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ON UNIFORM DISTRIBUTION MODULO 1 AND FUNCTIONAL CONVERGENCE

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ABSTRACT. In this note, we study the convergence of functional sequences. A criterion for uniform distribution mod 1 is derived. Then we study the partitions, block sequences and the uniform distribution preserving mappings. In the last part, we prove that to each one to one sequence dense in $[0, 1]$ a regular matrix summation method such that this sequence is uniformly distributed mod 1 with respect to this method exists.

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Introduction

In this note, we derive a criterion for uniform distribution mod 1. In the second part, we prove that to each one to one sequence dense in $[0, 1]$ a regular matrix summation method that is considered uniformly distributed with respect to this method exists.

For a set S the symbol $\mathcal{X}_S$ will be the indicator function of the set S and $|S|$ will be the cardinality of the finite set S.

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We start with the basic notions and facts. If $S \subset \mathbb{N}$, then the value, if it exists,

$$d(S) = \lim_{N\to\infty} \frac{|S \cap [1,N]|}{N}$$

is called the *asymptotic density* of S. In this case, we say that S has an asymptotic density. The family of all such sets we shall denote $\mathcal{D}$.

A sequence $\{v(n)\}$ of elements belonging to the interval $[0,1]$ is called *uniformly distributed modulo* 1 if for each $x \in [0,1]$ the set $v^{-1}([0,x))$ belongs to $\mathcal{D}$ and $d\big(v^{-1}([0,x))\big) = x$. This notion was firstly introduced by Herman Weyl in his famous work [15]. In this work he proved also the following

Weyl criterion:

The sequence $\{v(n)\}$ is uniformly distributed modulo 1 *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f\big(v(n)\big) = \int_0^1 f(x)\,\mathrm{d}x$$

for each continuous real function f defined on $[0,1]$.

Since the set of polynomials is dense with respect to the supremum metric in the metric space of all real continuous functions defined on $[0,1]$, we have:

Weyl criterion I:

The sequence $\{v(n)\}$ is uniformly distributed modulo 1 *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} v^m(n) = \frac{1}{m+1} \qquad \text{for } m = 1, 2, 3, \dots$$

For the proof of these criteria we refer to [15, 7, 2, 14, 13].

The notion of *star discrepancy* of given sequence is defined as

$$D_N^* = \sup\left\{ \left| \frac{1}{N} \sum_{\substack{n\le N\\ v(n)\le x}} 1 - x \right|, x \in [0,1] \right\}, \quad \text{for } N = 1, 2, 3, \dots$$

1. The sequence $\{v(n)\}$ is uniformly distributed modulo 1 if and only if $\lim_{N\to\infty} D_N^* = 0$, see [15, 7, 2, 14, 13].

An important role in this note will play the polynomial functions. Let us remark that in the paper [3], a notion of polynomial discrepancy is introduced and studied. In the paper [6], it is generalized for the finite dimensional case and its connection to D_N^* is studied.

1. Criterions

Suppose that $\{v(n)\}$ is a sequence of elements of $[0,1]$. Let us denote the elements of the set $\{1,\dots,N\}$ as $j_1^N,\dots,j_N^N$ for $N\in\mathbb{N}$ in such a way that

$$v\big(j_1^N\big)\le\dots\le v\big(j_N^N\big).$$

It holds (see [7, Ch. 2, Theorem 1.4]).

2.
$$D_N^*=\max\left\{\max\left\{\left|\frac{n}{N}-v\big(j_n^N\big)\right|,\left|\frac{n-1}{N}-v\big(j_n^N\big)\right|\right\},n=1,\dots,N\right\}.$$

for $N=1,2,3,\dots$

Now we formulate the criteria. The proofs will be given in the 2nd part.

Theorem 1. *The sequence $\{v(n)\}$ is uniformly distributed modulo 1 if and only if*

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N}v(n)^2-\frac{2}{N^2}\sum_{n=1}^{N}v\big(j_n^N\big)n\right)=-\frac{1}{3}. \tag{1}$$

