get:

Corollary 6.2. *There are depth 6 trees where any binary subtree with the additional property of being locally optimal (for its terminal depth-4 clusters) uses at least 14 path labels, nevertheless the weight of the tree is 10.*

Proof. See Fig. 16. $\square$

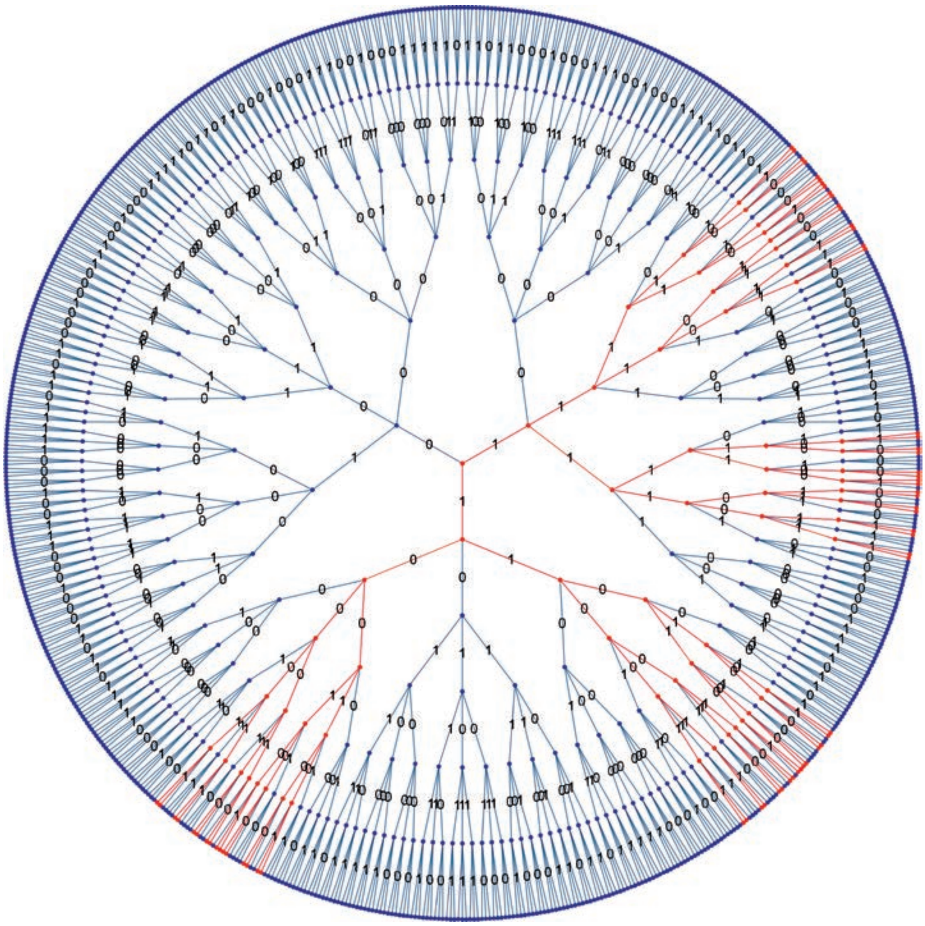


FIGURE 15. A tree of depth 6 and weight 12, with an optimal binary subtree highlighted using 12 path labels:

$$\{1,0,0,0,1,1\}^4,\quad \{1,0,0,0,1,0\}^4,\quad \{1,0,0,1,0,1\}^4,\quad \{1,0,0,1,0,0\}^4,$$
$$\{1,1,0,0,1,1\}^4,\quad \{1,1,0,0,1,0\}^4,\quad \{1,1,0,1,0,1\}^4,\quad \{1,1,0,1,0,0\}^4,$$
$$\{1,1,1,0,1,1\}^8,\quad \{1,1,1,0,1,0\}^8,\quad \{1,1,1,1,0,1\}^8,\quad \{1,1,1,1,0,0\}^8.$$

In the last layer, every terminal triple of edges have the same labels, for readability we show only one label in each group. There are 543 edges labeled 0 and 549 labeled 1, and 56 path-labels used 9, 12, 18, or 24 times.

```
0 1 1 1 0 0 0 0 1 1 1 1 0 1 0 1 0 1 1 0 1 0 1 0 1 1 0 1 1 0 1 0 0 1 0 0 1 1 0
0 0 0 0 0 1 1 0 1 1 0 1 1 1 1 0 1 1 1 1 0 1 0 0 0 1 1 1 0 1 1 0 0 0 0 0 1 0 0 1
0 1 1 1 1 0 0 0 1 0 0 0 0 0 0 0 0 1 1 1 0 1 0 1 0 1 0 0 0 1 0 0 1 1 1 0 1 1 0 0

4, {{1, 0, 1, 1}, {1, 0, 1, 0}, {1, 1, 1, 1}, {1, 1, 0, 1}, {1, 1, 0, 0}}
1 1 0 1 0 0 0 1 1 0 1 1 1 1 1 0 0 1 0 0 1 0 0 0 1 0 0 1 1 1 0 0 0 1 0 0 1 1 1 1
1 1 0 1 0 1 1 0 1 1 1 0 1 0 0 1 1 1 0 1 0 0 1 0 0 0 1 1 0 0 1 0 1 0 1 1 1 0 0 1
0 1 1 0 0 1 1 0 0 0 0 0 0 1 1 0 0 1 0 1 0 1 1 1 0 0 1 0 1 1 0 0 1 1 0 0 0 0 0 0

3, {{1, 0, 0, 1}, {1, 0, 0, 0}, {1, 1, 1, 0}}
{8, 7, 7}, {6, 5, 5}

0 0 1 0 0 0 1 1 1 0 0 0 0 1 1 0 1 1 0 1 1 1 1 1 0 0 0 1 0 1 1 1 1 0 0 0 1 0 1
1 1 1 0 0 0 1 0 1 0 0 0 0 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 1 1 0 1 1 1 0 0 0
0 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 1 1 0 1 1 1 0 0 0 0 1 1 1 1 1 1 1 1 1 0 0
0 0 0 0 0 0 1 1 0 1 1 1 0 1 0 1 0 1 1 0 1 0 1 0 1 1 0 1 1 0 1 0 0 1 0 0 1 1 0
1 0 1 0 1 0 0 1 0 1 0 1 0 0 1 0 0 1 0 1 1 0 1 1 0 0 1 1 1 0 0 1 0 0 1 0 0 0 1
0 0 1 1 1 0 0 0 1 0 0 1 1 1 1 0 1 0 1 0 1 1 0 1 0 1 0 1 1 0 1 1 0 1 0 0 1 0 0
1 1 0 1 0 1 0 1 0 0 1 0 1 0 1 0 0 1 0 0 1 0 1 1 0 1 1 0 0 1 1 1 0 0 1 0 0 1 0
0 0 1 0 0 1 1 1 0 0 0 1 0 0 1 1 1 1 0 1 0 1 0 1 1 0 1 0 1 0 1 1 0 1 1 0 1 0 0
1 0 0 1 1 0 1 0 1 0 1 0 0 1 0 1 0 1 0 0 1 0 0 1 0 1 1 0 1 1 0 0 1 1 1 0 0 1 0
0 1 0 0 0 1 0 0 1 1 1 0 0 0 1 0 0 1 1 1 1 0 0 0 0 0 1 1 0 1 1 0 1 1 1 1 0 1 1
1 1 0 1 0 0 0 1 1 1 0 1 1 0 0 0 0 0 1 0 0 1 0 1 1 1 1 0 0 0 1 0 0 0 0 0 0 0 0
1 1 1 0 1 0 1 0 1 0 0 0 1 0 0 1 1 1 0 1 1 0 0 1 1 1 1 1 0 0 1 0 0 1 0 0 0 0 1
0 0 0 0 1 0 1 1 1 0 0 0 1 0 0 1 1 1 1 1 0 1 1 0 1 0 0 0 0 1 1 1 0 1 1 1 1 1 1
1 1 0 0 0 1 0 1 0 1 0 1 1 1 0 1 1 0 0 0 1 0 0 1 1 1 1 0 1 0 1 1 0 1 1 1 0 1 0
0 1 1 1 0 1 0 0 1 0 0 0 1 1 0 0 1 0 1 0 1 1 1 0 0 1 0 1 1 0 0 1 1 0 0 0 0 0 0
1 1 0 0 1 0 1 0 1 1 1 0 0 1 0 1 1 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 1 0 1
1 1 1 0 1 1 1 1 0 1 0 0 0 1 1 1 0 1 1 0 0 0 0 0 1 0 0 1 0 1 1 1 1 0 0 0 1 0 0
0 0 0 0 0 0 1 1 1 0 1 0 1 0 1 0 0 0 1 0 0 1 1 1 0 1 1 0 0 1 1 1 1 1 0 0 1 0 0
1 0 0 0 0 1 0 0 0 0 1 0 1 1 1 0 0 0 1 0 0 1 1 1 1 1 0 1 1 0 1 0 0 0 0 1 1 1 0
1 1 1 1 1 1 1 1 0 0 0 1 0 1 0 1 0 1 1 1 0 1 1 0 0 0 1 0 0 1 1 1 1 0 1 0 1 1 0
1 1 1 0 1 0 0 1 1 1 0 1 0 0 1 0 0 0 1 1 0 0 1 0 1 0 1 1 1 0 0 1 0 1 1 0 0 1 1
0 0 0 0 0 0 1 1 0 0 1 0 1 0 1 1 1 0 0 1 0 1 1 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 1
1 0 1 1 0 1 1 1 1 0 1 1 1 1 0 1 0 0 0 1 1 1 0 1 1 0 0 0 0 0 1 0 0 1 0 1 1 1 1
0 0 0 1 0 0 0 0 0 0 0 0 1 1 1 0 1 0 1 0 1 0 0 0 1 0 0 1 1 1 0 1 1 0 0 1 1 1 1
1 0 0 1 0 0 1 0 0 0 0 1 0 0 0 0 1 0 1 1 1 0 0 0 1 0 0 1 1 1 1 1 0 1 1 0 1 0 0
0 0 1 1 1 0 1 1 1 1 1 1 1 1 0 0 0 1 0 1 0 1 0 1 1 1 0 1 1 0 0 0 1 0 0 1 1 1 1
0 1 0 1 1 0 1 1 1 0 1 0 0 1 1 1 0 1 0 0 1 0 0 0 1 1 0 0 1 0 1 0 1 1 1 0 0 1 0
1 1 0 0 1 1 0 0 0 0 0 0 1 1 0 0 1 0 1 0 1 1 1 0 0 1 0 1 1 0 0 1 1 0 0 0 0 0 0
```

FIGURE 16. Labels for depth-4 trees of weight 4 and 3 where any minimal set of pathlabels is a subset of the mentioned disjoint sets of pathlabels. These with the dual of the weight-4 (with opposite edge-labels) when used as terminal depth-4 clusters give a tree of depth 6 and weight 10, with an optimal (but not locally optimal) binary subtree using 10 path labels: $\{0,1,1,0,0,1\}^8, \{0,1,1,0,0,0\}^8, \{0,1,1,1,1,0\}^8, \{0,1,0,1,0,0\}^4, \{0,1,0,1,0,1\}^4$ $\{1,0,1,0,0,1\}^8, \{1,0,1,0,0,0\}^8, \{1,0,1,1,1,0\}^8, \{1,0,0,1,0,0\}^4, \{1,0,0,1,0,1\}^4$. For this depth-6 tree with the last array as edge-labels, to make the binary subtree optimal we must allow such clusters to be non-optimal.

7. Ramsey-Mahler-Prime diamonds: apartness within

We present some concurrencies involving the weight and versions of Mahler. Recall that $a(g) \leq 1 + b(g) = 1 + c(g) \leq d(g)$, and $a(g) < d(g)$. Our examples g will have $a(g) \geq 230$. We also obtain numbers g that additionally satisfy $\max\{d(g) - c(g), d(g) - a(g)\} \geq 23$ (note that $d(g) - c(g) \leq 1 + d(g) - a(g)$). We give an example of a diamond: quadruples of numbers (g_1, g_2, g_3, g_4) below 2^{363} (referred to as the bottom, left, right, and top) such that they all are $(\geq 230, \geq 23)$-Mahler, g_2 has weight $\neq 3, 4, 5$, g_3 is prime, and g_4 is either prime of weight $\neq 3, 4, 5$ or a lower twin prime (of any weight). Note that it cannot be a higher twin prime since such primes are of the form $6k + 1$.

We point to 10 examples included in the following collections of NFTs (half of the 10 we present here with further details, paper remains self-sufficient):

MR345: 300 items, $a(g) \geq 230$, and weight 3,4, or 5.

MR62: 900 items, $a(g) \geq 230$ and weight 2 or 6.

PM345: 150 item, $a(g) \geq 230$, weight 3,4, or 5, and g a prime.

PM62TM345: 460 items, $a(g) \geq 230$, prime w=2,6, or a lower twin w=3,4,5.

First, here is a not-quite diamond pick from those collections:[2]

$d(g) - c(g) = 21$, g: lower twin prime, w4 (item 385 in PM62TM345)

$d(g) - a(g) = 23$, w6 (124MR6) $\approx$ D $d(g) - a(g) = 22$, g: prime, w5 (146PM345)

$d(g) - a(g) = 23$, w4 (2MR345)

Theorem 7.1. (a) *There is a number of weight 6 with apartness greater than 32.*

(b) *There is a diamond with mean of the four apartness values greater than 24.*

Proof. (a) For an example of a g of weight 6 with $d(g) - c(g) = 33$, see Fig. 17.

(b) We show the following diamond is realizable for four appropriate numbers g:

$d(g) - a(g) = 25$, g: prime, w6 (302PM62TM345)

$d(g) - c(g) = 24$, w2 (96MR2) Our D $d(g) - c(g) = 23$, g: prime, w4 (45PM345)

$d(g) - c(g) = 25$, w5 (19MR345)

See Fig. 18-.21. [3]

In the captions for these figures, the True/False columns indicate whether at that row, we have $\{g(\frac{2}{3})^k\} < \frac{1}{2}$, $\lfloor g(\frac{2}{3})^k \rfloor \neq 1 (\mathrm{mod} 3)$, respectively $\lfloor g(\frac{2}{3})^k \rfloor = f_g(k)$. $\square$

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[2] Direct links for these four: top left right bottom; and a lower twin with $w = 3, d - a = 17$.

[3] Links to the examples in this theorem within the collections: High, DB, DL, DR, DT.

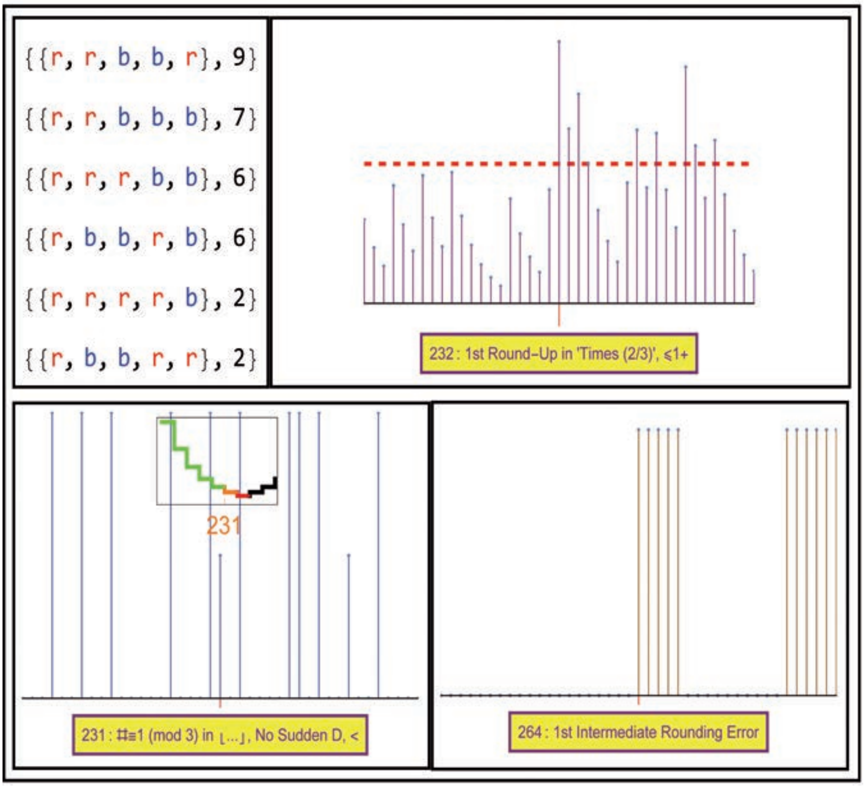

FIGURE 17. $d(g) - c(g) = 33$, and g is of weight 6.

Here g is: $g = 1560865380047521261855269353794795895427332731943766$ $5954506279836580309493471118441669227277204980248449460739$.

In multiplying this number g by powers of $2/3$, the first round up is at exponent 232, the first 1 mod 3 for the floor occurs at exponent 231, but the first intermediate rounding error does not occur until exponent 264.

Here are the floors for changed status iterations 231–264:

328310374281510715013132676218069962707105238470177766401885322012069	TFT	231,
218873582854340476675421784145379975138070158980118510934590214674712	FTT	232,
432342879712277484790956610657540691630755869590357552463388807836980	TTT	236,
8540106265921530563771982432741544526039622115365087456066925004835	FTT	240,
5693404177281020375847988288494363017359748076910058304044616669890	TTT	241,
3795602784854013583898658858996242011573165384606705536029744446593	FTT	242,
2530401856569342389265772572664161341048776923071137024019829631062	TTT	243,
1686934571046228259510515048442774227365851282047424682679886420708	TFT	244,
1124623047364152173007010032295182818243900854698283121786590947138	FTT	245,
499832465495178743558671125464525696997289268754792498571818198728	TTT	247,
333221643663452495705780750309683797998192845836528332381212132485	FTT	248,
222147762442301663803853833539789198665461897224352221587474754990	TTT	249,
19502684219900283242039294028184511268298438165100880907542365321	FTT	255,
3852382068128451010773193882110520744355247044958198697786146236	TTT,	259,
2568254712085634007182129254740347162903498029972132465190764157	FTT	260,
1141446538704726225414279668773487627957110235543169984529228514	TTT	262,
760964359136484150276186445848991751971406823695446656352819009	FFT	263,
507309572757656100184124297232661167980937882463631104235212672	TTF	264.

