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ANOTHER EULER CONSTANT: THE VALUE OF AN EULER INTEGRAL

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ABSTRACT. We show many occurrences in various contexts of the constant $E_u := \frac{\pi^2}{4}\log 2 - \frac{7}{8}\zeta(3) = 2.507907542\ldots$ Since this constant is related to an integral studied by Euler, we propose to name it after Euler.

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1. A constant that we name after Euler

At least three real numbers are called after Euler, namely

$$
\begin{aligned}
e &:= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828\ldots\\
\gamma &:= \lim_{n\to\infty}\left(\sum_{1\le j\le n}\frac{1}{j} - \log n\right) = 0.57721\ldots\\
\delta &:= \int_0^\infty \frac{\exp(-u)}{1+u}\,\mathrm{d}u = 0.59634\ldots
\end{aligned}
$$

the *Euler number*, e, is the base of natural logarithms; γ is called the *Euler (or Euler-Mascheroni) constant*; δ is called *the Euler or the Euler-Gompertz constant* (see [23, Section 6.2] and, for the name "Gompertz", see [23, Section 6.2.4]; also see [32, Section 2.5]).

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We would like to name a fourth constant after Euler. Namely, a nice equality due to Euler [22, p. 188] (also see, e.g., [27]) reads

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \cdots = \frac{\pi^2}{4}\log 2 + 2\int_0^{\pi/2} \varphi \log\sin\varphi \, \mathrm{d}\varphi.$$

Let $\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3}$ (also called the Apéry constant). Then Euler's equality can also be written

$$-2\int_0^{\pi/2} \varphi \log\sin\varphi \, \mathrm{d}\varphi = \frac{\pi^2}{4}\log 2 - \frac{7}{8}\zeta(3).$$

We will call E_u the (positive) constant appearing on the right side of the equality above, i.e.,

$$E_u := \frac{\pi^2}{4}\log 2 - \frac{7}{8}\zeta(3) = 2.507907542\ldots \tag{1}$$

We will see that "simple" expressions involving this constant appear in various contexts. The next section is a catalog of some occurrences of E_u that we have found, while the following sections give more context as well as references.

2. A partial catalog

We begin by displaying some of the occurrences of the constant E_u that we found in the literature.

$$E_u = \int_0^{\pi/2} \varphi^2 \cot\varphi \, \mathrm{d}\varphi = \frac{1}{3}\int_0^{\pi/2} \frac{\varphi^2}{\sin^2\varphi} \, \mathrm{d}\varphi = \int_0^1 \frac{(\arcsin t)^2}{t} \, \mathrm{d}t,$$

$$E_u = \frac{1}{4}\sum_{n \geq 1} \frac{4^n}{\binom{2n}{n} n^3},$$

$$E_u = \frac{\pi^2}{4} - \frac{1}{2}\sum_{n \geq 0} \frac{4^n}{\binom{2n}{n}(n+1)^2},$$

$$E_u = -\frac{\pi^2}{4}\sum_{k \geq 0} \frac{\zeta(2k)}{(k+1)2^{2k}} \quad \text{(recall that } \zeta(0) = -1\text{)},$$

$$E_u = \sum_{k\geq 1} \frac{1}{k(2k+1)} \sum_{1\leq j\leq k} \frac{1}{(2j-1)^2},$$

$$E_u = \frac{1}{4}\sum_{n\geq 1} \frac{h_{n-1}}{(n-1/2)^2}, \quad \text{where } h_n = \sum_{1\leq k\leq n} \frac{1}{k-1/2},$$

$$E_u = \frac{1}{4}\sum_{n\geq 1} \frac{4^n H_n^{(2)}}{n(2n+1)}, \quad \text{where } H_n^{(2)} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2},$$

$$E_u = \pi^2 \log \mathcal{S}_3(1/2), \quad \text{where the multiple sine } \mathcal{S}_3 \text{ is defined by}$$

$$\mathcal{S}_3(x) = e^{\frac{x^2}{2}} \prod_{n\geq 1} \left(e^{x^2} \left(1 - \frac{x^2}{n^2}\right)^{n^2} \right),$$

$$E_u = \frac{\pi^2}{16}\left(1 + \sum_{n\geq 1} \frac{4n+1}{(2n)(2n+1)} \binom{2n}{n}^4 \frac{1}{2^{8n}}\right),$$

$$E_u = -\frac{1}{2\pi}\int_0^1 K(\sqrt{x})K(\sqrt{1-x}) \log x \, \mathrm{d}x$$

where $K(x) := \int_0^{\pi/2} \left(1 - (x\sin\theta)^2\right)^{-1/2} \mathrm{d}\theta$ is the *complete elliptic integral of the first kind.*