In the paper [5], the notion of uniformly distributed partitions is introduced and studied. This definition has the following equivalent form (see [10]): A finite sequence $\{v(1)\le v(2)\le\dots\le v(r-1)\le v(r)\}$ in $[0,1]$ will be called a *partition* of $[0,1]$. Let

$$V_N=\big\{v_N(1)\le v_N(2)\le\dots\le v_N\big(r(N)-1\big)\le v_N\big(r(N)\big)\big\},$$

where $r(N)<r(N+1)$ for $N=1,2,3\dots$ be a sequence of partitions of $[0,1]$. This sequence of partitions will be called *uniformly distributed* if for each $x\in[0,1]$ we have

$$\lim_{N\to\infty}\frac{|\{j\le r(N);v_N(j)<x\}|}{r(N)}=x.$$

Theorem 2. *The sequence of partitions $V_N, N=1,2,3,\dots$ of $[0,1]$ is uniformly distributed if and only if*

$$\lim_{N\to\infty}\left(\frac{1}{r(N)}\sum_{n=1}^{r(N)}v_N^2(n)-\frac{2}{r^2(N)}\sum_{n=1}^{r(N)}v_N(n)n\right)=-\frac{1}{3}. \tag{2}$$

In 1988, in the paper [12], the authors studied such functions $f:[0,1]\to[0,1]$ that for each sequence $\{v(n)\}$ which is uniformly distributed modulo 1 the sequence $\big\{f\big(v(n)\big)\big\}$ is uniformly distributed modulo 1 too. In this case we say that f is *uniform distribution preserving.*

Theorem 3. *Let $f : [0,1] \to [0,1]$ be a continuous function. To each natural number N we can associate a permutation $k_n^N, n = 1, \dots, N$ of $\{1, \dots, N\}$ in such a manner that*

$$f\left(\frac{k_1^N}{N}\right) \le \dots \le f\left(\frac{k_N^N}{N}\right).$$

The function f is a uniform distribution preserving if and only if

$$\int_0^1 f^2(x)\,\mathrm{d}x = \frac{1}{3} \tag{3}$$

and

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{n=1}^{N} f\left(\frac{k_n^N}{N}\right) n = \frac{1}{3}. \tag{4}$$

Corollary 1. *Let $f : [0,1] \to [0,1]$ be a continuous uniform distribution preserving function.*

a) *If f is non-decreasing, then $f(x) = x$ for $x \in [0,1]$.*

b) *If f is non-increasing, then $f(x) = 1 - x$ for $x \in [0,1]$.*

Proof. In the case of non-decreasing continuous f, the condition (4) has the form

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f\left(\frac{n}{N}\right) \frac{n}{N} = \frac{1}{3}.$$

Considering the Riemann integrability, we obtain

$$\int_0^1 f(x)x\,\mathrm{d}x = \frac{1}{3}.$$

Together with the condition (3) we get

$$\int_0^1 \big(f(x) - x\big)^2\,\mathrm{d}x = 0$$

and the assertion follows. The proof of the part b) is a full analogy. □

2. Step functions associated to the sequence

For the proof of this statements we shall use some properties of functional sequences and convergence with respect to the measure and uniform convergence. Theorem 1 will be then a consequence of Theorem 5.

Let us define the step functions $a_N(x), x \in [0,1]$ as

$$a_N(x) = \sum_{n=0}^{N-1} v\left(j_{n+1}^N\right) \mathcal{X}_{\left[\frac{n}{N}, \frac{n+1}{N}\right)}(x), \quad \text{for} \quad N = 1,2,3,\dots$$

By a simple calculation we obtain that for each real function f defined on $[0,1]$ the equality

$$\int_0^1 f\big(a_N(x)\big)\,\mathrm{d}x = \frac{1}{N}\sum_{n=1}^{N} f\big(v(n)\big). \tag{5}$$

Let $f_N, N = 1,2,3,\dots$ be a sequence of real measurable functions defined on $[0,1]$. We say that this sequence *converges with respect to the measure* to given measurable real function f defined on $[0,1]$ if for each $\varepsilon > 0$ we have

$$\lim_{N\to\infty} \lambda\big(\{x \in [0,1];\ |f_N(x) - f(x)| \geq \varepsilon\}\big) = 0. \tag{6}$$

We shall note this fact in the following text as $f_N \rightsquigarrow f$. If f_N and h_N are sequences of uniformly bounded measurable real functions defined on $[0,1]$ and $h_N \rightsquigarrow h$ for given bounded measurable real function h, then

$$f_N + h_N \rightsquigarrow f + h, \qquad f_N h_N \rightsquigarrow fh. \tag{7}$$

Inspired by the proof of Korovkin theorem on approximation [1, 8, 9], we prove

Theorem 4. *The following statements are equivalent:*

a) $a_N \rightsquigarrow \iota$,

b) *the sequence* $\{v(n)\}$ *is uniformly distributed modulo* 1,

c) a_N *converges uniformly to the identity* ι.