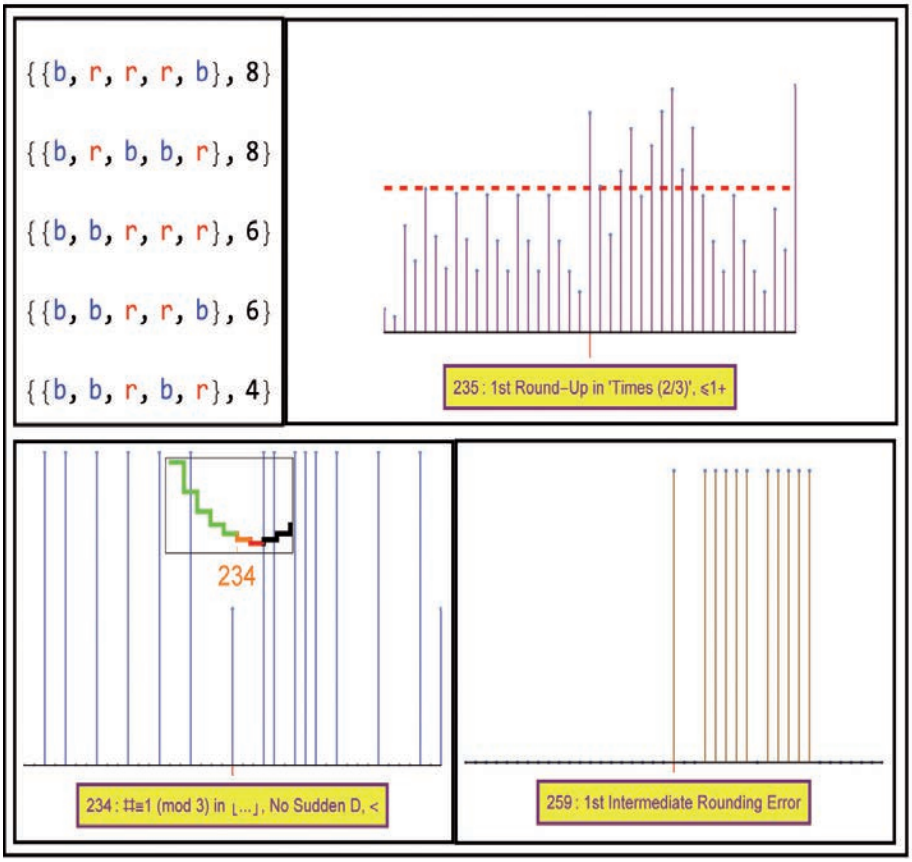

FIGURE 18. d(g)-c(g)=25, the bottom of our diamond.

Here g is: $g = 98067103171696235122342420725653416753145820867548304185199476285502691710604101088158662417373141339289728 17$.

In multiplying this number g by powers of $2/3$, the first round up is at exponent 235, the first 1 mod 3 for the floor occurs at exponent 234, but the first intermediate rounding error does not occur until exponent 259.

Here are the floors for changed status iterations 234–259:

611179428076321248297309672503391291720651639560974308407393624 10424 TFT 234,
4074529520508808321982064483355941944804344263739828722715957494 0282 FTT 235,
1810902009115025920880917548158196419913041894995479432318203330 6792 TTT 237,
1207268006076683947253945032105464279942027929996986288212135553 7861 FTT 238,
53656355825630397655730890315798412441867907999866057253872691279 38 TTT 240,
35770903883753598437153926877198941627911938666577371502581794186 25 FTT 241,
471057170485644094645648419782043675758511126473446867523710869942 TTT 246,
18379917031599586683323578031428620788626558204115591843632065646 TFT 254,
12253278021066391122215718687619080525751038802743727895754710430 FTT 255,
5445901342696173832095874972275146900333795023441656842557649080 TFT 257,
3630600895130782554730583314850097933555863348961104561705099386 FFT 258,
2420400596753855036487055543233398622370575565974069707803399590 TFF 259.

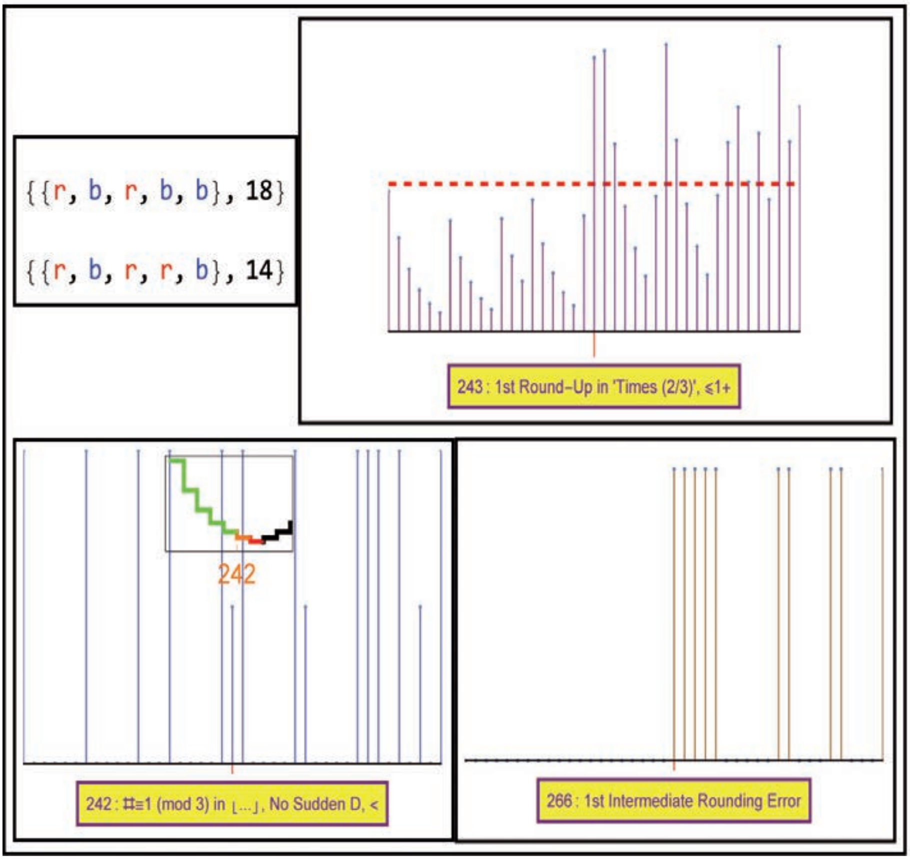

FIGURE 19. $d(g) - c(g) = 24$, w is not 3,4,5, the left of our diamond.

Here g is: $g = 12461566749206550645580330683591293833868038957206937539283625238988998954045198554004544013211001829573607008$.

In multiplying this number g by powers of $2/3$ the first round up is at exponent 243, the first 1 mod 3 for the floor occurs at exponent 242, but the first intermediate rounding error does not occur until exponent 266.

Here are the floors for changed status iterations 242–266:

30303162631163307738586412824511421369821023370003860908290479055 89 TFT 242,
2020210842077553849239094188300761424654734891333590727219365270392 FTT 243,
5985809902452011405152871669039293110088103381729157710279600801 16 TTT 246,
177357330443022560152677679082645721780388248347530598823099282997 TFT 249,
118238220295348373435118452721763814520258832231687065882066188664 FTT 250,
52550320131265943748941534543006139786781703214083140392029417184 TTT 252,
10380310149385865431889685835655533785043299400312719089783588579 FTT 256,
2050431634446590702595493498401093093341886301296339573290585398 TFT 260,
1366954422964393801730328998934062062227924200864226382193723598 FTT 261,
270015688486793837378830419542530777724034410047254594013575031 FFT 265,
180010458991195891585886946361687185149356273364836396009050020 TTF 266.

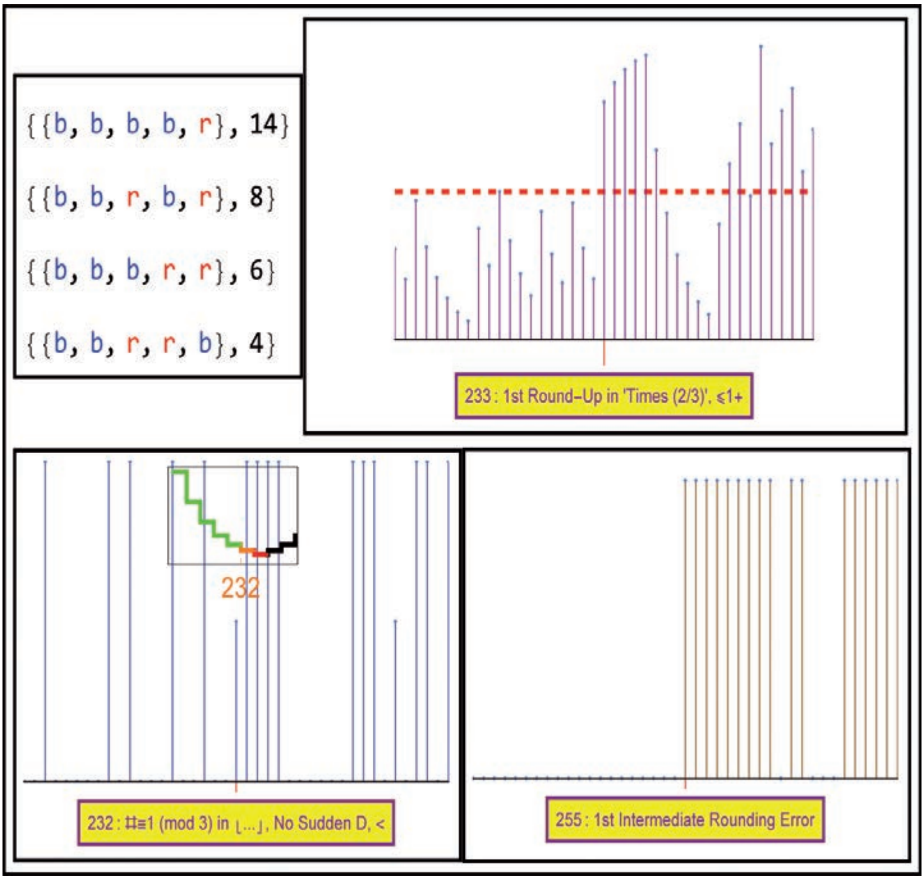

FIGURE 20. $d(g) - c(g) = 23$, g is prime, the right of our diamond.

Here g is $g = 132709122629774542051582174017237904156650820257650478344693893552219145905729945706714391721491481616871865$1.

In multiplying this number g by powers of $2/3$, the first round up is at exponent 233, the first 1 mod 3 for the floor occurs at exponent 232, but the first intermediate rounding error does not occur until exponent 255. Here are the floors for changed status iterations 232–255:

18609241718559004947180236930658877288358459111657443853149023449236	TFT 232,
12406161145706003298120157953772584858905639407771629235432682299490	FTT 233,
1089155436660060646199849257944369589807902499447715049475571559900	TTT 239,
95618584288400385948957959545184710216331632324627933012944553955	FTT 245,
42497148572622393755092426464526537873925169922056859116864246202	TFT 247,
28331432381748262503394950976351025249283446614704572744576164134	FTT 248,
2487258809920286420051134242093917168661372542306025590744683819	FFT 254,
1658172539946857613367422828062611445774248361537350393829789212	TTF 255.

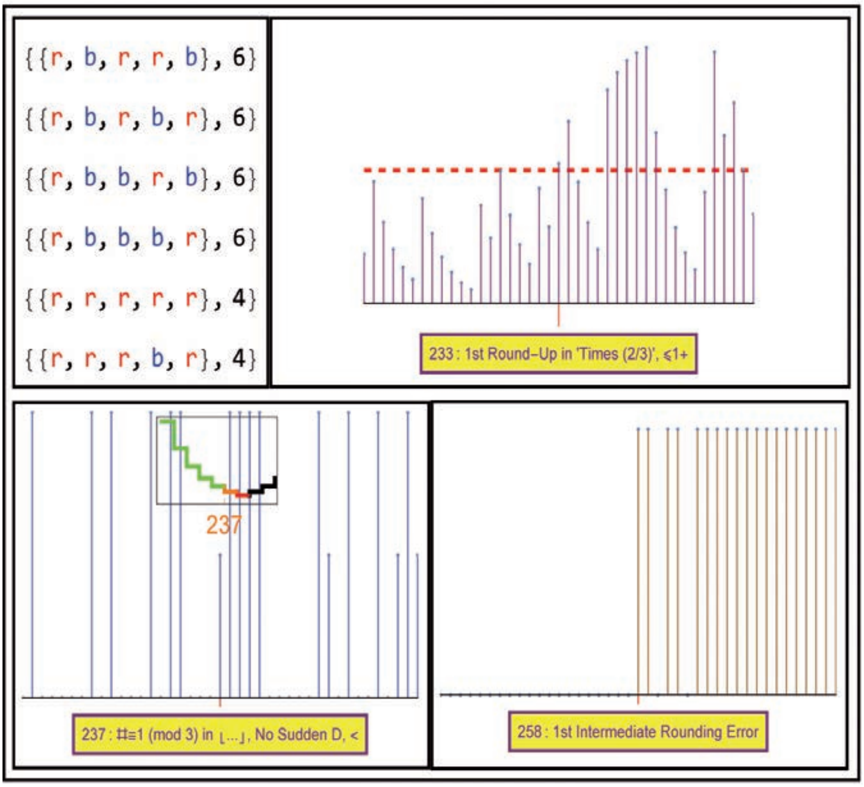

FIGURE 21. $d(g) - a(g) = 25$, g is prime, and w $\neq 3, 4, 5$; the top of our diamond.

Here g is $g = 1600214652100269722804051284394303203788480435404806883986182582091695590806470937557839945850898704174696574 7$.

In multiplying this number g by powers of $2/3$ the first round up is at exponent 233, the first 1 mod 3 for the floor occurs at exponent 237, but the first intermediate rounding error does not occur until exponent 258. Here are the floors for changed status iterations 233–258:

1495942437737261899374435907024570192909653184320802971713471143345 37	FTT 233.
664863305661005288610860403122031196848734748587023542983764952597 94	TTT 235,
295494802516002350493715734720902754154993221594232685770562201154 64	TFT 237,
196996535010668233662477156480601836103328814396155123847041467436 42	FTT 238,
172946203575895294299019725854026303300590454339560053857484964553 2	TTT 244,
341622130520287001084483409094372944791289786349748254533303633685	TFT 248,
227748087013524667389655606062915296527526524233165503022202422456	FTT 249,
44987276447115983681907280209958577091857091206551210473521466164	TTT 253,
29991517631410655787938186806639051394571394137700806982347644109	FTT 254,
19994345087607103858625457871092700929714262758467204654898429406	TFT 255,
13329563391738069239083638580728467286476175172311469769932286270	FTT 256,
8886375594492046159389092387152311524317450114874313179954857513	FFT 257,
5924250396328030772926061591434874349544966743249542119969905008	TTF 258.

8. How long can Ramsey-Collatz LoH successions be?

For $i = 1, \ldots, 6$, it is of interest to construct long LoH-iSeq's and long Outer LoH-iSeq's. Let m_i denote the maximum possible length of a LoH-iSeq, and m'_i for the outer ones. These exist since we start with $\gamma_\infty(l_4) < 6.945 - 4 \times 10^{-i}$ (considering the l_8 in that LoH-iSeq) and it decreases by more than 10^{-i} each time going to the next LoH-i and cannot go below $\frac{1}{\ln 2} \approx 1.44$ (since powers of 2 have this smallest possible value of γ_∞). For example, for $i = 3$, there is the obvious upper bound of about 5500 for the length of any LoH-3Seq. Our mining established the following lower bounds:

Theorem 8.1. *We have*

(i) $m_1 \geq 19$, *and* $m'_1 \geq 16$,

(ii) $m_2 \geq 124$, *and* $m'_2 \geq 99$,

iii) $m_3 \geq 380$, *indeed* $m'_3 \geq 380$ *as well,*

(iv) $m_4 \geq 800$, *and* $m'_4 \geq 615$,

(v) $m_5 \geq 1110$, *and* $m'_5 \geq 688$.

Proof. Here we consider $i = 1, 3, 5$, where we have made the data available on Zenodo (an open repository developed under the European OpenAIRE program).

Zenodo link: A LoH-1Seq of length 19, the last 16 terms of which are outer.

Zenodo link: A LoH-3Seq of length 380, all its terms are outer.

Zenodo link: A LoH-5Seq of length 1110, the last 688 terms of which are outer.

A Mathematica code that for any number $n < 2^{363}$ gives the weight of n as well as $\gamma_\infty(n)$ is linked here. To illustrate just a few LoH-i's here, we consider $i = 5$ and present our first and second, then the three indexed with multiples of 360, then the last of our LoH-5's in the constructed LoH-5Seq of length 1110. See Fig. 23 through 28.[4] □

[4] For cases $i = 2, 4, 6$, see the collection of minted NFTs: (also showing $m_6 \geq 1394$, and $m'_6 \geq 720$). On the left, Traits can be selected from 1 to 1, or 2 to 2, or ..., or 6 to 6, and then sorted. For the first four groups, some LoH-i's have their main 16 numbers redacted (again, for $i = 1, 3$ they are on Zenodo — open to everyone — and for $i = 2, 4$ the owner — by which we mean the owner of the value of the involved mining as proof of work gets these from that platform's built-in features). For the last two groups (of sizes 1110 and 1394), all numbers are shown and they are grouped in 74, respectively 93, sets of 15 (with the last set in LoH-6 of size 14 having a few stronger lows and highs — which lead to the following result).

Before the six figures, we give the following result with a similar apartness theme.