3. Some first occurrences of the constant E_u in integrals

A first occurrence is obtained by integrating Euler's integral above by parts, one or two times for example, respectively yielding

$$\int_0^{\pi/2} \varphi^2 \cot\varphi \, \mathrm{d}\varphi = E_u \qquad \text{and} \qquad \int_0^{\pi/2} \frac{\varphi^3}{\sin^2\varphi} \, \mathrm{d}\varphi = 3E_u.$$

For the first integral, also see Remark 2 in Section 5 below. For the second integral, see, e.g., Identity (5.8) in [36, p. 2334].

Another equality involving E_u is given by:

$$\int_0^1 \frac{(\arcsin t)^2}{t}\,\mathrm{d}t = E_u$$

and can be found, e.g., in [5, Eq. 6.6.25, p. 122]. It is also the particular case $p = 2$ of [40, Theorem 1, p. 5]. Of course this equality can also be obtained from $\int_0^{\pi/2} \varphi^2 \cot\varphi\,\mathrm{d}\varphi = E_u$ by the change of variable $t = \sin\varphi$.

4. Some occurrences of the constant E_u in the sum of infinite series

The probably most famous identity involving E_u is

$$\sum_{n\geq 1} \frac{4^n}{\binom{2n}{n} n^3} = \pi^2 \log 2 - \frac{7\zeta(3)}{2} = 4E_u. \tag{2}$$

It was known to Ramanujan (see Examples (i) [4, p. 269]), and can be found in several places, e.g., Identity (3.169) in [43, p. 23], Identity (12) in [12, p. 367]; Identity 5.11 in [6, p. 10]; Example 8 in [19, p. 116]; Example 3.1 in [46, p. 18]; Identity (2.3) in [40, p. 5].

Another identity, resembling (2), is given in [43, p. 27] (Identity (3.201))

$$\sum_{n\geq 0} \frac{4^n}{\binom{2n}{n}(n+1)^2} = \frac{7\zeta(3)}{4} - \frac{\pi^2}{2}\log 2 + \frac{\pi^2}{2} = -2E_u + \frac{\pi^2}{2}\cdot$$

Three more identities worth citing are:

* Identity (2.12) in [49, p. 1585] (also see, for example, [20, p. 202]; Identity (2.14) in [15, p. 183] or (3.19) in [15, p. 191], and Identity (3.5) in [45, p. 836]):

$$-\sum_{k\geq 0} \frac{\zeta(2k)}{(k+1)2^{2k}} = \log 2 - \frac{7\zeta(3)}{2\pi^2} = \frac{4}{\pi^2} E_u \qquad \text{(recall that } \zeta(0) = -1/2\text{)}$$

* the first Identity (2.5) in Theorem 2.5 of [36]:

$$\sum_{k\geq 1} \frac{1}{k(2k+1)} \sum_{1\leq j\leq k} \frac{1}{(2j-1)^2} = \frac{\pi^2}{4}\log 2 - \frac{7\zeta(3)}{8} = E_u$$

* the Identity given in Example 12 of [48, p. 987] (also see Example 2.1 in [47, p. 438])
$$T_{1,2} := \sum_{n\geq 1} \frac{h_{n-1}}{(n-1/2)^2} = 4E_u,$$
where $h_n := \sum_{1\leq k\leq n} \frac{1}{k-1/2}$. As noted in [48] this identity is equivalent to Identity (33) in [13], where h_n there has the slightly different definition $h_n := 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$.
* the Identity given in [40, p. 17]:
$$\sum_{n\geq 1} \frac{4^n H_n^{(2)}}{n(2n+1)} = \pi^2 \log 2 - \frac{7\zeta(3)}{2} = 4E_u,$$
where $H_n^{(2)} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{n^2}$.