Proof. If $a_N \rightsquigarrow \iota$, then (7) provides $(a_N)^m \rightsquigarrow \iota^m$ for each $m \in \mathbb{N}$. It yields

$$\lim_{N\to\infty} \int_0^1 \big(a_N(x)\big)^m\,\mathrm{d}x = \frac{1}{m+1}.$$

Considering (5) we get

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} v(n)^m = \frac{1}{m+1}, \quad m = 1,2,3,\dots$$

Together with the Weyl criterion, it proves the implication $a) \Rightarrow b)$.

If $x \in \left[\frac{n}{N}, \frac{n+1}{N}\right)$, then

$$|a_N(x) - x| \leq \max\left\{\left|v\big(j_{n+1}^N\big) - \frac{n}{N}\right|, \left|v\big(j_{n+1}^N\big) - \frac{n-1}{N}\right|\right\}.$$

It implies

$$|a_N(x) - x| \leq D_N^* \quad \text{for } x \in [0,1].$$

A uniform distribution of the sequence $\{v(n)\}$ provides

$$\lim_{N\to\infty} D_N^* = 0.$$

It proves the implication b) $\Rightarrow$ c). The implication c) $\Rightarrow$ a) is trivial. □

Since $([0,1],\lambda)$ is a probability space, the convergence with respect to the measure can be characterized by the following manner:

3. Suppose $\alpha > 0$ is given. A sequence f_N of uniformly bounded measurable functions defined on $[0,1]$ converges with respect to the measure to given bounded measurable function f if and only if

$$\lim_{N\to\infty} \int_0^1 |f_N(x) - f(x)|^\alpha \,\mathrm{d}x = 0.$$

This implies

Theorem 5. *The sequence* $\{v(n)\}$ *is uniformly distributed modulo* 1 *if and only if*

$$\lim_{N\to\infty} \int_0^1 \bigl(a_N(x) - x\bigr)^2 \,\mathrm{d}x = 0. \tag{8}$$

Proof of Theorem 1. By the calculation of the term $\bigl(a_N(x) - x\bigr)^2$ and its integration we get Theorem 1. □

The same arguments provide Theorem 2.

Example 1. Consider the well-known sequence of partitions (see [14, 13])

$$V_N = \left\{\frac{n}{N}; n = 1, \dots, N\right\}, \quad N = 1, 2, 3, \dots$$

Then Theorem 2 provides, after a simple calculation, that this sequence of partition is uniformly distributed.

From a sequence of partitions of $[0,1]$ a block sequence

$$\{v(n)\} = \{v_1(1), \dots, v_1\bigl(r(1)\bigr),\, v_2(1), \dots, v_2\bigl(r(2)\bigr), \dots\} \tag{9}$$

can be constructed. Applying Theorem 2 we prove

Theorem 6. *If*

$$\lim_{N\to\infty} \frac{r(N+1)}{\sum_{n=1}^N r(n)} = 0, \tag{10}$$

then the block sequence given by (9) *is uniformly distributed mod* 1 *if and only if the condition* (2) *holds.*

P r o o f. One implication is trivial. If (2) holds, then Theorem 2 implies that the sequence of partitions of $[0,1]$ V_N is uniformly distributed. Let

$$s(N) = \sum_{n=1}^{N} r(n),$$

for each $N = 1,2,3,\dots$ such that k_N exists that $s(k_N) \le N < s(k_N+1)$. This yields

$$\{1,\dots,N\} = \{1,\dots,s(k_N)\} \cup \{s(k_N)+1,\dots,N\}.$$

The set $\{s(k_N)+1,\dots,N\}$ contains at most $r(k_N+1)$ elements. Thus for each $x \in [0,1]$ we have

$$|\{k \in \mathbb{N}, k \le N \wedge v(k) < x\}| = \sum_{n \le s(k_N)} |V_n \cap [0,x)| + \mathcal{O}\big(r(k_N+1)\big).$$

Clearly,

$$k_N \to \infty \quad \text{for } N \to \infty.$$

Therefore by a standard procedure (10) we can derive

$$\lim_{N\to\infty} \frac{1}{N} |\{k \in \mathbb{N}, k \le N \wedge v(k) < x\}| = x. \qquad \square$$

Remark 1. The condition (10) can be provided by some simpler conditions, for instance.