Theorem 8.2. *There is a pair of numbers l_6 and h_6 below 2^{363} whose Collatz orbits contain numbers l_5, l_4, l_3, l_2, l_1, respectively h_5, h_4, h_3, h_2, h_1 (indices need not be sorted as the order of appearance in the orbit), these 10 numbers also below 2^{363}, such that:*

(i) *l_j and h_j are of weight j, for $j = 1, \dots, 6$,*

(ii) *For suitable $s_j, t_j \in \{-9, -8, \dots, 8, 9\}$, for $1 \le j \le 6$, the 12 weights do not change, however $\gamma_\infty(l_j) < \gamma_\infty(h_j + s_j) < \gamma_\infty(l_j + t_j) < \gamma_\infty(h_j)$ and the mean of $\{\gamma_\infty(h_j) - \gamma_\infty(h_j + s_j),\ \gamma_\infty(l_j + t_j) - \gamma_\infty(l_j); 1 \le j \le 6\}$ exceeds 5.3.*[5]

(iii) *The mean of the two numbers $\gamma_\infty(h_6) - \gamma_\infty(l_6)$ and $\gamma_\infty(h_5) - \gamma_\infty(l_5)$ exceeds 8,*

(iv) *The mean of the six numbers $\gamma_\infty(h_j) - \gamma_\infty(l_j), 1 \le j \le 6$, exceeds 9.*

(v) *We have $\gamma_\infty(h_j) > 12$ for $j = 2, \dots, 6$, and $\gamma_\infty(h_1) > 14$.*

Proof.

Let $l_6 = 2622441567633348756060229718185339914943852744384044120$
$6013905987646266392224930459106215090633974235593333 83$.

Take the 6th, 5th, 11th, 200th, and 1004th Collatz iterate of l_6 as $l_5, \dots, l_1$.

Let $h_6 = 3808097834144861522973977429574661701126083749826310170$
$9597706391590555571254507735897355557086007380946088 87$.

Take the 5th, 63rd, 158th, 153rd, and 1605th Collatz iterate of h_6 as $h_5, \dots, h_1$.

Take the s's as $-6, 1, 6,\ 5, 1, 1$; and the t's as $1, -1, -1, 6, -9, -9$. See Fig. 22.

The mentioned means are 5.36189, 8.00544, and 9.00543, link to code in pdf. $\square$

One might want to see the numbers l_j and h_j, for j for some $j < 6$ right here:

$l_4 = 663805521807191403877745647415664165970162725922211168 0$
$272269953122961180531935522461260694816724728384562628$,

$l_2 = 1582621650709857325527713514938570503347126631$
$293888164789478653762701721980307542787480744$, and $l_1 = 64$.

$h_4 = 2550217765475607974761046408173816824484747946608 78$
$087449199284472524821231402973167998035516051626998813$,

$h_2 = 608603042952343640933942621364377327892059241 07$
$4041460467922287049171645156923289096883710911505575$,

l_j stp t & scld	shftd h_j s.t. & s.s.t.	shftd l_j s.t. & s.s.t	h_j s.t. & s.s.t
{1010, 4.0458}	{1686, 6.7436}	{1760, 7.0501}	{3011, 12.0432}
{1004, 4.0179}	{1681, 6.7281}	{1754, 7.0194}	{3006, 12.0314}
{1005, 4.0108}	{1623, 6.7511}	{1755, 7.0040}	{2948, 12.2626}
{999, 4.0183}	{1528, 6.6904}	{1749, 7.0351}	{2853, 12.4919}
{810, 3.9000}	{1525, 6.7045}	{1560, 7.5111}	{2858, 12.5649}
{6, 1.4427}	{940, 9.4095}	{71, 17.7175}	{1406, 14.0742}

Figure 22. The σ_∞ and γ_∞ values for our numbers and their mentioned shifts.

[5] For other suitable non-zero single-digit numbers s'_j, t'_j this time for $1 \le j \le 5$, the weights still remain the same and the 10 numbers $\{\sigma_\infty(h_j) - \sigma_\infty(h_j + s'_j),\ \sigma_\infty(l_j + t'_j) - \sigma_\infty(l_j);$ $\le j \le 5\}$ are all 0: take the s''s as $1, 2, 1, 2, -8$; and the t''s as $-1, 1, 1, 4, 2$. For the other two indices we have $\sigma_\infty(l_1) = \sigma_\infty(l_1 - 43) = 6$ (and for numbers greater than 64, $\sigma_\infty > 6$). Also, $\sigma_\infty(h_1) = \sigma_\infty(h_1 - 1638) = \sigma_\infty(h_1 + 265358) = 1406$.

8-tuple of pairs of numbers of weights 1-8 as indicated.
Collatz lows and highs are truncated to six decimals (not rounded) and are
differently monotone towards the w4 low and high, where w4 has the lowest low and highest high.
The last column shows the difference between the high and low Collatz values in each row.
Apartness is more than 10^-5: within the LoH, the least and the averagre of the 14
absolute value of differences (all required to be > 10^-5) are the first two rounded numbers below.
This is the first LoH in the sequence, so one of the conditions on apartness from the predecesso does not apply here.
The average high minus low for the first LoH (starting at the opposite sides
of the expected Collatz value) in our LoH_5 sequence exceeds 1.99 and for our last LoH_5
(LoH_5 number 1110) it is more than 7.18; here for this <u>LoH_5 number 1</u>,
the average difference is the other boxed rounded number below.

{$\boxed{0.000565}$, $\boxed{0.691586}$, $\boxed{\text{No predecessor in our LoH\_5Seq}}$, $\boxed{1.994915}$}

7 341 751 071 703 620 291 365 541 518 144 017 106 \ 754 651 172 750 183 781 378 102 999 145 061 137 128 \ 083 562 807 290 504 770 390 636 062 766 815	1	6.438673	7.365866	1	17 931 939 493 217 965 102 278 504 462 112 570 519 \ 343 319 813 484 669 114 030 621 483 368 788 067 214 \ 480 296 934 819 894 643 730 202 459 549 625	0.927193
10 743 040 375 955 015 867 098 482 816 792 394 680 \ 194 813 414 817 945 050 145 729 663 587 233 696 221 \ 834 236 970 447 969 445 487 841 313 129 541	2	6.393060	7.646024	2	2 699 475 425 701 330 210 794 460 659 119 943 629 \ 665 009 380 721 166 887 173 863 124 459 433 794 117 \ 517 741 545 145 364 561 326 434 523 508 010	1.252964
881 281 745 718 118 983 244 295 261 579 987 656 259 \ 995 297 588 368 788 209 096 865 215 285 189 482 089 \ 817 405 693 600 249 133 709 043 094 385	3	6.340704	10.139804	3	2 299 121 295 085 340 614 988 544 262 981 856 148 \ 784 852 536 415 492 400 191 256 922 029 018 681 335 \ 847 721 811 784 450 501 380 560 717 196 271	3.799100
12 053 684 992 049 139 466 777 407 499 531 186 250 \ 528 381 499 706 571 634 300 607 604 251 158 899 006 \ 206 698 897 947 837 417 555 613 634 902 693	4	4.996644	10.355368	4	5 597 516 191 115 384 519 252 186 945 897 203 339 \ 062 626 226 707 018 590 702 011 511 873 350 581 207 \ 363 388 264 540 608 278 552 689 713 862 256	5.358724
10 799 320 406 340 707 208 334 910 163 395 265 684 \ 448 746 697 396 382 634 920 493 316 840 433 124 287 \ 138 939 942 507 982 418 768 418 607 686 729	5	6.436742	8.208492	5	6 008 061 845 215 890 061 980 422 386 955 714 819 \ 125 081 923 691 321 524 838 001 331 106 019 883 434 \ 479 145 261 490 707 717 402 558 824 368 391	1.771750
4 159 370 774 358 497 534 131 306 988 531 041 140 \ 910 140 473 452 607 251 261 270 707 488 913 090 114 \ 420 377 130 081 739 337 926 464 545 921 826	6	6.437307	7.897879	6	9 720 676 720 713 612 797 290 597 498 700 425 491 \ 903 471 654 801 371 947 995 725 031 502 572 541 655 \ 692 023 505 725 156 190 378 814 309 255 580	1.460572
10 731 970 675 939 239 380 213 083 580 499 702 464 \ 141 389 061 752 414 931 121 925 871 911 163 465 016 \ 106 010 307 055 588 995 938 684 015 486 983	7	6.608181	7.889154	7	3 181 859 497 435 116 498 510 611 666 842 803 160 \ 884 644 302 951 685 792 961 656 655 813 344 081 654 \ 406 432 867 527 886 456 826 253 644 155 798	1.280973
2 606 958 982 045 808 630 096 311 635 517 292 297 \ 847 335 002 667 212 644 454 525 138 456 270 701 257 \ 737 141 452 029 801 029 791 531 873 020 330	8	6.942069	7.050114	8	2 617 300 026 270 084 136 517 724 824 370 312 242 \ 727 432 629 097 699 612 163 097 137 178 617 302 677 \ 499 679 423 060 754 654 079 106 737 049 052	0.108045

Figure 23. Our first LoH5, included in Zenodo file LoH5-NoRedact.zip.

8-tuple of pairs of numbers of weights 1-8 as indicated.
Collatz lows and highs are truncated to six decimals (not rounded) and are
differently monotone towards the w4 low and high, where w4 has the lowest low and highest high.
The last column shows the difference between the high and low Collatz values in each row.
Apartness is more than 10^-5: within the LoH, the least and the averagre of the 14 absolute
value of differences (all required to be > 10^-5) are the first two rounded numbers below.
Also compared to the previous LoH_5 in our onloine sequence, for the absolute values of each Collatz value minus the
corresponding one in the predecessor LoH (the successor has lower lows and highjer highs), the least and the
average of the 16 positive differences (all required to exceed 10^-5) are the third and fourth rounded numbers.
The average high minus low for the first LoH (starting at the opposite sides of the expected Collatz value) in
our LoH_5 sequence exceeds 1.99 and for our last LoH_5 (LoH_5 number 1110) it is more than 7.18; here
for this **LoH_5 number 2**, the average difference is the fifth boxed rounded number below.

$\{\boxed{0.000154}, \boxed{0.691075}, \boxed{0.000055}, \boxed{0.002465}, \boxed{1.999845}\}$

10 127 260 082 511 952 571 231 168 794 968 229 083 \ 629 070 045 220 841 899 462 698 779 768 957 332 240 \ 277 527 956 062 966 744 401 134 422 325 358	1	6.438390	7.366415	1	17 599 164 992 575 993 149 453 724 958 084 887 898 \ 945 111 391 282 043 089 477 689 225 032 043 677 365 \ 485 893 683 534 555 111 875 500 454 646 320	0.928025
4 358 293 666 300 685 835 354 255 067 537 432 014 \ 138 882 372 653 483 305 750 409 217 848 325 025 952 \ 651 504 272 437 474 234 166 283 858 024 182	2	6.392132	7.647562	2	1 171 486 338 517 415 468 916 004 017 610 063 514 \ 828 738 066 112 697 694 678 732 765 189 634 321 974 \ 146 872 496 479 621 190 617 774 608 516 305	1.255430
16 128 749 842 484 578 815 934 904 235 730 657 290 \ 695 190 062 808 579 317 436 116 506 656 305 399 265 \ 340 766 245 439 484 291 852 994 414 511 575	3	6.335008	10.140076	3	2 283 755 891 571 550 831 945 606 353 976 748 682 \ 277 455 574 222 334 671 541 454 565 550 801 023 924 \ 310 599 209 418 839 645 116 709 506 726 780	3.805068
12 629 921 837 763 353 222 541 145 715 918 290 611 \ 894 308 733 498 061 800 069 123 499 798 785 233 766 \ 416 239 858 624 840 061 513 704 965 829 101	4	4.995715	10.355700	4	6 735 844 568 307 220 571 736 036 381 459 920 098 \ 511 936 301 783 660 884 332 494 210 637 278 761 386 \ 003 957 192 328 359 712 842 856 373 039 381	5.359985
7 968 352 771 228 311 657 157 943 412 476 281 326 \ 709 045 087 119 834 676 338 988 071 512 303 423 388 \ 861 876 255 018 301 161 564 019 422 191 933	5	6.436570	8.216945	5	13 883 894 089 868 710 135 968 153 657 427 957 141 \ 070 835 405 838 585 550 735 218 145 058 813 177 685 \ 393 241 184 216 205 077 737 853 145 222 501	1.780375
5 805 313 706 179 836 887 929 747 811 652 309 473 \ 253 130 367 797 317 343 414 592 923 062 697 282 728 \ 934 519 961 728 397 715 474 529 016 379 833	6	6.436723	7.899434	6	11 917 705 494 737 787 616 774 051 059 337 025 193 \ 535 879 579 243 503 395 089 533 818 868 688 787 292 \ 752 689 340 665 903 453 737 155 745 304 373	1.462711
11 057 426 428 430 163 103 488 912 400 632 162 735 \ 130 046 025 074 014 096 264 753 959 349 774 308 747 \ 883 957 204 635 682 678 326 867 955 051 892	7	6.607395	7.897666	7	5 897 481 597 920 247 450 940 280 601 360 260 333 \ 488 842 011 494 761 550 214 203 468 397 753 759 294 \ 294 933 372 472 643 856 128 163 237 118 970	1.290271
2 612 119 747 436 795 116 027 460 054 778 326 519 \ 706 092 368 493 246 981 989 828 186 207 552 208 103 \ 645 736 475 191 088 800 541 143 148 043 738	8	6.942014	7.058908	8	6 862 948 064 212 371 304 973 479 188 976 701 748 \ 659 649 218 068 936 749 775 676 853 868 101 053 201 \ 500 238 734 391 837 655 867 288 414 580 942	0.116894

Figure 24. Our second LoH5, included in Zenodo file LoH5-NoRedact.zip.

8-tuple of pairs of numbers of weights 1-8 as indicated.
Collatz lows and highs are truncated to six decimals (not rounded) and are
differently monotone towards the w4 low and high, where w4 has the lowest low and highest high.
The last column shows the difference between the high and low Collatz values in each row.
Apartness is more than 10^-5: within the LoH, the least and the averagre of the 14 absolute
value of differences (all required to be > 10^-5) are the first two rounded numbers below.
Also compared to the previous LoH_5 in our onloine sequence, for the absolute values of each Collatz value minus the
corresponding one in the predecessor LoH (the successor has lower lows and highjer highs), the least and the
average of the 16 positive differences (all required to exceed 10^-5) are the third and fourth rounded numbers.
The average high minus low for the first LoH (starting at the opposite sides of the expected Collatz value) in
our LoH_5 sequence exceeds 1.99 and for our last LoH_5 (LoH_5 number 1110) it is more than 7.18; here
for this **LoH_5 number 360**, the average difference is the fifth boxed rounded number below.

$\{\boxed{0.000086}, \boxed{0.632347}, \boxed{0.000036}, \boxed{0.001318}, \boxed{2.982409}\}$

1 761 038 215 093 745 886 397 022 778 767 430 241 \ 843 399 815 902 547 196 419 078 910 974 861 327 749 \ 700 047 180 893 485 342 573 966 395 750 096	1	6.054281	7.745894	1	16 591 455 667 917 348 545 631 345 398 162 276 389 \ 400 049 322 474 167 350 471 643 655 018 017 344 792 \ 051 970 051 984 499 982 889 757 342 384 712	1.691613
959 065 586 303 778 940 145 299 635 841 579 380 190 \ 180 692 370 843 505 860 084 935 198 291 178 203 349 \ 867 289 196 944 961 062 268 409 711 113	2	5.956465	8.165141	2	1 735 034 118 658 054 557 020 976 055 669 960 435 \ 235 926 134 421 411 859 236 633 383 504 762 910 381 \ 283 903 532 306 502 737 502 624 555 258 518	2.208676
17 976 989 859 527 668 924 049 064 324 995 938 647 \ 717 777 352 809 926 127 897 768 184 485 600 530 448 \ 807 718 346 271 900 067 089 047 933 836 636	3	5.879119	10.225845	3	12 768 399 086 261 628 650 155 708 003 651 384 988 \ 928 957 911 541 581 080 857 205 722 791 641 356 835 \ 881 215 753 300 244 363 445 510 975 505 139	4.346726
6 490 049 236 576 110 469 007 411 612 734 074 712 \ 486 086 567 357 314 194 042 403 360 455 962 852 635 \ 757 757 985 015 161 076 239 197 143 023 869	4	4.669736	10.468856	4	11 549 108 794 279 986 522 936 709 740 442 092 552 \ 357 427 463 494 030 763 059 859 549 185 656 326 354 \ 666 746 368 866 608 275 145 648 441 627 548	5.799120
8 901 462 553 311 663 466 299 874 700 731 194 460 \ 653 089 348 242 350 820 611 416 468 792 173 503 350 \ 321 809 433 756 091 337 468 043 962 614 955	5	6.086929	10.192220	5	13 341 494 421 002 556 173 597 108 445 526 973 806 \ 543 724 578 902 780 487 400 383 057 751 936 910 007 \ 511 949 185 370 058 040 352 280 064 479 086	4.105291
3 310 025 362 179 528 566 426 457 855 064 853 042 \ 712 685 417 579 450 885 694 446 497 974 904 052 909 \ 495 249 677 315 057 093 727 350 191 046 096	6	6.087015	8.578325	6	2 772 927 670 364 724 025 235 831 648 675 771 707 \ 732 088 732 453 728 506 645 897 549 738 744 040 207 \ 766 312 064 466 886 001 544 807 428 933 474	2.491310
6 359 767 867 757 111 690 590 930 707 234 968 655 \ 351 826 557 760 937 395 850 871 932 377 423 842 452 \ 496 986 226 261 510 119 370 036 153 198 538	7	6.087115	8.249887	7	10 533 210 167 218 125 708 869 438 174 445 306 166 \ 186 032 289 976 747 244 448 212 740 924 650 411 869 \ 750 844 404 775 523 260 337 534 346 754 552	2.162772
4 411 989 627 224 531 346 747 093 079 612 128 978 \ 615 015 981 841 879 262 330 731 244 291 163 183 064 \ 029 281 157 802 222 734 824 767 280 218 312	8	6.511740	7.565506	8	6 902 728 983 051 221 443 011 315 757 906 118 827 \ 470 572 239 457 622 643 228 277 485 714 122 496 448 \ 990 846 263 041 817 248 495 829 693 798 455	1.053766

FIGURE 25. Our LoH5 number 360, included in Zenodo file LoH5-NoRedact.zip.