Remark 1. The identity cited above, namely
$$\sum_{k\geq 0} \frac{\zeta(2k)}{(k+1)2^{2k}} = \frac{7\zeta(3)}{2\pi^2} - \log 2$$
is but one of the numerous identities for series involving values of the zeta function. A huge source of such identities is the book [44], where, in particular, the above identity can be found as Identity (532) on page 318, in the form
$$\sum_{k\geq 1} \frac{\zeta(2k)}{(k+1)2^{2k}} = \frac{1}{2} + \log\bigl(2^{-1}B^{14}\bigr)$$
correcting a misprint in (4.34) [16, p. 111], where $1/2$ had been replaced with $7/6$. Furthermore the value of B is given, e.g., as (3.9) in [17, p. 92]: $\log B = -\zeta'(-2)$, hence $\log B = \zeta(3)/4\pi^2$ (use the functional equation of ζ). We cannot resist to cite one of the other identities in the book, which can be compared with the one at the beginning of this remark (this is Identity (494) in [44, p. 313])
$$\sum_{k\geq 1} \frac{\zeta(2k+1)}{(k+1)2^{2k}} = -2 - \gamma - \frac{1}{3}\log 2 + 12\log A,$$
where γ is the Euler constant and A is the Glaisher-Kinkelin constant ($\log A = \frac{1}{12} - \zeta'(-1)$).

5. The multiple sine functions

Recall that the multiple sines are defined in [28,29] by $\mathcal{S}_1(z) := 2\sin(\pi z)$, and for $r \geq 2$,

$$\begin{aligned}\mathcal{S}_r(z) &:= \exp\left(\pi\int_0^x t^{r-1}\cot(\pi t)\,\mathrm{d}t\right)\\ &= \exp\left(\frac{z^{r-1}}{r-1}\right)\prod_{n\geq 1}\left(P_r\left(\frac{z}{n}\right)P_r\left(-\frac{z}{n}\right)^{(-1)^{r-1}}\right)^{n^{r-1}},\end{aligned}$$

where

$$P_r(z) = (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^r}{r}\right).$$

In particular $\mathcal{S}_3(x)$ is defined by

$$\mathcal{S}_3(x) = e^{\frac{x^2}{2}}\prod_{n\geq 1}\left(e^{x^2}\left(1-\frac{x^2}{n^2}\right)^{n^2}\right).$$

In [28, p. 62] (also see Formula (1.5) in [30, p. 840], and [27, p. 1259]) we find that

$$\log\mathcal{S}_3(1/2) = \frac{1}{4}\log 2 - \frac{7}{8\pi^2}\zeta(3) = \frac{1}{\pi^2}E_u.$$

Remark 2. The value of $\log\mathcal{S}_3(1/2)$ was also obtained as Identity (3.22) in [37, p. 287]; also see Identity (1.2) in [38, p. 1463]) in the form

$$\zeta(3) = \frac{2\pi^2}{7}\log 2 - \frac{8}{7}\int_0^{\pi/2} x^2\cot x\,\mathrm{d}x.$$

For related formulas, the reader can also consult, e.g., [1,2,26] and the references therein.

6. Mahler measures

A proof of the nice equality below is given in Theorem 3.1 of [41, p. 1158], using Mahler measures.

$$1 + \sum_{n\geq 1}\frac{4n+1}{(2n)(2n+1)}\binom{2n}{n}^4\frac{1}{2^{8n}} = -\frac{14}{\pi^2}\zeta(3) + 4\log 2 = \frac{16}{\pi^2}E_u.$$

Note that it was proved by Smyth (Equality (8) cited in [8, p. 456]) that the logarithm of the Mahler measure of $1 + z_1 + z_2 + z_3$ is equal to $7\zeta(3)/2\pi^2$. In the same vein one can also read results on the Mahler measure of

$$1 + X + X^{-1} + Y + Y^{-1},$$

e.g., in Theorem 6 of [31, p. 123]) and in the main theorem of [42, p. 2324].

Remark 3. An interesting discussion about Theorem 3.1 of [41, p. 1158] (see the beginning of this section) can be found in the recent paper [9], where there is but another expression for E_u, namely,

$$E_u = -\frac{1}{2\pi}\int_0^1 K(\sqrt{x})K(\sqrt{1-x})\log x \, \mathrm{d}x,$$

where K is the *complete elliptic integral of the first kind* defined by

$$K(x) := \int_0^{\pi/2} \left(1 - (x\sin\theta)^2\right)^{-1/2} \mathrm{d}\theta$$

Remark 4. As written in Section 3 of [41, p. 1158]: *"It turns out that there are very few instances, where hypergeometric functions with more than three binomial coefficients have been explicitly evaluated"*. Namely, we found only a few examples in the literature. They all involve powers of π and possibly values of the gamma function. None involves $\zeta(3)$, except the one given at the beginning of this section. The most ancient such identities that we discovered, thanks to [39], can be found in a paper of Glaisher [24]:

* (ix) in [24, p. 194]

$$\sum_{n\geq 0} \binom{2n}{n}^4 \frac{(2n+1)(4n+3)}{(n+1)^3}\frac{1}{2^{8n}} = \frac{32}{\pi^2}$$

which can also be written

$$8\sum_{n\geq 0}\binom{2n}{n}^4 \frac{1}{(n+1)}\frac{1}{2^{8n}} - 6\sum_{n\geq 0}\binom{2n}{n}^4 \frac{1}{(n+1)^2}\frac{1}{2^{8n}} + \sum_{n\geq 0}\binom{2n}{n}^4 \frac{1}{(n+1)^3}\frac{1}{2^{8n}} = \frac{32}{\pi^2}.$$

* (xii) in [24, p. 195]

$$\sum_{n\geq 0}\binom{2n}{n}^4 \frac{4n+3}{2^{8n+4}(n+1)^4} = 1 - \frac{8}{\pi^2},$$

note that this identity is also given as $s(1/2, 3/2)$ in [34, p. 12]), and that it can be written

$$\sum_{n\geq 0}\frac{1}{2^{8n}}\binom{2n}{n}^4\frac{1}{(n+1)^3}-\frac{1}{4}\sum_{n\geq 0}\frac{1}{2^{8n}}\binom{2n}{n}^4\frac{1}{(n+1)^4}=4-\frac{32}{\pi^2}\cdot$$

* (xiii) in [24, p. 195]

$$\sum_{n\geq 0}\binom{2n}{n}^4\frac{(4n+3)^3}{2^{8n+4}(n+1)^4}=1+\frac{8}{\pi^2}\cdot$$

We also discovered – thanks to [18] – the paper of Dougall [21] (see, in particular, Identity (17) in [21, p. 124], with, e.g., $c=-1/2$). Then we can cite Identity (3) in [33, p. 222], Theorem 8 in [33, p. 227] and the expressions (23) (24) proposed in the Closing Remark Section in [33, p. 229]. Interesting such series can be found in [18] (see Identity (3) on p. 252 and Identity (7) on p. 254) and in [35] (see in particular Identities (7.1) and (7.2) on p. 1063 and Identities (7.3) and (7.4) on p. 1064). Also see Theorems 1 to 6 in [3], Identity (91) in [39, p. 85], Formulas (4) and (5) in [11], and Theorem 15 in [14, p. 12] as well as Identities (58)–(60) of Theorem 18 in [14, p. 13]. Here we give only three such identities:

* an identity that can be found in [11, 14, 33]

$$\sum_{n\geq 0}\frac{3(4n+1)}{32(n+1)^2(2n-1)^2}\left[\frac{1}{4^n}\binom{2n}{n}\right]^4=\frac{1}{\pi^2}$$

* an identity that occurs in the theory of uniform random walks in the plane (see [7, p. 5])

$$W_4(-1)=\frac{4}{\pi}\sum_{n\geq 0}\frac{\binom{2n}{n}^4}{4^{4n}}\sum_{k\geq 2n+1}\frac{(-1)^{k+1}}{k},$$

where

$$W_n(s)=\int_{[0,1]^n}\left|\sum_{1\leq k\leq n}e^{2\pi x_k i}\right|^s \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n$$

* an identity given in [10, Theorem 2 (4)] (ArXiv version)

$$K(x)K(1-x)=\frac{\pi^3}{8}\sum_{m\geq 0}\left[\frac{1}{4^m}\binom{2m}{m}\right]^4(4m+1)\ P_{2m}(2x-1),$$

where P_n is the nth Legendre polynomial,

$$K(x)=\int_0^{\pi/2}\frac{\mathrm{d}u}{\sqrt{1-x\sin^2 u}}\quad\text{and}\quad x\in(0,1).$$

7. Arithmetic nature of the constant E_u

To the best of our knowledge the arithmetic nature of E_u is not known, although it is certainly a transcendental number. Nevertheless we cite two irrationality results of Gutnik [25] which involve the three constants $\zeta(3), \pi^2 = 6\zeta(2)$ and $\log 2$:

- *For any rational q, at least one of the two real numbers $3\zeta(3) + q\zeta(2)$ and $\zeta(2) + 2q\log(2)$ is irrational.*
- *Either $\zeta(2) - \big(6\zeta(3)\log(2)/\zeta(2)\big)$ is irrational, or the two real numbers $\zeta(3)/\zeta(2)$ and $\log(2)/\zeta(2)$ are irrational.*

8. Conclusion

This paper that proposes to introduce a new Euler constant E_u and gives a (certainly not exhaustive) catalog of expressions involving this constant, has some series involving $\binom{2n}{n}^4$. These occurrences might suggest to look in the literature for all the(relatively rare) occurrences of closed forms of series involving $\binom{2n}{n}^4$.

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