P r o o f o f T h e o r e m 3. We use Theorem 3 of [12]. This result gives that a function $f : [0,1] \to [0,1]$ preserves a uniform distribution if and only if for each continuous function g defined on $[0,1]$ we have

$$\int_0^1 g\big(f(x)\big)\,\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x.$$

It yields that the given function f preserves a uniform distribution if and only if such a uniformly distributed sequence of partitions of $[0,1]$

$$V_N = \big\{v_N(1) \le v_N(2) \le \dots \le v_N\big(r(N)-1\big) \le v_N\big(r(N)\big)\big\}, \quad N = 1,2,3\dots$$

exists that the sequence of partitions $f(V_N)$ consisting from the sets

$$\big\{f\big(v_N(i)\big); i = 1,\dots,r(N)\big\}, \quad N = 1,2,3\dots$$

in natural order is uniformly distributed. If we apply this fact to the partitions of $[0,1]$ from the Example 1 and consider Theorem 2 we get Theorem 3. $\square$

3. Matrix summability methods

In this part we suppose that the given sequence $\{v(n)\}$ is dense in $[0,1]$. Let $\mathbf{M} = (M_{N,j})$ be an infinite matrix with the rows

$$(M_{N,j})_{j=1}^{\infty}, \quad N = 1,2,3,\dots$$

We say that a sequence of real numbers $\{\alpha(n)\}$ is $\mathbf{M}$ *summable* to some α if

$$\lim_{N\to\infty} \sum_{j=1}^{\infty} M_{N,j}\alpha(j) = \alpha.$$

In this case we say that a set $S \subset \mathbb{N}$ has $\mathbf{M}$-density if its indicator function is $\mathbf{M}$ summable to $d_{\mathbf{M}}(S)$. For the details we referee to [7, 2, 11]. Let us remark that the asymptotic density can be considered as $\mathbf{M}$-density, where

$$M_{Nn} = \frac{1}{N} \text{ for } n \le N \quad \text{and} \quad M_{Nn} = 0, n > N.$$

Such a matrix summation is known as "Cesaro's means".

Put

$$\Delta_N = \max\left\{ v\left(j_{i+1}^N\right) - v\left(j_i^N\right);\ i = 1,\dots,N-1 \right\}.$$

We suppose that the sequence $\{v(n)\}$ is dense in $[0,1]$ and so

$$\lim_{N\to\infty} \Delta_N = 0. \tag{11}$$

Since each continuous real function is Riemann integrable, from the last equality we obtain:

4. For each function f that is continuous on $[0,1]$ we have

$$\lim_{N\to\infty} \sum_{i=1}^{N-1} \Big(v\big(j_{i+1}^N\big) - v\big(j_i^N\big) \Big) f\Big(v\big(j_i^N\big) \Big) = \int_0^1 f(x)\,\mathrm{d}x.$$

To the sequence $\{v(n)\}$ we can associate the infinite matrix $\mathbf{M} = (M_{Nn})$, where $M_{Nn} = v(j_{k_n+1}^N) - v(j_{k_n}^N)$ for $1 \le n \le N$ and where k_n is such an index that $j_{k_n}^N = n$, $n = 1,\dots,N-1$. $M_{Nn} = 0, n \ge N$. This matrix method is "regular". Since for matrix density an analogy of the Weyl criterion holds, (see [4, 7, 2]). In this notation, **4** can be written as:

$$\lim_{N\to\infty} \sum_{n=1}^{N-1} M_{Nn} f\big(v(n)\big) = \int_0^1 f(x)\,\mathrm{d}x,$$

and so

Theorem 7. *The sequence $\{v(n)\}$ is uniformly distributed with respect to the summation method* $\mathbf{M}$.

Let us conclude with the following remark. Define the step functions

$$s_N = \sum_{i=1}^{N-1} v\bigl(j_i^N\bigr)\mathcal{X}_{[v(j_i^N),v(j_{i+1}^N))} \qquad \text{for } N = 1, 2, 3, \dots$$

Theorem 8. *For each continuous function f defined on the interval $[0,1]$ the sequence of step functions $f \circ s_N$ converges uniformly to f, for $N \to \infty$.*

P r o o f. Each $x \in [0,1)$ belongs to suitable interval

$$\Bigl[v\left(j_i^N\right),\, v\left(j_{i+1}^N\right)\Bigr)$$

In this case we have

$$(f \circ s_N)(x) = f\bigl(s_N(x)\bigr) = f\Bigl(v\bigl(j_i^N\bigr)\Bigr).$$

This yields

$$(f \circ s_N)(x) - f(x) = f\Bigl(v\bigl(j_i^N\bigr)\Bigr) - f(x).$$

Clearly $|v(j_i^N)-x| \le \Delta_N$. The continuous function on closed interval is uniformly continuous and so the assertion follows from (11). $\square$

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