8-tuple of pairs of numbers of weights 1-8 as indicated.
Collatz lows and highs are truncated to six decimals (not rounded) and are
differently monotone towards the w4 low and high, where w4 has the lowest low and highest high.
The last column shows the difference between the high and low Collatz values in each row.
Apartness is more than 10^-5: within the LoH, the least and the averagre of the 14 absolute
value of differences (all required to be > 10^-5) are the first two rounded numbers below.
Also compared to the previous LoH_5 in our onloine sequence, for the absolute values of each Collatz value minus the
corresponding one in the predecessor LoH (the successor has lower lows and highjer highs), the least and the
average of the 16 positive differences (all required to exceed 10^-5) are the third and fourth rounded numbers.
The average high minus low for the first LoH (starting at the opposite sides of the expected Collatz value) in
our LoH_5 sequence exceeds 1.99 and for our last LoH_5 (LoH_5 number 1110) it is more than 7.18; here
for this LoH_5 number 720, the average difference is the fifth boxed rounded number below.

{ 0.000109 , 0.395787 , 0.000012 , 0.001524 , 4.579667 }

18 739 183 159 791 056 085 550 580 190 222 993 670 \ 096 361 862 170 060 362 386 510 567 216 356 775 893 \ 467 404 417 607 128 531 001 395 037 226 564	1	4.801084	9.436413	1	10 692 761 845 539 960 921 706 352 717 146 155 846 \ 154 955 678 153 929 318 758 079 324 172 892 006 117 \ 309 506 313 000 718 106 633 085 064 577 913	4.635329
17 906 190 571 925 772 834 606 948 276 225 041 699 \ 373 874 805 543 519 515 967 037 330 673 799 970 941 \ 044 815 894 815 172 841 560 108 843 892 319	2	4.770151	9.703910	2	17 617 215 076 869 466 367 167 372 796 878 731 561 \ 550 731 460 415 802 710 980 623 174 143 619 618 377 \ 589 554 776 402 903 617 444 815 838 944 154	4.933759
18 009 783 894 857 839 915 959 897 098 688 232 063 \ 973 207 983 319 503 349 099 577 147 152 313 274 286 \ 916 760 018 857 546 855 316 177 709 370 730	3	4.770042	10.404628	3	17 175 420 531 829 321 115 146 624 214 635 738 226 \ 708 493 836 111 782 334 763 969 942 996 405 131 643 \ 449 873 305 857 352 227 737 029 617 419 578	5.634586
10 459 030 005 286 901 396 860 568 999 474 216 188 \ 408 288 369 043 384 971 161 557 639 004 335 644 849 \ 388 344 996 262 410 357 837 035 745 975 698	4	4.493546	10.664426	4	4 914 214 997 519 460 364 759 203 438 302 942 502 \ 247 655 369 335 760 703 627 863 383 544 710 954 615 \ 625 976 624 942 357 540 891 579 888 925 803	6.170880
12 107 989 852 253 743 352 084 791 867 142 549 701 \ 994 507 255 005 580 893 628 643 188 895 785 946 783 \ 147 246 207 294 884 880 084 272 327 740 245	5	5.072200	10.362954	5	3 842 190 218 425 693 122 414 698 009 590 244 360 \ 432 697 709 017 678 379 705 021 846 392 069 112 865 \ 946 716 329 235 596 621 113 641 947 319 929	5.290754
10 153 031 408 225 699 896 390 138 295 258 042 017 \ 717 016 368 673 721 116 213 185 204 196 444 472 738 \ 900 861 132 859 249 974 460 759 807 074 544	6	5.251059	10.182156	6	10 466 472 950 707 263 000 919 462 157 551 592 964 \ 036 012 162 118 218 275 166 649 752 938 326 813 493 \ 731 454 028 726 529 127 450 789 125 276 150	4.931097
9 462 298 618 486 952 612 948 074 576 759 480 598 \ 294 953 345 120 052 321 897 170 369 395 178 495 914 \ 075 213 123 861 445 905 255 023 921 008 961	7	5.663012	8.538526	7	11 223 436 050 626 590 104 040 002 926 144 470 315 \ 934 935 240 604 071 178 132 183 150 510 520 833 268 \ 777 771 476 867 731 968 490 083 267 351 511	2.875514
4 915 699 995 469 025 143 701 222 233 007 713 768 \ 760 935 599 049 504 471 165 325 687 355 255 435 823 \ 151 672 092 912 555 436 417 102 142 012 089	8	5.965518	8.130931	8	4 951 203 444 048 145 305 877 238 801 633 729 981 \ 173 365 461 023 332 083 051 652 019 719 622 914 581 \ 129 226 824 845 400 749 416 871 086 514 412	2.165413

FIGURE 26. Our LoH5 number 720, included in Zenodo file LoH5-NoRedact.zip.

8-tuple of pairs of numbers of weights 1-8 as indicated.
Collatz lows and highs are truncated to six decimals (not rounded) and are
differently monotone towards the w4 low and high, where w4 has the lowest low and highest high.
The last column shows the difference between the high and low Collatz values in each row.
Apartness is more than 10^-5: within the LoH, the least and the averagre of the 14 absolute
value of differences (all required to be > 10^-5) are the first two rounded numbers below.
Also compared to the previous LoH_5 in our onloine sequence, for the absolute values of each Collatz value minus the
corresponding one in the predecessor LoH (the successor has lower lows and highjer highs), the least and the
average of the 16 positive differences (all required to exceed 10^-5) are the third and fourth rounded numbers.
The average high minus low for the first LoH (starting at the opposite sides of the expected Collatz value) in our
LoH_5 sequence exceeds 1.99 and for our last LoH_5 (LoH_5 number 1110) it is more than 7.18; here for
this **LoH_5 number 1080**, the average difference is the fifth boxed rounded number below.

{0.060615, 0.307564, 0.000180, 0.008528, 5.970304}

12 000 207 655 501 710 512 026 034 071 538 337 488 \ 539 466 140 046 641 389 880 812 740 717 454 574 107 \ 207 769 824 968 288 015 970 126 104 075 191	1	4.554790	10.478596	1	17 835 942 239 917 551 434 295 037 593 471 799 617 \ 567 418 617 050 193 460 516 977 279 298 569 465 375 \ 457 496 157 924 004 321 824 937 615 614 016	5.923806
15 757 772 148 902 680 346 560 614 657 138 042 515 \ 452 365 139 140 451 634 825 444 279 591 225 498 365 \ 330 387 682 312 573 355 865 993 542 245 125	2	4.494176	10.683346	2	9 700 068 336 367 644 235 215 798 007 569 856 283 \ 375 492 381 237 729 702 696 393 589 976 205 954 379 \ 459 566 437 698 550 402 075 574 481 150 450	6.189170
17 668 447 771 999 405 147 097 790 746 417 406 234 \ 585 515 787 056 477 358 721 717 811 376 583 125 578 \ 460 601 033 177 453 442 398 162 403 016 779	3	4.341068	11.020925	3	9 090 121 007 868 240 026 490 556 025 788 214 940 \ 104 559 925 348 167 317 302 399 931 393 519 557 782 \ 415 800 419 009 000 745 920 171 618 852 536	6.679857
13 834 795 443 955 704 918 547 509 534 123 420 021 \ 706 829 465 846 659 892 431 307 685 697 761 670 435 \ 446 823 324 422 471 610 635 917 110 412 148	4	4.249792	11.212815	4	9 812 435 343 601 946 410 911 839 856 765 047 900 \ 040 913 744 695 947 480 916 493 875 759 813 682 713 \ 430 172 172 617 002 378 189 653 635 709 006	6.963023
6 626 842 829 956 227 941 971 010 911 093 539 815 \ 519 927 779 674 686 139 198 101 370 706 099 580 263 \ 127 379 179 640 347 464 768 391 322 281 984	5	4.346085	11.036198	5	3 729 638 210 198 146 278 613 744 336 057 736 417 \ 219 918 653 404 118 972 866 410 632 374 672 423 059 \ 121 370 399 200 587 720 202 234 185 403 843	6.690113
12 142 816 895 995 416 488 606 702 585 754 475 208 \ 491 101 886 193 349 017 051 825 511 248 339 484 791 \ 632 588 664 155 755 195 133 623 143 079 426	6	4.478932	10.759812	6	11 477 927 233 722 536 296 784 296 425 268 992 097 \ 447 255 076 236 840 107 363 394 351 782 109 668 496 \ 595 014 920 403 035 951 539 054 313 716 707	6.280880
6 053 888 306 835 262 516 971 863 299 633 865 102 \ 779 090 036 962 639 485 885 578 604 109 404 023 105 \ 831 553 061 854 489 250 703 355 820 980 725	7	4.746888	10.086126	7	15 725 563 001 440 468 402 576 126 048 768 784 282 \ 813 790 483 882 736 701 259 633 685 123 408 616 123 \ 414 783 871 009 868 569 443 591 590 463 044	5.339238
4 404 550 331 341 087 923 467 459 458 120 865 888 \ 309 800 428 412 885 278 405 486 203 691 754 428 969 \ 730 938 128 790 768 533 191 635 756 403 160	8	5.308563	9.004905	8	2 618 255 646 843 087 065 372 787 466 469 147 872 \ 924 832 504 255 904 254 631 392 626 031 604 796 412 \ 175 874 687 292 932 531 795 931 053 335 087	3.696342

Figure 27. Our LoH5 number 1080, included in Zenodo file LoH5-NoRedact.zip.

8-tuple of pairs of numbers of weights 1-8 as indicated.
Collatz lows and highs are truncated to six decimals (not rounded) and are
differently monotone towards the w4 low and high, where w4 has the lowest low and highest high.
The last column shows the difference between the high and low Collatz values in each row.
Apartness is more than 10^-5: within the LoH, the least and the averagre of the 14 absolute
value of differences (all required to be > 10^-5) are the first two rounded numbers below.
Also compared to the previous LoH_5 in our onloine sequence, for the absolute values of each Collatz value minus the
corresponding one in the predecessor LoH (the successor has lower lows and highjer highs), the least and the
average of the 16 positive differences (all required to exceed 10^-5) are the third and fourth rounded numbers.
The average high minus low for the first LoH (starting at the opposite sides of the expected Collatz value) in our
LoH_5 sequence exceeds 1.99 and for our last LoH_5 (LoH_5 number 1110) it is more than 7.18; here for
this **LoH_5 number 1110**, the average difference is the fifth boxed rounded number below.

$\{\boxed{\boxed{0.013978}}, \boxed{\boxed{0.245424}}, \boxed{\boxed{0.002912}}, \boxed{\boxed{0.146361}}, \boxed{7.182521}\}$

16 698 897 875 348 715 859 680 393 115 677 128 904 \ 848 707 893 764 943 209 878 128 359 028 174 760 996 \ 606 253 998 309 522 124 406 265 050 494 821	1	4.254565	11.595677	1	84 859 097 972 808 254 077 691 126 946 629 326 596 \ 170 128 070 132 214 913 805 344 852 799 124 313 712 \ 128 414 317 112 399 247 533 098 740 751	7.341112
20 206 123 632 500 694 475 967 931 638 439 408 296 \ 165 502 132 530 886 674 146 460 437 838 422 203 641 \ 953 616 283 384 142 282 166 435 940 621	2	4.240587	11.746962	2	1 068 198 246 991 848 122 105 951 307 663 749 376 \ 552 162 569 860 580 517 956 848 222 695 524 617 084 \ 295 816 259 794 180 828 708 421 197 051 383	7.506375
2 528 047 379 029 616 788 811 817 147 020 026 482 \ 009 854 635 776 439 906 479 432 207 065 179 333 057 \ 305 298 154 414 103 457 636 295 494 745 005	3	4.034347	11.889790	3	8 366 436 140 340 581 094 809 111 194 193 561 186 \ 216 881 012 739 698 780 001 796 567 584 546 095 159 \ 021 336 361 584 635 475 176 534 057 008 679	7.855443
16 634 952 899 448 255 878 453 373 737 651 313 304 \ 755 331 905 097 406 937 449 761 540 233 245 543 981 \ 230 359 703 970 860 194 168 440 958 581 557	4	3.845072	11.918922	4	4 167 466 255 483 785 485 286 344 881 611 456 268 \ 710 622 333 009 080 882 105 302 834 082 207 936 636 \ 304 659 509 932 670 856 713 672 952 022 456	8.073850
6 811 555 882 303 431 693 383 546 976 242 506 992 \ 378 506 208 022 156 705 228 676 634 435 070 892 581 \ 609 032 975 533 399 373 309 008 156 580 149	5	4.138104	11.744530	5	13 285 858 838 872 112 847 866 251 746 166 417 660 \ 655 396 399 510 193 259 550 026 452 326 184 883 262 \ 670 188 114 579 632 791 853 106 000 559 575	7.606426
4 725 275 928 832 561 643 765 544 954 647 191 122 \ 435 506 143 410 824 179 096 138 648 072 719 680 277 \ 316 786 597 698 501 044 274 238 135 947 009	6	4.180118	11.301880	6	13 550 659 867 104 659 112 489 934 101 747 336 488 \ 561 212 745 161 013 661 251 779 808 851 249 752 819 \ 138 432 576 819 584 383 924 843 057 358 959	7.121762
7 897 747 954 717 429 315 118 622 020 998 200 340 \ 030 118 358 339 135 015 771 978 768 940 902 010 594 \ 357 126 685 627 655 498 153 105 693 899 621	7	4.307151	10.891703	7	5 969 634 570 308 540 055 354 815 859 823 152 309 \ 271 673 272 775 594 156 811 534 448 519 692 478 181 \ 284 340 437 326 401 283 256 762 224 084 537	6.584552
4 416 478 252 342 904 608 064 225 382 838 236 425 \ 295 767 112 512 423 610 262 570 876 464 701 537 598 \ 988 966 994 352 607 286 588 713 265 070 364	8	4.984719	10.355366	8	6 790 392 383 418 066 248 894 512 543 072 614 365 \ 381 334 571 397 729 665 986 782 078 241 904 181 669 \ 894 731 275 179 490 057 286 802 472 075 210	5.370647

FIGURE 28. Our LoH5 number 1110, included in Zenodo file LoH5-NoRedact.zip.

9. Recap, more extremal combinatorics, and correlation

Result 1.

(a) We approximated shares of weights for three types of depth-5 trees:

Table 1. Approximate shares of weights 1–8 for R and, IndAB, and LogSA trees/numbers.

Method \Weight	1	2	3	4	5	6	7	8
Random	0 %	1.04 %	23.6 %	55.0 %	18.8 %	1.54 %	0 %	0 %
IndAB	0	0	0 % 0?	0.46^- %	23^- %	76^- %	0.59^- %	0 %
LogSA	0	0	0	0	0	92.9 %	7.10 %	0 %

(b) We found and linked to some 3000 weight-8, and 2900 weight-7 trees.

Result 2.

(a) We found Mahler-rare numbers below 2^{363} (various stopping times exceeding 230) that are Ramsey-uncommon (weight 2 or 6).

(b) Also, Mahler-Ramsey rare and Prime, or Mahler-rare and a Lower Twin Prime.

(c) We identified a diamond from our four Ramsey-Mahler collections where the differences between various Mahler-type sopping times are "large".

Result 3.

(a) We constructed sizable (Outer) LowHigh_i sequences, $i = 1, \ldots, 6$. This can be viewed as a result in extremal combinatorics (and we mention other results of this general type below in this section).

(b) We presented a pair of weight-6-seed Collatz orbits with extremity for apartness.

Result 4. We originally did not anticipate a correlation between the weight and Collatz stopping time. Our codes that were designed to affect $w(n)$ do not involve Collatz (directly, anyway). However we spotted and tested affirmatively correlation between intervals containing Collatz values (scaled short cut) and either of: being of weight 1, being listable by our code IndAB, or being listable by our code LogSA. This involved 96,000,000 numbers below 2^{363} as follows. For each of the four Ramsey classes (including Rand), for 240 samples each of size 100,000, we gathered the frequencies of LowCollatz < 6.5 and HighCollatz > 7.5. So we made the following:

Claims (L1-L3): For the low Collatz ($c < 6.5$) numbers, the LogSA proportion is less than the other three.

Claims (H1-3): For the high Collatz ($c > 7.5$) numbers, the LogSA proportion is greater than the other three.

The p-values, and conclusion: The p-values for these two-sample proportion hypothesis testing (as well as for several "inter-comparisons" among the other three) were extremely small (and the z-stats extremely large), see Table 2. This very strongly supports a correlation between Collatz and LogSA-listability. The boxplots and histograms of LogSA in comparison with the other three illustrate the matter, see Fig. 29.

TABLE 2. Considering the four listability classes LogSA, IndAB, W1, and Rand, there are six pair-comparisons for each of the three cases of high, low, or mid Collatz (separated by Collatz values of 6.5 and 7.5).
Top frame: Class with the smaller $c < 6.5$-proportion, its proportion, approximate decimal; Class with the larger $c < 6.5$-proportion, its proportion, approximate decimal; p-value, z-stat.
Mid frame: Similar for the high Collatz ($c > 7.5$): Class with the smaller $c > 7.5$-proportion, its proportion, approximate decimal; Class with the larger $c > 7.5$-proportion, its proportion, approximate decimal; p-value, z-stat.
Bottom frame: Similar for the mid Collatz ($6.5 < c < 7.5$): Class with the smaller $6.5 < c < 7.5$-proportion, its proportion, approximate decimal; Class with the larger $6.5 < c < 7.5$-proportion, its proportion, approximate decimal; p-value, z-stat. [Row 2 in the first and second frame shows that if LogSA-listability of a number $< 2^{363}$ was not correlated to its Collatz value, such apart proportions (by over 0.2 % in populations of size 24,000,000) would be expected to happen with probability less than 10^{-98}.]

LCLogSA	$\frac{5876493}{24000000}$	0.244854	LCIndAB	$\frac{5962071}{24000000}$	0.24842	6.8×10^{-181}	28.6557
LCLogSA	$\frac{5876493}{24000000}$	0.244854	LCRand	$\frac{5939653}{24000000}$	0.247486	1.1×10^{-99}	21.1625
LCW1	$\frac{5905801}{24000000}$	0.246075	LCIndAB	$\frac{5962071}{24000000}$	0.24842	2.3×10^{-79}	18.8263
LCW1	$\frac{5905801}{24000000}$	0.246075	LCRand	$\frac{5939653}{24000000}$	0.247486	4.5×10^{-30}	11.3331
LCLogSA	$\frac{5876493}{24000000}$	0.244854	LCW1	$\frac{5905801}{24000000}$	0.246075	4.2×10^{-23}	9.8295
LCRand	$\frac{5939653}{24000000}$	0.247486	LCIndAB	$\frac{5962071}{24000000}$	0.24842	3.4×10^{-14}	7.4932

HCIndAB	$\frac{4510977}{24000000}$	0.187957	HCLogSA	$\frac{4587482}{24000000}$	0.191145	6.2×10^{-175}	28.1736
HCRand	$\frac{4528880}{24000000}$	0.188703	HCLogSA	$\frac{4587482}{24000000}$	0.191145	1.9×10^{-103}	21.5645
HCW1	$\frac{4547811}{24000000}$	0.189492	HCLogSA	$\frac{4587482}{24000000}$	0.191145	1.7×10^{-48}	14.5866
HCIndAB	$\frac{4510977}{24000000}$	0.187957	HCW1	$\frac{4547811}{24000000}$	0.189492	2.4×10^{-42}	13.5872
HCRand	$\frac{4528880}{24000000}$	0.188703	HCW1	$\frac{4547811}{24000000}$	0.189492	1.5×10^{-12}	6.9779
HCIndAB	$\frac{4510977}{24000000}$	0.187957	HCRand	$\frac{4528880}{24000000}$	0.188703	1.9×10^{-11}	6.6093

MCIndAB	$\frac{13526952}{24000000}$	0.563623	MCW1	$\frac{13546388}{24000000}$	0.564433	7.7×10^{-9}	5.65727
MCRand	$\frac{13531467}{24000000}$	0.563811	MCW1	$\frac{13546388}{24000000}$	0.564433	7.0×10^{-6}	4.34318
MCLogSA	$\frac{13536025}{24000000}$	0.564001	MCW1	$\frac{13546388}{24000000}$	0.564433	0.00127846	3.01652
MCLogSA	$\frac{13536025}{24000000}$	0.564001	MCIndAB	$\frac{13526952}{24000000}$	0.563623	0.00413621	2.64074
MCLogSA	$\frac{13536025}{24000000}$	0.564001	MCRand	$\frac{13531467}{24000000}$	0.563811	0.0923103	1.32666
MCIndAB	$\frac{13526952}{24000000}$	0.563623	MCRand	$\frac{13531467}{24000000}$	0.563811	0.0944093	1.31408

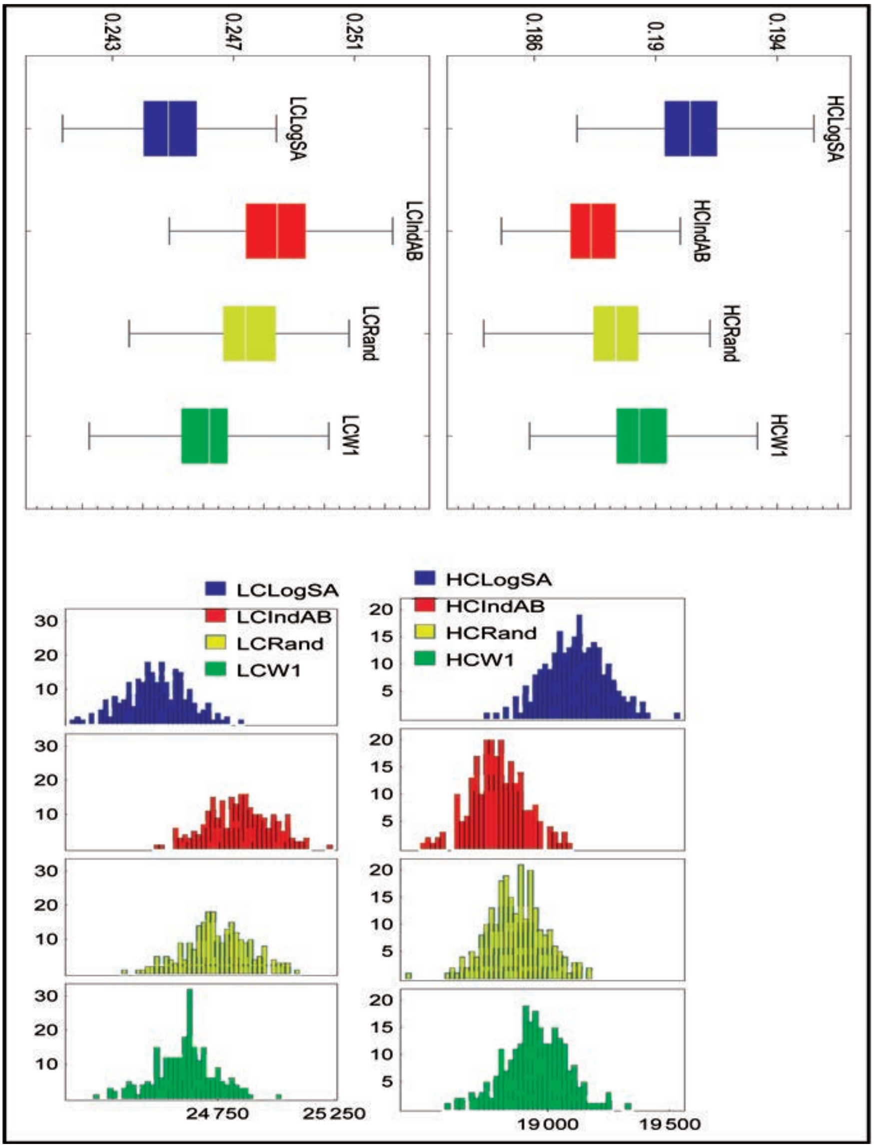

FIGURE 29. Boxplots and histograms of proportions (not pooled) of Low ($c < 6.5$, left of both subfigures) and High ($c > 7.5$, right of both subfigures) Collatz for our 240 samples of size 100,000 of each of the four classes. The MidCollatz ($6.5 < c < 7.5$) did not have as small p-values as the other tests, so we do not include the illustration for that. The Mathematica codes we used to generate the samples are included in this file (link).

As we were going through the gradual construction of 240 samples of size 100,000, we recorded some extreme Collatz values that are notable especially for the two classes of IndAB and LogSA (for the other two classes W1 and Rand, we got even further apartness but W1 has further apart values in the literature for small numbers and rand has no constraints). See the following Example, Fig. 30 and Fig. 31.

Example.

(i) There are IndAB weight-6 numbers m, n such that $\gamma_\infty(m) - \gamma_\infty(n) > 7.3$.
(ii) There are LogSA weight-6 numbers p, q such that $\gamma_\infty(p) - \gamma_\infty(q) > 7.07$.
(iii) There are LogSA weight-7 numbers r, s such that $\gamma_\infty(r) - \gamma_\infty(s) > 6.66$.
Here are our two IndAB-listable numbers of weight 6 and four LogSA-listable, two of weight 6 followed by two of weight 7, witnesses:

(i) Take $m = 62504853942671488724252975466633418518152859102186057862301303470798879733626179735037575382865218948998561 39$ and

(i) Take $m = 625048539426714887242529754666334185181528591021860578$
$623013034707988797336261797350375753828652189489985613 9$ and
$n = 102417771806170140786859728976752652542763795707536843$
$979740475037339725164583991883246951827776765222244182 20$
(their approximate Collatz values are 11.4845 and 4.18317, respectively).
(ii) Take $p = 149155922472550272842426424509659099476169754341451452$
$917702261091113092378817005777641254462203884632446050 27$ and
$q = 138471167793684524767366633280532674025015072039358759$
$480189762974297350499094316591715168080056527081035535 99$
(their approximate Collatz values are 11.1344666 and 4.054797, respectively).
(iii) Take $r = 380364946897578976485955832782585736175984243078861867$
$150776516959448479642277570183899151659922634555472782 9$ and
$s = 122091815639727737058423553457771593169365090834029166$
$255399672316260421193534917042075807434168641049002244 41$
(their approximate Collatz values are 11.0353 and 4.3713, respectively).

Result 5. We gathered 25,315 numbers below 2^{363} that are of weight 8, these are the ones we saved just when their Collatz value was in $(4.5, 6.5) \cup (7.5, 9.5)$. We got them with an extensive search for W8-numbers with Collatz values below 6.5 or above 7.5 (and then set aside a dozen or so we found that were over 9.5, all were below 10.3). Subdividing each of these two intervals into 10^j equal-length subintervals for $j = 1, \ldots$ $\ldots, 12$, Fig. 40 shows the number of subintervals met (at least once) by these Collatz values for each mesh and either of the two intervals. This lower-bound type of result is along the lines of part (b) (but just considering weight 8, not lower weights, and mentioned subintervals not apartness). One of these counts (20) is sharp, the others call for extension. These 25,315 Collatz values (together with a code to count how many subintervals are met for each mesh, but not the actual numbers below 2^{363} themselves) are in this file (link), the corresponding actual numbers with those Collatz values are here on Zenodo (link). For the case of $j = 2$, we present some witnesses in Fig. 32 through Fig. 39.

5662189387471799231656518260252294576021154330399714774420616708007194820312055340234192788723742535770586948, 4.45600, {weight, 7}
11217694110074033477777994502033169815165082567992745845021805798360982021207225679417793639155963567911365888, 4.46000
4915089386131623265110210320462577859948478584537497227361319594106294351785638588184992712818692864556811725, 4.47100
15507377467389566535192973133581072481851316858639433618562149625939220849373484417981727695248545172831330314, 4.48200
14371876009887395642503389918074393187771275238860998470780978537539047301193434357137563629819597160295609585, 4.49500
14379495313789996639239871572116341547082759030064638263726424323479199653647668137606499129439957677254400085, 4.50700
12541622826383054199947964731214663785925111726724762759410981798461130942291451080531700807599692099652210080, 4.51000, {weight, 7}
11851186789222389429758733139309844656011321465723503200133346487625738310803722755171157237297242335250644345, 4.52300
5520268238700868553278139198897724715638922658582935032840647126711438102884218530088226658690691038591593568, 4.53200
3783016584121391341577344815702176084173896105087623441758706792362524818502096753917286046236838423974895845, 4.54300
9917862655591844204059242319279759061950579759496116773366760698952783050816955905438681412496431761328327453, 4.55000
4496029637411455497495440510584342622518114343954037944471601287622478465303585990430346227418035995933678973, 4.56000, {weight, 7}
15405772503506247412515419258144011759112296450769649150278089117819966957487452750026746243610722462733988353, 4.57000
7153166508094799148466675509153241497609271583723691013891506274266736027662391508777420532643095697059119767, 4.58000
8627395144040503399772909636632697948604767050070898397263985983602329178939178038801762595613675414201845727, 4.59600
16175774357211412346046978127067537245755925868311485509633817189257929357823050899630936633272314525009170948, 4.60100
11868373572653250125558309008687162795762617449655331754804890868006922595311681900622556047800295886603984477, 4.61000
4840317475339578598318821757505117319737904771295179325175734015580792400899293039701043758531371701659922768, 4.62300
11772783353455686444438648410988864418837502806058048605558846901537761185041551339729378923639149287748896243, 4.63000
8717922358756501515534926778394417786489696275182685866907834801144413491499572191704584330841899442064529846, 4.64000
6278253766267505805932187768292152392233261892327649900862146707293625416768632865860260821616953941043657400, 4.65000
12621361725398452754137533535084893006576618539868941519702493433129993512790258094591288249739218922748618245, 4.66100
8281361482049739124437838104675613510189123514747446404719503480531005231743257773361180506655966376605008437, 4.67300
4479881099420194091568894833110482545204286815472392428191694933957352000643753887294794904598400266523560961, 4.68000
9716781510976188426530078314712140758282846358625940612109919087098280586915757617267618978279875255305166490, 4.69000
10457674089220202647501579503580287972245745722778271988915982547379560875866245093500732130344104819461694989, 4.70000
3319384931218041997287086131319171286782545802318749106291957084637265804988550738592798206733714337039272271, 4.71000
10347525201951715687907840487387497127032718106611218522403533885926272659645078665538446785144120795747055638, 4.72000
4969621271715267650500855175906645199420633017215451801819311156003393173675582002802707659305596248768925962, 4.73000
16295564637291112305252912055343329096690710359833003760107122036950777900376303152714028138389040537740238849, 4.74000
11915539261103581487262846084166298476121622809326099268903570323332282090340429264354458972214088669000432407, 4.75000
7910075636055391543558913333492359333869174212522681718644107526766390009356244419463559135077966265469547010, 4.76100
3712147017858075304397699694544398928600367505140689755146091859263626972027281631147483517300979807581224243, 4.77200
11188576797078629091787315884522473648053063600995050049060752879480490175922109520029180605343937223962068609, 4.78300
13948290185266416503718165691518947367587890602248322412168651934330373995218496356857997834078059134360980999, 4.79800

FIGURE 30. Some witnesses of LogSA with low Collatz value (truncated after three decimal digits) in the partition of the interval (4.45,4.80) of mesh=0.01. Three of the numbers are of weight 7 as indicated, the others are of weight 6.

7897406351796692776451480944749693905593321620973432069907428311839089208705827831354214276306842062758661823, 9.95800
2777873489651475565010351174644155256701491410835641800068100594270205179541106636168801202743346990532351486, 9.96700
7750563551130126454240103343518559759415441246834616469656486029099369925100601503792449675998597091151339052, 9.97100
11709264790504884414840753735892706769282218175155955018635603490648091939474096472756533008104835632281257471, 9.98200
13818591886324655448858715641348395030705233278581547967801489441979638987690168785268742661218407208018018110, 9.99500
6168683504381384068624714874876278311898877368563524476198927336595106156119953646418780202613863855398861266, 10.00400
42585735437910736064291736617773326388846765527971327525336444056381054537291754466175808251918556442276094211, 10.01000
6800490804614378196361802400169010952778220498918773962792807812534430451482157916295057736037937608457776655, 10.02000
8087606665372034534711556621689016457040316122719883880508606058213804649544229450567749331285589004677840376, 10.03300
4570211818209247107120735003433888789173972687990455197106632973780579054558054874857201823017699985212419711, 10.04000
15521668234127246421571311659278832641103349320235582298782832283365927337173242918843523902227454900758908159, 10.05400
9084947501141721178807202269919110587615094421632287702844820323460520408837803172083523078870585563897426159, 10.06000
13734990117158371225318254635818204334865885460744781006423686071277554681119879104633856617647654619627499640, 10.07100
7232169691119222356400262410554026885476754344969011494896360919192259913644560099975642908063769367911952884, 10.08100
8823531901856519347677619263484081661485908059735376550681799982539224401703836356864610429528842287829375965, 10.09300
14876667088822250934334357946776488138553079119600888705309544145269683537662209826968274295614081204423867526, 10.10000
4489260121088828659306505969192447995197554620880419092454028410668574528965547592217783008518032051132198767, 10.11600
3387304514648630092682583243322073875731137151692758792486462339862208168495385883805717761319324310184935356, 10.12000
15038217496948767604674827651884237766853309852249076462986726501101645549062728046442849488827653824915151623, 10.13100
6757322018879317854717496621827456512420143664851039614629920130575400674573759426315355371538037157925010945, 10.14000
7523405295047665391806604241458708733742121848015433108028660837129303687410079021666353173278959827501774279, 10.15500
16221254536520657130562560138880848060545487371751551013164621783785607369251755750146240950423569059351130545, 10.16000
6649092581368678467893483349801136270611573990283717841377032091535074396213487010131757508828965555924803079, 10.17200
10459727572604924572386754960232063003564265888706338792598021903516399974012881344507027052527793756833658469, 10.18200
12583753501042905178337165365060359913681594448607006935853235733682117278664180033492567555586755495536291879, 10.19400
5346489333099774545748872747757784330042438398974211495197078104462235754929979165056071177546310452202682919, 10.20500
3790834698651527637472492714396020617588993540209945641783479506678101642634467323158452900757921337565743875, 10.21100
13525010132430854675781666635197222976594395427887626470693935168877872079150094944968122318610490813923745258, 10.22300
9465022157271133912224198291635056185629878949034115084318270918659796844388361678362467537068164969956126683, 10.23000
4115364642669218947468177067738240296660077352959494552430270553332952368042349837342835994998065066653809279, 10.24000
4858904743292700497599201545219029334845583366007858239218818178595221199375887042051082138618567759371509711, 10.25300
2638989078851712157625832299692235524049978759069689632677684625912210519189599235146419066023869443802433423, 10.26200
4213642101360114259489924854164940902596377292839643078315860567270765827397390284919596366692027926183905856, 10.27100
14966597458957140829798769158221509780597444344262600627724316376496692846285197300009703302172337606444199450, 10.28300, {weight, 7}
2930363082708729917782877411472731406498495951181332381182968295537493152735383934623552231974712202272952218, 10.29000

FIGURE 31. Some witnesses of LogSA with high Collatz value (truncated after three decimal digits) in the partition of the interval (9.95,10.30) of mesh = 0.01. One of the numbers is of weight 7 as indicated, the others are of weight 6.

2 620 558 839 244 931 346 265 614 337 383 788 183 841 088 942 828 716 311 677 025 684 276 100 698 149 679 834 709 069 627 643 005 606 124 028 313, 5.1073
2 619 877 912 822 424 275 858 286 648 373 776 781 595 180 742 838 625 138 003 714 968 360 455 485 763 071 019 812 421 332 650 187 641 215 059 143, 5.1193
4 418 324 064 719 218 100 124 461 632 867 517 300 019 974 533 680 466 478 093 264 750 261 543 306 454 246 870 548 056 541 035 754 878 959 710 670, 5.1246
5.13
6 900 767 087 139 090 894 404 963 263 464 716 058 277 986 019 108 458 042 678 370 895 837 924 196 346 653 696 464 438 929 868 527 113 472 266 345, 5.1434
2 617 555 903 170 295 989 732 313 599 896 229 098 178 836 543 283 883 966 303 297 072 900 277 614 006 589 348 174 638 422 432 020 556 247 457 696, 5.1513
4 415 158 246 991 692 443 810 540 005 328 584 417 977 559 676 828 420 400 508 372 887 867 324 381 575 136 487 537 338 619 683 622 205 010 464 327, 5.1686
6 864 546 179 860 444 181 522 112 762 865 838 297 464 871 349 307 004 044 426 436 913 820 951 856 074 832 311 526 208 655 707 476 681 766 341 488, 5.1754
2 606 163 163 839 369 786 002 627 033 493 287 516 484 887 481 085 556 273 780 778 983 397 318 264 548 864 244 513 311 565 236 545 499 303 991 276, 5.1835
2 612 673 138 757 421 977 447 065 148 287 781 004 827 946 967 587 371 414 932 221 031 928 472 518 191 977 283 546 198 758 824 846 467 843 718 274, 5.1954
4 425 030 875 645 603 909 569 497 523 862 426 172 467 320 746 643 494 055 174 584 903 521 824 122 134 200 224 207 960 956 256 534 686 907 997 114, 5.2005
6 865 903 041 395 198 670 828 541 198 499 070 945 577 206 089 732 177 333 582 269 715 715 603 466 777 600 280 202 874 335 104 646 978 057 160 665, 5.2193
2 622 925 409 229 693 396 909 774 692 087 370 500 815 055 769 672 073 508 253 477 328 508 267 841 859 895 520 059 546 619 065 882 707 182 713 079, 5.2274
4 419 112 988 879 569 452 489 608 604 935 954 088 011 784 228 895 700 597 462 227 025 729 885 994 883 309 928 583 425 035 036 692 533 182 465 990, 5.2325
5.24
2 617 689 866 726 112 618 514 000 329 319 650 215 854 765 723 385 843 559 553 625 952 900 460 403 351 833 280 057 594 154 284 655 997 353 962 880, 5.2595
4 914 010 274 383 843 356 160 897 995 608 358 656 419 755 058 587 214 199 697 135 273 057 719 422 735 350 240 081 267 405 600 900 174 017 549 955, 5.2622
2 632 245 053 533 064 349 436 057 270 969 477 244 857 202 592 406 919 620 631 895 626 141 497 842 504 859 626 075 841 213 561 431 688 309 634 427, 5.2714
4 423 930 583 619 573 043 323 933 561 752 695 432 115 820 384 181 397 257 955 615 968 505 709 838 538 715 518 548 705 289 311 811 166 940 323 724, 5.2884
4 876 782 837 941 875 709 013 064 476 143 680 630 937 968 678 784 303 599 749 001 044 616 671 713 009 205 442 969 139 881 988 603 571 165 853 172, 5.2944
2 611 850 541 057 740 844 707 538 668 853 076 290 484 720 007 119 657 856 278 485 136 956 531 920 923 192 211 259 802 946 644 682 497 757 761 914, 5.3036
2 621 082 324 921 315 140 285 731 188 973 129 702 602 572 032 821 762 093 188 383 737 246 441 421 751 164 221 017 027 759 181 827 471 675 672 097, 5.3155
4 411 145 052 745 480 118 720 938 974 655 947 355 347 718 946 729 553 887 085 515 270 966 376 882 111 640 555 146 914 546 873 730 416 921 873 861, 5.3205
2 607 444 168 606 585 377 311 265 843 810 956 223 728 792 542 517 724 683 047 166 199 363 350 076 045 490 809 686 388 142 025 604 192 186 158 181, 5.3357
2 611 841 721 323 844 178 707 998 691 685 385 112 595 730 658 880 677 959 038 977 521 683 793 428 500 408 800 647 431 179 844 032 932 822 317 068, 5.3477
4 407 304 220 521 708 824 906 747 647 321 915 672 474 954 245 685 219 122 283 438 616 934 345 611 051 775 041 549 258 085 847 485 768 971 815 992, 5.3525
2 638 587 907 579 335 023 329 609 677 490 933 716 471 646 002 251 662 858 019 008 786 563 252 152 651 745 640 310 951 796 094 120 307 592 162 872, 5.3675
2 607 389 640 521 654 335 923 508 146 833 399 624 846 615 834 756 361 762 180 157 664 118 379 353 979 399 741 058 937 157 813 947 847 020 444 421, 5.3797
4 410 500 194 070 053 946 898 255 261 251 468 201 193 632 753 179 298 438 170 981 024 551 518 577 365 741 699 875 288 399 913 067 465 400 380 676, 5.3844
10 015 025 988 627 747 773 897 107 890 788 858 608 911 847 336 152 399 678 493 884 813 408 825 923 698 380 476 955 672 580 092 540 690 007 592 556, 5.3907
6 826 730 892 762 601 908 657 704 437 472 757 351 773 145 643 564 574 771 124 615 713 240 535 028 293 758 800 064 100 624 089 436 001 235 172 911, 5.4030
2 611 540 771 329 058 380 043 075 703 170 268 198 292 643 404 577 644 232 607 051 787 648 910 045 576 319 155 859 421 348 676 341 214 723 675 472, 5.4118
4 416 315 007 394 640 965 002 879 195 431 139 704 714 869 545 322 168 688 092 558 786 812 183 020 248 709 271 884 772 164 274 535 247 667 277 843, 5.4284
6 973 044 019 280 166 057 659 118 323 447 134 200 209 412 428 463 182 834 990 881 650 608 346 895 975 359 408 540 250 660 390 488 554 401 201 657, 5.4344
2 616 629 088 006 501 657 308 679 929 453 473 395 914 647 411 419 125 266 288 926 049 432 246 009 747 169 422 087 523 692 980 427 192 616 088 460, 5.4438

FIGURE 32. Some witnesses of weight 8 and Collatz value (truncated after four decimal digits) in the partition of (5.10,5.45) of mesh 0.01 (33 of the 35 subintervals).

2 608 482 154 952 348 020 953 425 342 467 292 010 724 294 734 262 791 576 464 567 201 450 594 743 004 088 135 466 840 655 017 864 798 641 819 193, 5.4559
4 406 478 031 923 703 158 410 481 994 719 676 489 930 055 025 278 717 696 599 208 571 856 630 081 447 516 158 172 058 375 325 219 084 543 200 711, 5.4604
4 879 738 349 825 263 271 891 526 520 140 295 896 857 933 177 132 973 461 624 395 954 308 184 923 865 112 964 757 546 681 374 258 820 192 936 455, 5.4782
2 608 748 626 503 099 783 063 675 955 951 606 423 460 339 833 877 558 518 470 171 989 073 678 072 439 957 415 541 229 815 257 014 259 721 196 085, 5.4879
4 404 286 190 512 425 337 738 283 852 101 561 976 412 362 029 749 651 004 731 410 629 589 497 550 918 981 696 476 672 831 749 866 531 057 405 304, 5.4924
4 405 411 687 041 141 189 737 344 664 833 806 957 817 344 837 885 625 363 799 551 578 965 166 394 053 861 302 758 559 197 404 348 834 996 540 418, 5.5044
2 608 042 965 815 472 378 611 374 608 835 765 522 227 329 371 995 062 014 265 677 490 066 647 820 286 687 230 923 688 550 548 948 409 746 487 375, 5.5199
4 730 148 725 460 129 520 819 331 257 547 296 099 161 665 682 459 514 278 350 918 547 745 061 804 588 449 154 855 876 796 335 095 063 391 740 906, 5.5228
2 606 953 315 117 245 700 289 325 318 052 936 713 415 552 382 965 058 582 640 876 347 347 341 880 795 475 509 643 264 509 623 424 094 473 064 319, 5.5320
9 723 406 429 231 289 791 616 632 705 575 248 351 128 444 037 878 247 314 390 957 073 817 769 109 476 235 219 369 014 975 221 211 730 020 140 793, 5.5428
4 841 126 163 324 612 550 326 387 756 015 413 294 576 960 214 157 780 092 721 707 426 788 594 853 477 482 572 019 784 355 716 390 924 226 344 961, 5.5543
2 608 028 569 642 830 839 697 881 264 534 422 851 187 230 627 073 173 549 486 323 374 738 918 252 247 975 209 718 144 296 028 064 289 070 366 267, 5.5640
4 731 145 292 390 932 197 349 656 922 726 827 568 965 754 264 877 484 645 302 367 965 303 239 339 123 469 369 633 819 774 187 798 024 362 100 824, 5.5787
4 434 081 803 401 846 591 520 406 515 009 015 747 212 489 383 590 645 660 849 565 013 012 831 044 658 792 876 923 115 114 396 808 128 992 863 673, 5.5802
2 613 545 170 968 170 279 000 164 663 992 754 083 918 099 045 016 662 262 590 106 183 003 980 255 710 553 756 124 411 642 902 249 903 184 696 894, 5.5960
2 625 954 261 514 311 948 688 582 593 717 764 443 294 808 690 225 494 885 838 524 366 892 687 056 736 674 144 626 317 738 022 989 685 844 747 032, 5.6079
4 406 228 083 902 225 558 654 684 307 343 525 198 619 213 472 046 919 378 313 687 365 228 438 427 986 318 635 800 968 311 952 843 473 686 468 152, 5.6123
4 414 606 925 633 770 982 992 087 079 357 916 018 173 234 366 382 958 268 487 192 292 936 545 292 361 363 729 881 179 802 479 109 872 333 033 800, 5.6243
2 640 092 057 545 908 945 179 816 413 969 692 166 935 018 580 826 067 086 818 262 691 274 736 076 818 037 265 837 465 142 884 086 978 851 955 075, 5.6398
2 611 395 835 792 497 427 988 656 075 978 774 412 143 421 065 641 436 964 031 069 748 230 955 909 912 467 931 398 330 294 747 896 173 844 203 041, 5.6401
2 615 652 889 540 312 510 133 830 446 850 653 136 820 079 909 924 632 537 956 419 562 690 285 245 682 593 735 075 772 805 987 743 455 689 434 319, 5.6521
6 789 567 433 841 020 152 631 033 342 305 679 661 173 006 394 641 393 750 147 442 767 418 650 035 810 982 852 869 263 120 764 312 824 515 186 918, 5.6625
2 607 306 420 057 457 660 494 733 049 023 940 331 407 496 750 982 504 723 397 223 706 262 540 005 776 387 082 076 359 729 237 894 812 451 959 271, 5.6722
2 606 218 245 277 231 726 660 098 835 425 775 245 557 784 855 160 863 618 636 105 240 458 738 626 965 884 999 243 498 670 805 066 264 168 438 120, 5.6842
9 720 520 624 568 265 062 211 755 310 472 441 571 857 381 427 633 086 086 223 106 323 935 888 399 519 940 776 774 298 976 759 243 899 798 032 377, 5.6942
2 635 908 164 612 812 068 829 913 362 094 737 387 915 811 408 649 399 334 874 196 743 429 674 542 108 987 279 959 527 551 179 257 991 080 475 064, 5.7040
2 608 045 507 639 486 483 550 963 228 917 299 032 882 416 861 306 948 619 579 961 128 061 360 224 484 596 024 271 107 877 217 335 658 596 122 371, 5.7162
4 404 338 835 916 588 330 708 782 327 178 129 951 883 615 122 651 952 131 375 820 198 139 716 165 433 569 217 546 923 512 349 908 860 808 132 642, 5.7202
4 951 783 978 679 038 183 758 182 107 965 044 336 994 939 129 628 815 029 332 516 141 084 561 915 376 442 746 013 305 944 144 582 500 073 398 294, 5.7375
2 606 115 015 766 465 545 975 712 160 032 412 714 648 568 390 158 054 502 277 657 565 391 605 295 076 932 504 080 656 881 979 897 176 405 510 048, 5.7483
4 404 327 188 344 866 036 190 322 789 298 060 496 967 743 377 372 109 815 392 755 394 262 558 640 352 170 385 424 693 221 699 388 860 088 283 928, 5.7522
4 406 766 410 526 835 762 295 182 663 240 658 572 450 324 711 380 150 469 836 918 996 846 254 800 059 236 166 948 932 527 271 366 479 337 975 828, 5.7642
5.77
2 607 052 314 556 673 388 696 455 677 789 045 716 101 380 615 530 030 508 701 565 065 815 302 767 197 813 859 129 887 984 950 890 435 990 007 349, 5.7803
2 606 952 663 358 072 378 223 357 421 429 428 135 064 282 759 369 825 455 865 501 255 156 051 253 447 573 502 616 669 887 450 954 652 980 315 064, 5.7924

FIGURE 33. Some witnesses of weight 8 and Collatz value (truncated after four decimal digits) in the partition of (5.45,5.80) of mesh 0.01 (34 of the 35 subintervals).

9 947 273 708 146 291 069 721 606 997 102 067 064 254 809 911 973 020 081 628 022 235 193 004 589 731 004 511 238 956 162 557 128 456 105 390 556, 5.8013
4 877 365 127 589 516 950 420 122 005 045 653 225 075 022 020 025 500 081 839 659 134 678 485 261 414 201 677 460 031 703 269 426 790 475 427 208, 5.8138
2 607 014 372 756 015 960 234 719 306 867 863 256 546 129 652 877 320 158 922 430 118 320 458 385 649 820 481 706 786 133 048 266 773 885 644 747, 5.8244
4 767 792 202 771 855 637 408 083 069 296 646 487 368 004 293 269 170 887 307 836 501 304 058 526 941 399 851 984 536 614 414 956 336 410 509 076, 5.8383
4 430 002 075 142 471 313 836 389 291 457 468 594 832 988 166 424 042 296 070 467 153 096 070 798 779 495 614 485 202 168 295 379 820 742 646 832, 5.8400
2 610 848 500 460 722 089 722 515 317 289 227 385 354 464 183 468 467 768 305 073 732 794 780 266 707 152 483 473 594 111 494 961 264 068 905 702, 5.8564
2 606 404 894 403 691 010 772 709 692 892 065 909 145 893 703 749 351 043 061 690 499 828 868 068 629 489 309 866 120 000 010 655 550 998 021 973, 5.8685
4 404 889 718 295 242 556 621 922 403 951 792 016 737 700 820 891 956 530 933 089 115 076 614 873 105 862 990 362 238 672 328 446 590 488 867 844, 5.8721
4 840 029 145 852 461 322 234 280 711 735 375 242 162 343 865 471 433 080 526 859 187 109 540 518 071 207 906 281 910 518 833 961 091 476 711 607, 5.8899
4 803 975 283 380 976 043 108 980 602 751 748 339 217 148 839 546 319 071 133 268 831 592 958 025 720 029 684 156 703 491 517 149 179 276 992 263, 5.8901
2 606 816 004 183 122 911 389 065 866 876 889 300 937 798 335 893 933 247 853 925 467 568 091 770 450 120 827 276 406 882 131 182 144 383 520 249, 5.9005
4 422 695 882 573 656 674 021 507 639 278 035 909 414 401 645 765 792 725 729 025 800 943 557 136 702 267 902 933 644 620 683 089 587 187 266 435, 5.9160
6 826 272 278 282 444 989 795 805 020 255 250 489 824 055 438 795 421 252 222 940 649 435 835 852 670 807 911 025 461 744 887 887 956 076 375 800, 5.9217
2 606 441 008 218 919 084 802 430 036 229 767 411 476 201 185 301 527 626 909 410 649 608 243 019 312 532 354 004 579 363 394 194 747 575 498 122, 5.9326
2 620 083 429 769 907 070 440 216 219 337 387 580 439 310 492 653 834 393 118 554 008 945 510 657 919 066 893 519 336 870 825 400 281 165 304 120, 5.9445
4 431 852 535 411 498 883 065 215 986 787 027 650 083 307 297 741 730 219 765 272 227 234 124 656 349 599 100 546 358 209 881 036 205 306 608 012, 5.9599
4 405 539 338 282 585 687 751 794 717 836 440 377 098 344 810 118 557 623 256 799 234 059 616 239 899 669 108 615 248 564 796 354 653 894 234 605, 5.9601
2 606 361 388 842 520 993 313 488 178 483 322 090 899 237 346 873 494 273 520 713 455 656 178 733 773 168 721 680 014 431 351 639 940 035 326 970, 5.9766
2 611 338 228 123 519 871 344 752 693 678 646 764 224 286 748 560 578 019 728 234 935 507 760 797 400 960 897 573 969 122 970 684 111 820 881 966, 5.9886
6 862 851 339 709 229 091 778 007 512 406 462 620 992 982 837 863 146 756 327 981 715 961 286 507 620 192 874 028 655 782 563 715 053 023 981 971, 5.9974
2 606 091 367 246 138 362 807 738 733 816 109 906 638 278 565 529 880 907 364 396 923 666 349 543 532 295 219 189 967 077 617 891 719 549 821 919, 6.0087
4 434 007 468 192 957 935 295 743 284 732 772 455 063 299 332 788 435 068 174 704 772 641 865 690 083 039 377 785 231 481 188 995 562 762 677 222, 6.0119
4 404 496 482 074 157 322 450 951 881 571 396 352 483 782 600 655 377 790 440 805 918 107 977 136 199 827 174 441 636 154 051 855 998 103 263 838, 6.0241
10 307 830 183 430 847 464 391 686 420 179 049 008 280 014 680 376 813 400 612 387 372 984 981 302 851 813 992 414 390 885 209 440 521 965 769 223, 6.0395
2 629 114 791 462 629 699 049 072 513 864 397 623 571 716 502 538 087 010 810 454 130 720 009 020 871 561 246 227 031 899 215 209 307 072 717 800, 6.0405
2 608 651 336 388 964 051 019 896 916 591 825 636 704 754 391 706 134 328 130 961 160 756 299 390 614 306 081 986 540 136 964 926 252 970 614 358, 6.0527
6.06
4 951 214 030 444 983 398 394 010 518 920 252 169 748 669 933 129 084 221 627 548 353 761 447 827 827 709 386 114 431 749 467 573 046 776 279 023, 6.0732
2 606 105 928 608 402 989 652 172 182 474 838 001 150 417 510 623 061 372 030 953 687 052 803 641 996 228 858 205 543 253 541 510 695 117 285 320, 6.0848
4 414 030 032 606 418 561 093 652 265 396 989 641 331 987 588 783 342 146 747 308 511 372 604 996 827 516 650 766 404 001 401 153 480 275 275 839, 6.0999
4 404 358 163 888 117 354 800 973 950 814 429 630 876 435 708 924 123 379 356 162 901 169 781 434 901 700 860 041 244 541 271 441 160 147 010 064, 6.1000
2 606 879 741 817 865 930 986 443 726 552 840 243 046 079 086 886 555 192 768 899 339 252 396 549 955 520 663 116 118 409 986 235 057 039 229 147, 6.1168
2 607 374 859 679 472 090 006 547 912 248 738 673 813 981 968 128 271 297 762 769 918 333 334 083 432 555 108 180 549 627 691 548 199 979 359 761, 6.1288
4 409 356 691 452 136 432 968 214 198 497 183 673 216 930 860 021 400 884 991 498 751 446 010 337 469 801 334 554 399 450 880 217 263 162 981 671, 6.1320
4 876 743 093 820 124 127 814 331 876 885 913 998 337 279 648 285 358 834 381 192 619 490 591 888 763 649 351 652 819 241 288 117 093 922 511 635, 6.1495

FIGURE 34. Some witnesses of weight 8 and Collatz value (truncated after four decimal digits) in the partition of (5.80,6.15) of mesh 0.01 (34 of the 35 subintervals).

6.15
2 606 217 558 769 396 262 907 793 701 678 895 378 304 347 288 096 351 515 036 546 420 172 666 257 249 279 040 538 056 721 580 233 346 653 577 221, 6.1609
4 420 805 693 223 246 862 228 319 039 119 615 461 576 526 109 327 616 008 156 360 631 633 426 376 492 189 304 602 318 219 272 913 275 768 167 754, 6.1759
6 826 197 216 468 475 009 402 407 920 313 375 800 604 972 764 853 430 562 749 892 289 534 455 177 366 376 814 523 696 983 788 120 319 699 336 720, 6.1811
2 616 115 027 989 339 737 388 299 305 918 404 310 816 839 215 805 101 732 812 993 793 001 706 359 282 693 761 321 014 432 594 757 360 653 671 049, 6.1928
2 619 925 895 054 365 385 932 491 614 251 270 365 758 888 903 685 093 013 728 715 154 940 346 903 673 609 003 577 279 256 613 055 427 400 145 924, 6.2048
9 720 255 443 224 352 983 715 563 566 260 551 245 734 668 293 751 375 485 880 311 925 982 405 558 339 880 468 241 875 005 643 228 000 594 968 324, 6.2123
4 841 258 910 873 816 005 730 734 718 280 252 159 299 372 116 419 375 999 952 344 857 099 479 891 411 895 564 248 688 050 890 084 409 638 944 696, 6.2256
2 606 092 605 216 107 337 782 325 025 982 770 760 990 532 530 655 365 344 449 314 378 358 488 881 893 562 729 705 907 547 186 905 509 777 670 875, 6.2370
2 640 294 591 290 279 347 131 461 619 363 904 801 731 664 173 877 179 052 609 210 194 511 257 275 466 890 894 257 262 138 943 387 912 798 401 943, 6.2487
4 413 467 030 632 198 900 486 182 223 460 745 893 449 553 341 742 434 975 246 008 451 219 428 217 628 399 768 164 362 490 378 562 017 077 041 742, 6.2519
2 606 215 340 265 976 516 185 603 147 514 719 429 050 321 469 610 892 745 409 696 969 808 238 591 281 019 082 409 022 529 853 310 299 564 883 000, 6.2691
4 729 869 685 735 193 291 417 591 542 970 288 079 105 654 473 685 645 601 546 018 384 759 283 849 103 900 512 110 065 400 484 717 533 301 274 921, 6.2701
2 615 290 917 607 913 190 962 208 896 856 708 913 362 751 552 526 378 769 209 584 569 365 467 263 540 333 570 161 515 264 339 508 576 653 381 335, 6.2810
4 404 190 879 009 440 301 285 429 238 936 104 490 328 084 833 987 021 035 526 461 933 982 878 231 403 998 564 259 096 504 377 079 284 354 122 810, 6.2959
4 730 002 299 037 509 362 095 960 088 211 166 510 005 533 030 511 240 698 548 862 872 618 852 970 794 291 028 009 105 992 704 720 829 214 522 311, 6.3021
2 606 260 582 492 905 866 647 513 562 350 178 856 026 932 685 916 662 734 669 213 608 648 204 612 135 677 538 834 340 433 578 595 704 318 426 130, 6.3131
2 606 320 663 289 806 836 890 599 781 096 506 336 569 343 597 258 458 643 745 179 742 956 533 452 782 420 585 171 719 762 664 425 786 666 874 374, 6.3251
6 789 804 614 236 427 908 999 384 569 171 258 885 693 885 769 582 460 716 423 116 545 550 834 088 044 954 698 018 598 128 648 506 179 254 983 104, 6.3329
2 606 181 292 218 097 162 071 170 694 922 697 796 055 135 775 353 316 943 424 730 155 891 902 022 390 498 873 430 405 116 468 059 205 629 133 303, 6.3452
4 405 034 177 790 481 536 917 631 390 946 621 465 682 399 349 014 886 571 615 641 342 574 087 457 392 545 606 701 422 578 826 105 305 478 443 984, 6.3598
7 009 760 730 621 293 269 696 832 089 618 764 466 744 776 160 149 891 804 104 083 123 978 835 073 422 113 807 439 077 031 331 301 974 883 696 321, 6.3640
2 622 341 973 261 610 135 205 738 204 487 738 587 731 632 025 251 655 195 088 577 811 269 934 555 069 367 128 257 209 850 183 275 334 777 997 788, 6.3770
2 606 334 004 153 197 027 630 309 925 323 636 151 622 112 661 506 801 058 863 830 742 135 611 059 467 011 831 111 622 027 939 359 235 189 253 530, 6.3892
4 404 319 633 170 140 878 897 795 613 402 606 866 990 751 715 278 099 998 373 846 018 725 498 858 951 853 478 793 957 488 330 763 408 702 635 393, 6.3918
4 913 626 683 276 128 404 213 951 206 635 242 597 033 915 543 621 579 874 879 747 647 218 094 607 099 417 031 928 190 928 752 701 248 170 627 640, 6.4090
9 721 449 016 782 224 864 794 057 729 696 832 949 774 379 065 506 007 149 222 458 748 367 149 638 622 672 912 617 436 098 285 582 852 952 163 335, 6.4195
2 606 179 040 822 777 278 287 031 169 349 469 416 084 082 262 618 532 181 663 466 478 126 264 727 597 633 207 970 516 735 481 067 886 438 490 085, 6.4213
4 405 334 516 577 745 088 620 928 257 909 847 787 742 847 549 528 373 752 683 487 895 758 469 518 576 869 683 747 431 753 381 308 854 812 139 841, 6.4358
6 899 863 525 087 111 677 113 328 566 282 772 676 627 211 125 866 037 676 353 407 403 034 261 658 290 253 148 503 815 964 059 460 001 390 845 992, 6.4402
2 606 663 967 371 802 381 722 622 897 120 617 874 977 886 726 842 592 311 064 430 543 463 967 723 737 008 028 085 421 689 606 991 124 703 449 077, 6.4533
2 606 225 448 515 414 647 005 838 793 684 373 562 937 131 425 062 649 703 140 673 662 082 451 829 599 705 099 464 067 873 263 139 082 132 620 599, 6.4653
9 945 500 740 796 601 484 389 537 856 208 789 458 965 187 291 909 291 298 351 204 809 127 997 658 297 704 973 734 081 824 680 314 540 350 501 312, 6.4707
4 876 725 269 888 346 486 905 250 750 245 227 605 703 801 095 414 555 013 638 710 236 100 059 572 318 953 290 593 295 120 242 977 610 962 797 044, 6.4851
2 607 839 328 667 765 092 099 298 267 705 673 214 454 702 485 969 156 147 009 018 786 949 101 649 081 169 348 618 584 450 771 126 105 558 486 914, 6.4974

Figure 35. Some witnesses of weight 8 and Collatz value (truncated after four decimal digits) in the partition of (6.15,6.50) of mesh 0.01 (34 of the 35 subintervals).

6 936 496 284 795 341 580 294 596 993 021 519 392 602 046 580 378 930 224 900 113 882 218 029 872 097 974 306 418 461 059 272 510 549 380 690 025, 7.6531
10 897 904 359 440 801 546 452 775 474 986 548 537 215 507 389 186 604 011 868 764 230 529 514 161 515 813 692 192 664 143 241 772 096 557 811 231, 7.6672
9 948 409 960 607 660 847 466 961 130 944 536 361 029 970 919 971 081 311 365 310 489 953 837 516 473 209 599 287 571 664 444 687 427 039 924 063, 7.6700
9 723 730 172 460 078 512 385 167 660 158 552 649 893 618 604 226 251 549 150 787 345 228 228 470 159 653 041 250 908 701 168 368 632 937 983 800, 7.6826
2 641 284 406 278 663 944 064 822 520 121 099 317 153 089 824 422 777 206 216 205 869 434 561 294 455 368 123 250 344 471 019 084 186 764 518 614, 7.6907
4 843 168 153 984 734 634 597 944 709 165 228 984 177 782 730 918 676 129 758 167 686 635 727 121 274 675 126 123 732 500 849 924 492 350 101 916, 7.7000
10 240 939 922 403 691 129 644 626 030 924 441 380 973 074 418 105 084 062 599 112 247 836 013 370 705 073 388 332 444 699 589 504 969 905 993 791, 7.7129
4 843 193 352 516 993 900 634 711 169 606 586 050 401 181 857 029 273 074 224 374 133 455 133 846 795 892 577 216 987 675 943 452 164 396 780 030, 7.7200
10 898 013 144 179 312 048 789 159 026 133 080 022 238 251 425 634 040 665 153 594 786 893 992 800 468 285 440 849 203 255 094 908 107 561 443 891, 7.7309
9 723 586 051 108 821 477 464 733 365 335 169 147 269 565 216 162 613 675 255 029 983 241 807 295 344 568 987 269 807 477 660 460 756 213 602 752, 7.7464
4 952 915 701 118 358 829 388 548 486 834 358 674 122 046 112 430 278 235 962 766 346 953 003 595 700 722 684 017 081 605 019 064 777 439 383 557, 7.7513
11 416 051 991 471 064 546 641 153 672 288 359 204 111 999 295 870 828 103 125 757 197 937 486 449 034 879 625 150 595 216 511 787 915 567 386 448, 7.7614
11 415 490 350 423 452 564 894 488 605 763 377 831 730 705 919 083 732 910 217 133 515 919 190 899 824 151 461 825 671 598 475 130 031 168 917 637, 7.7733
7 012 503 540 576 775 907 931 667 595 057 933 972 447 069 811 132 515 367 574 032 066 845 835 368 999 698 975 860 717 088 610 612 900 597 859 855, 7.7804
7 012 169 779 599 235 770 016 161 862 152 366 501 484 493 318 180 372 461 526 433 595 763 697 324 908 803 096 632 921 720 522 118 893 795 857 207, 7.7924
10 829 205 887 210 548 904 075 421 289 483 171 361 242 116 002 654 757 522 348 968 859 208 568 430 420 241 776 243 049 892 045 072 983 477 582 084, 7.8068
9 723 539 601 599 617 701 241 910 038 772 105 581 702 576 756 477 828 416 693 917 305 559 950 452 939 290 418 682 886 803 843 302 726 098 338 808, 7.8102
6 976 146 046 213 973 107 987 464 872 817 493 658 393 147 332 029 809 193 164 692 333 693 397 752 907 315 213 462 904 683 260 401 169 616 367 432, 7.8245
11 415 219 323 814 069 381 214 717 250 931 847 341 051 210 451 587 118 437 268 685 890 138 923 690 824 659 196 226 219 412 278 766 625 114 037 512, 7.8370
4 805 779 248 238 418 728 939 223 470 614 752 697 136 323 265 829 993 549 148 049 211 337 539 377 692 888 434 391 295 583 895 181 101 111 746 557, 7.8401
10 239 742 875 343 395 160 499 499 318 281 943 346 864 334 073 639 065 354 954 957 913 146 502 066 677 214 613 312 388 139 874 771 207 467 836 685, 7.8524
9 948 412 827 058 651 724 730 969 352 370 989 459 159 190 227 011 882 234 612 505 371 869 572 389 821 548 209 270 616 339 682 929 662 700 818 360, 7.8652
10 828 093 454 880 402 760 166 348 987 434 448 443 912 676 029 801 230 279 986 029 201 530 308 407 339 527 750 772 435 780 792 306 718 554 173 304, 7.8705
11 122 439 646 586 906 995 294 305 630 605 751 336 914 317 103 227 186 606 687 164 641 887 400 377 885 428 327 753 077 686 649 884 031 359 144 483, 7.8816
9 723 344 047 113 667 553 131 084 743 616 918 687 515 243 826 300 697 402 843 251 212 637 109 279 883 098 892 935 681 994 261 265 426 666 102 846, 7.8978
7 012 610 193 279 315 451 054 784 196 787 120 485 889 897 544 226 380 235 483 597 354 627 329 135 768 173 012 524 905 109 959 502 229 808 316 817, 7.9001
7 012 670 415 070 846 750 676 185 785 372 053 524 796 784 341 667 488 468 905 892 736 156 197 427 238 921 447 160 836 162 884 260 224 438 121 090, 7.9121
11 120 839 574 368 289 749 370 714 843 311 253 736 193 088 550 781 333 892 110 376 296 528 829 262 123 542 186 689 699 561 339 677 088 761 252 373, 7.9255
2 636 058 849 774 794 796 474 400 335 098 597 144 530 963 881 078 549 797 029 282 530 565 711 477 619 614 151 531 773 834 426 657 817 135 023 101, 7.9311
6 902 377 911 711 822 071 695 177 240 225 396 080 350 165 452 324 629 324 543 915 988 946 503 787 228 770 149 125 705 198 552 118 260 794 937 902, 7.9445
4 438 023 091 189 334 311 175 113 633 332 083 551 864 866 743 299 017 110 991 011 695 623 061 267 948 940 380 330 374 589 753 733 867 303 592 967, 7.9506
10 016 987 703 175 030 259 195 844 730 594 393 638 992 094 143 484 480 145 565 907 840 918 454 939 463 226 864 230 065 745 760 480 377 131 790 727, 7.9606
10 310 723 402 003 617 015 331 465 856 800 536 380 102 715 080 976 980 551 694 859 431 801 298 984 150 562 261 734 891 696 068 725 363 694 672 935, 7.9717
4 806 181 640 825 107 106 184 937 781 231 735 053 516 615 938 573 375 428 591 329 011 729 673 533 848 272 153 609 427 761 445 038 720 665 043 205, 7.9800
10 534 410 479 989 825 338 858 880 639 126 105 287 082 747 797 878 500 974 316 176 453 283 419 922 964 143 309 202 525 673 630 592 818 505 874 535, 7.9909

FIGURE 36. Some witnesses of weight 8 and Collatz value (truncated after four decimal digits) in the partition of (7.65,8.00) of mesh 0.01.

10 897 349 558 984 589 610 862 716 127 847 209 226 218 277 878 814 315 644 535 390 394 974 016 299 927 374 501 638 390 284 805 700 873 617 711 492, 8.0018
4 952 637 142 902 247 048 759 520 267 797 041 702 051 886 996 763 611 336 879 463 834 601 293 785 534 399 081 240 268 199 332 256 250 781 500 251, 8.0110
11 191 499 861 447 353 212 181 078 861 332 649 377 620 905 412 947 300 613 176 199 774 600 584 127 751 307 785 441 284 691 579 863 933 301 587 656, 8.0208
10 828 836 246 012 763 546 988 642 129 181 988 098 296 141 738 678 119 137 166 618 056 192 968 474 710 642 698 638 581 912 255 760 342 884 659 399, 8.0338
6 976 119 233 109 391 972 574 639 477 576 435 257 917 379 996 964 451 640 989 595 423 727 079 938 721 665 064 280 654 177 609 325 482 200 894 974, 8.0400
6 974 975 109 369 130 297 786 160 831 262 131 544 769 460 959 749 474 050 853 714 411 000 869 786 364 537 881 020 137 536 970 756 414 706 449 446, 8.0519
11 415 919 802 304 955 965 811 656 977 748 402 015 507 704 679 635 993 135 910 444 409 882 385 507 398 714 960 847 666 657 951 571 581 492 757 212, 8.0640
4 439 233 806 876 983 062 565 417 420 024 618 065 694 072 870 526 590 888 260 122 728 011 494 320 657 783 654 995 790 919 712 634 512 718 398 016, 8.0705
9 720 685 546 460 644 251 954 733 885 331 343 581 573 054 055 572 484 629 259 885 016 043 401 767 356 398 124 153 895 967 853 013 396 450 742 266, 8.0811
11 190 422 563 754 324 372 156 828 987 238 450 018 037 519 308 573 930 874 601 444 403 082 580 595 900 215 215 547 061 385 592 555 955 468 923 716, 8.0965
4 732 162 566 065 786 134 042 962 879 988 260 054 889 514 978 476 012 825 673 862 707 303 045 194 253 027 820 094 230 008 774 720 488 210 685 999, 8.1004
10 310 473 723 084 326 403 151 429 227 250 725 752 286 546 530 923 023 934 712 384 564 869 144 702 938 822 339 370 391 574 792 960 820 544 413 759, 8.1111
11 414 395 892 813 334 406 593 188 743 918 545 141 515 050 569 412 047 006 374 515 634 859 025 525 860 165 641 751 971 637 768 235 937 473 725 942, 8.1277
4 953 265 361 977 162 287 035 701 046 776 295 064 926 034 881 419 828 724 452 891 276 458 145 232 872 555 909 588 958 072 392 677 269 974 816 378, 8.1309
10 896 512 497 255 751 922 472 443 317 449 051 693 121 627 043 818 483 019 047 069 828 691 687 014 314 861 382 340 262 791 156 285 041 044 084 228, 8.1412
10 604 319 315 501 915 171 676 978 479 463 253 095 437 366 575 553 493 155 994 822 267 648 717 751 628 257 874 717 954 253 758 188 949 025 619 496, 8.1540
6 863 120 847 376 826 557 947 679 832 588 284 497 602 554 992 730 378 719 742 730 048 221 282 815 688 757 754 968 250 097 663 045 070 697 918 065, 8.1602
11 122 550 241 141 734 135 023 040 963 172 855 399 302 513 633 998 466 638 566 234 605 448 250 546 860 368 279 043 476 337 454 669 884 085 451 927, 8.1724
11 191 456 102 701 221 519 885 332 479 257 445 695 397 592 295 459 280 511 358 384 047 401 830 872 663 638 680 362 388 747 446 282 980 211 950 076, 8.1841
4 418 878 274 983 947 689 362 499 355 131 172 576 527 411 482 055 394 832 472 079 217 208 681 664 153 372 628 473 986 990 218 362 093 980 600 375, 8.1905
11 415 196 417 255 941 801 264 496 770 473 531 011 079 375 002 482 522 458 887 930 712 252 052 594 819 061 170 706 849 691 358 091 956 550 263 675, 8.2034
2 639 587 920 847 170 609 643 403 369 723 513 483 414 525 961 704 151 350 812 258 369 340 035 330 703 218 681 749 627 234 581 468 765 924 794 495, 8.2115
9 723 456 138 493 571 618 381 766 994 994 863 254 829 368 608 187 137 023 543 254 017 932 123 684 856 073 448 733 812 321 451 290 264 504 122 911, 8.2206
4 438 747 072 104 949 207 741 827 719 854 162 145 302 788 244 211 500 943 767 714 169 319 636 961 394 290 073 440 403 071 838 303 905 536 344 512, 8.2344
4 769 658 917 555 056 013 163 875 823 955 816 113 341 122 139 302 598 710 511 887 091 927 847 117 778 617 106 217 879 683 529 590 619 838 275 527, 8.2400
10 310 534 287 494 699 685 403 206 403 393 594 826 947 972 373 078 955 572 966 454 963 994 676 331 834 436 052 237 752 195 174 574 781 044 913 112, 8.2505
9 946 449 222 409 931 071 923 572 269 373 785 176 128 953 581 305 615 657 349 034 302 111 103 501 317 149 519 372 546 871 691 491 109 546 785 401, 8.2637
11 415 690 808 082 123 304 938 477 211 141 643 107 879 043 145 215 806 781 567 945 583 402 172 357 208 245 744 760 446 861 375 932 723 092 503 604, 8.2791
10 897 330 461 842 700 063 703 766 119 915 509 423 940 728 595 189 988 356 849 313 036 097 398 773 157 008 090 412 457 916 240 477 383 387 120 183, 8.2806
10 310 641 710 381 629 956 749 887 548 015 974 869 406 145 853 101 552 878 626 973 942 174 059 962 696 576 396 088 007 652 944 975 029 698 602 403, 8.2944
10 309 511 707 314 502 460 873 220 023 703 545 493 714 783 371 899 198 959 115 402 367 572 968 586 184 450 897 505 862 033 006 288 225 434 142 206, 8.3063
11 415 753 131 256 727 802 093 425 102 142 994 382 138 189 861 661 858 714 700 912 634 972 598 307 811 011 305 590 756 873 001 776 253 190 544 980, 8.3109
11 188 514 758 697 138 653 302 768 345 897 817 985 246 309 844 070 157 955 251 443 340 611 073 851 576 135 214 326 668 566 905 779 305 708 802 793, 8.3235
4 438 867 814 307 530 624 176 738 067 707 581 080 406 288 373 114 824 093 355 009 353 582 348 274 370 044 131 598 714 710 416 273 806 962 705 384, 8.3303
7 012 707 530 483 920 093 698 859 030 099 714 182 397 005 615 230 051 939 210 276 538 215 657 237 359 637 194 071 901 340 164 564 197 520 357 508, 8.3430

FIGURE 37. Some witnesses of weight 8 and Collatz value (truncated after four decimal digits) in the partition of (8.00,8.35) of mesh 0.01.

11 413 986 499 787 394 849 134 863 599 310 051 846 710 388 670 444 549 535 512 061 531 750 621 588 432 805 259 151 377 789 855 787 881 511 096 262, 8.3547
4 733 104 019 109 139 080 233 564 196 707 836 439 186 888 779 811 221 187 947 211 945 562 839 920 579 507 582 245 789 610 849 735 491 501 651 576, 8.3601
10 308 323 082 326 515 711 023 803 623 882 794 516 849 454 226 070 833 964 834 845 665 144 649 716 954 801 194 376 654 912 827 551 065 238 599 215, 8.3701
10 240 134 035 991 515 670 210 224 575 506 786 389 933 108 839 773 528 072 283 105 490 542 016 277 878 252 938 293 416 263 428 639 295 885 020 609, 8.3822
4 914 498 401 949 746 957 647 669 907 991 462 089 845 124 426 612 107 666 157 969 385 929 915 016 331 138 631 736 270 091 153 131 246 434 091 702, 8.3908
6 792 074 322 712 950 776 866 289 289 153 232 442 360 523 342 743 942 811 893 450 190 501 977 319 226 232 979 020 093 013 419 733 929 560 359 583, 8.4000
9 948 178 842 289 708 462 060 320 987 969 831 121 605 567 577 966 883 575 403 833 590 407 445 653 999 575 407 135 125 530 693 090 832 998 459 848, 8.4151
4 916 439 948 006 641 784 527 896 736 306 705 388 425 456 248 437 552 298 737 660 206 923 995 198 634 140 238 222 595 268 196 761 489 552 732 280, 8.4228
11 415 410 587 614 040 894 515 082 978 448 521 209 918 737 333 436 789 203 297 498 911 988 048 930 835 727 095 145 361 849 369 418 200 088 565 385, 8.4304
2 625 394 853 367 910 768 863 748 969 150 628 211 971 345 046 108 798 217 811 453 872 228 017 087 978 221 083 355 034 770 051 708 947 137 306 520, 8.4400
4 420 527 462 914 584 169 140 263 029 315 508 703 472 849 433 034 047 646 572 478 051 906 617 844 548 137 235 331 316 561 704 256 888 658 234 936, 8.4504
11 416 338 876 792 800 167 243 490 740 660 533 289 373 660 561 581 632 401 273 739 103 892 356 362 855 100 407 809 670 900 154 271 436 669 130 561, 8.4622
2 640 287 513 192 293 911 385 430 086 729 157 424 164 145 909 165 912 257 104 429 328 787 430 308 649 384 802 315 955 039 166 356 727 504 474 044, 8.4718
4 422 812 489 118 124 248 012 461 400 081 942 334 216 491 169 721 014 319 580 050 032 823 023 294 908 742 214 651 102 272 082 299 784 185 220 992, 8.4823
9 722 966 630 174 182 343 598 627 354 686 799 581 603 817 594 205 892 509 497 953 597 717 479 029 606 951 937 403 774 896 149 946 766 479 896 575, 8.4916
11 413 967 409 868 280 130 738 555 115 218 688 884 681 654 831 900 313 325 582 733 890 507 429 483 834 655 050 639 382 942 036 068 755 031 786 038, 8.5060
4 766 918 559 578 479 827 508 742 208 777 214 188 323 226 534 896 794 466 260 742 420 876 652 233 067 659 340 567 753 379 355 142 963 858 700 738, 8.5117
10 241 343 443 341 522 401 635 062 352 517 755 901 652 029 312 852 601 671 951 007 644 069 866 609 188 985 819 842 959 749 954 772 761 973 770 104, 8.5217
7 011 912 359 595 724 757 035 265 739 498 936 075 917 913 944 821 833 890 430 125 895 756 995 905 716 884 794 528 151 514 944 037 464 110 396 037, 8.5385
4 952 321 338 021 685 279 818 909 930 762 695 140 543 183 871 190 197 917 591 357 138 083 548 193 303 372 928 453 445 796 543 102 561 313 646 983, 8.5424
10 603 510 664 576 247 618 290 443 230 558 005 178 150 871 010 507 853 981 626 260 541 381 272 240 439 405 791 765 064 678 396 057 845 367 956 261, 8.5524
2 629 938 832 535 615 516 813 990 169 779 556 499 636 681 041 444 302 207 729 282 900 278 535 359 692 187 640 418 730 292 249 150 633 591 393 805, 8.5601
4 415 435 422 886 722 278 595 055 843 740 316 084 051 949 048 788 597 514 456 367 392 220 723 744 815 031 465 534 391 695 155 317 883 396 687 935, 8.5703
6 975 975 789 239 149 402 149 009 252 898 517 082 387 284 768 653 460 854 672 812 771 116 820 866 337 778 701 916 100 871 162 940 666 534 209 992, 8.5826
4 439 405 277 241 982 632 503 613 294 696 549 606 407 199 292 962 223 399 047 598 576 265 276 214 918 086 499 472 710 574 411 586 874 466 558 271, 8.5901
7 011 600 045 090 605 634 593 228 933 143 678 242 319 253 402 141 444 252 289 066 220 544 160 353 310 557 777 943 217 833 733 461 213 481 403 119, 8.6024
11 416 286 465 942 049 151 242 919 114 256 513 178 351 021 720 845 100 024 561 253 720 283 999 280 110 909 718 991 271 541 588 684 421 484 474 431, 8.6136
4 427 298 561 065 455 152 132 962 581 223 507 347 837 480 427 364 615 290 292 393 416 319 380 803 304 791 384 053 959 712 244 254 094 625 358 391, 8.6222
4 915 850 512 735 544 007 312 125 428 754 058 119 729 201 175 377 583 452 298 552 184 424 815 334 083 845 754 807 006 225 205 555 329 715 392 969, 8.6306
11 416 263 200 176 961 289 753 811 586 022 218 560 527 702 072 029 881 874 820 644 289 633 866 753 457 651 142 783 175 479 646 316 873 586 065 533, 8.6454
4 953 004 425 044 564 743 632 366 218 704 061 751 491 101 780 554 987 214 430 006 198 251 816 711 454 042 052 689 322 750 470 346 078 709 861 885, 8.6503
10 533 803 455 418 923 138 480 333 803 068 712 407 747 613 144 506 987 078 814 642 541 241 329 900 741 082 736 932 335 665 095 422 138 907 066 402, 8.6601
10 828 434 806 764 474 058 676 494 503 522 098 257 385 017 569 268 789 502 859 481 785 697 464 174 514 633 924 924 541 834 348 446 784 698 361 164, 8.6711
4 952 941 713 668 750 829 806 548 776 958 263 536 796 031 321 788 448 838 743 186 332 230 003 828 658 422 878 706 896 142 257 646 759 820 299 466, 8.6823
10 896 173 770 789 798 504 688 153 328 275 692 739 393 089 342 354 087 680 877 409 311 781 831 304 249 830 820 846 403 309 913 205 608 417 009 146, 8.6908

FIGURE 38. Some witnesses of weight 8 and Collatz value (truncated after four decimal digits) in the partition of (8.35,8.70) of mesh 0.01.

2 631 030 011 133 748 243 110 132 873 863 035 042 940 925 429 704 619 618 591 640 667 779 847 679 365 882 652 088 978 650 365 631 098 840 023 036, 8.7002
4 429 635 480 432 741 985 099 689 280 541 872 490 013 606 803 042 728 530 003 319 370 763 711 796 675 299 054 532 107 350 275 126 718 433 578 938, 8.7101
11 415 471 007 441 482 116 778 321 342 531 638 053 430 813 534 369 103 613 445 899 238 910 389 212 791 471 558 567 801 734 036 172 312 361 704 959, 8.7211
2 622 388 988 733 054 054 847 738 413 891 588 490 058 034 076 360 198 006 421 235 801 031 837 744 846 279 555 577 148 497 959 674 548 684 992 897, 8.7324
4 428 326 797 448 258 291 688 398 171 720 186 110 261 858 011 197 379 241 753 558 595 710 820 042 430 956 440 243 365 325 466 323 538 133 284 643, 8.7421
6 936 878 388 553 867 935 299 899 905 034 471 177 147 087 539 742 415 093 835 323 926 883 500 925 133 522 349 667 592 759 900 044 081 661 493 752, 8.7544
11 414 481 007 186 996 018 833 636 877 741 208 937 978 626 964 797 213 466 931 497 861 199 750 559 332 260 590 896 918 160 747 477 286 452 764 071, 8.7649
4 841 187 902 497 686 254 945 709 875 564 971 873 053 975 443 213 633 526 075 826 446 049 914 963 737 493 425 678 254 138 024 629 576 011 382 313, 8.7710
10 016 384 651 649 280 652 358 093 059 327 124 521 490 438 070 340 834 914 736 749 506 998 724 610 303 678 387 138 070 505 902 274 610 682 917 221, 8.7814
10 014 682 402 034 553 039 710 886 434 432 312 783 226 236 909 191 771 402 193 509 249 262 908 222 645 464 245 597 113 512 586 196 106 344 431 065, 8.7934
4 951 254 918 567 484 816 023 469 806 358 532 126 305 241 109 712 713 175 067 572 020 474 546 033 496 884 377 902 590 378 171 137 767 315 173 249, 8.8021
10 535 502 596 303 899 495 533 410 438 889 618 372 634 156 651 926 524 294 570 264 671 268 774 304 008 122 434 847 934 019 496 020 166 352 781 086, 8.8115
2 617 331 911 707 088 272 057 136 450 011 989 264 601 035 315 447 476 552 982 026 282 319 849 932 854 611 843 760 436 983 774 412 084 857 014 782, 8.8206
6 974 356 418 370 272 538 102 270 862 789 489 041 062 674 053 117 989 975 375 374 693 083 601 285 212 786 761 751 640 785 030 145 550 086 351 655, 8.8300
10 897 042 776 761 785 408 252 149 698 268 899 907 010 647 218 721 778 118 168 440 298 322 067 098 830 494 930 006 647 372 345 616 718 375 980 465, 8.8422
4 433 680 705 251 457 831 230 223 134 990 959 382 609 682 637 842 945 004 442 167 266 688 806 596 645 446 438 288 039 567 540 261 015 465 224 688, 8.8500
7 010 995 915 537 805 578 644 341 529 395 620 009 253 390 897 625 457 713 640 242 262 759 723 928 968 660 444 921 673 017 614 595 678 456 279 278, 8.8617
11 414 874 419 500 702 831 365 115 821 093 117 446 039 319 590 357 311 696 011 222 908 103 702 620 044 542 451 794 903 103 383 825 445 777 398 927, 8.8724
2 641 069 247 516 291 701 795 564 193 726 109 932 968 618 412 292 650 278 827 741 186 413 434 961 532 852 862 506 243 655 120 968 931 158 048 754, 8.8844
4 406 570 737 691 481 496 288 727 693 409 471 254 563 890 396 841 807 817 718 624 185 026 354 587 231 144 411 815 406 992 606 808 198 527 598 367, 8.8942
11 414 490 020 607 828 079 122 156 406 616 541 290 790 064 260 548 775 253 606 972 078 281 974 860 372 478 967 201 701 795 943 925 944 871 519 806, 8.9043
4 915 445 914 697 101 694 024 248 553 012 105 928 868 287 995 572 923 696 800 668 368 777 739 197 900 580 715 712 854 800 203 498 691 145 666 591, 8.9103
4 914 299 751 846 807 450 117 278 166 772 126 000 917 481 436 880 095 689 777 481 224 460 733 298 711 980 597 274 413 943 926 268 801 191 612 410, 8.9223
9 947 920 545 932 269 609 159 035 929 136 145 724 869 390 842 839 834 268 678 062 791 733 858 846 128 057 932 714 270 787 847 029 929 130 815 906, 8.9331
11 412 868 108 227 907 309 493 076 035 848 749 227 743 761 520 056 348 805 628 963 751 912 147 043 465 267 608 782 379 982 761 284 441 061 326 338, 8.9481
10 827 834 506 426 922 299 243 501 698 432 345 344 699 150 884 738 036 775 643 013 209 011 693 173 165 847 661 984 425 121 631 864 815 503 010 296, 8.9500
2 639 543 285 652 505 123 093 025 862 707 461 422 264 739 592 335 002 763 594 546 150 646 347 136 480 083 398 336 757 083 695 315 509 622 578 748, 8.9605
10 308 537 673 740 176 777 460 872 424 992 861 254 742 429 967 952 582 871 606 757 117 366 778 737 122 135 791 764 359 154 407 183 261 690 171 867, 8.9756
6 901 241 788 176 646 590 286 941 627 607 170 511 627 874 091 868 548 842 476 451 482 060 057 247 744 543 532 117 300 192 238 160 725 124 812 788, 8.9820
2 630 467 417 037 866 924 959 150 137 242 625 291 479 530 362 321 241 950 294 303 561 484 414 828 485 057 978 985 072 998 548 726 308 707 745 822, 8.9927
4 430 363 865 142 811 192 925 422 580 810 990 586 645 184 555 200 457 583 282 132 492 821 945 577 475 836 240 755 623 337 016 325 847 071 399 879, 9.0019
6 976 050 752 886 656 247 464 620 249 095 159 140 253 419 339 159 362 545 830 924 334 440 797 805 457 165 757 942 691 746 847 159 947 717 982 235, 9.0135
10 897 861 501 617 212 396 184 025 618 714 433 464 881 915 012 344 595 802 704 856 148 309 517 343 457 881 690 541 799 691 618 034 845 424 286 309, 9.0254
4 878 726 845 175 469 712 276 983 880 397 308 125 425 405 700 001 560 196 829 647 473 105 760 195 668 827 822 559 197 101 644 464 036 484 751 335, 9.0304
9 948 394 823 465 255 494 985 382 154 754 017 454 112 439 289 412 993 775 043 023 656 479 515 696 919 866 654 325 749 606 599 176 769 693 745 279, 9.0406

FIGURE 39. Some witnesses of weight 8 and Collatz value (truncated after four decimal digits) in the partition of (8.70,9.05) of mesh 0.01.

"Mesh →, Interval↓"	0.1	0.01	0.001	0.0001	0.00001	0.000001	0.0000001	0.00000001	0.000000001	0.0000000001	0.00000000001	0.000000000001
{4.5`, 6.5`}	19	153	507	1209	2691	7239	11820	13576	13994	14055	14064	14066
{7.5`, 9.5`}	20	185	668	1438	3079	7001	10011	10963	11203	11234	11247	11249

FIGURE 40. The number of subintervals in the partition of the indicated intervals and meshes that are met by the Collatz function applied to our $14066 + 11249 = 25315$ numbers (they are below 2^{363} and are of weight 8). The Collatz values for these numbers are at Collatz values on Zenodo (link); and the actual numbers at (link) on Zenodo.

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