

ON JORDAN DOUBLE SUMS AND RELATED SUMMATORY FUNCTIONS

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ABSTRACT. We study the Jordan double sums

$$\sum_{[d,\delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon}},$$

which are connected with usual zeta approximating sums. Some other related summatory functions involving Jordan's function are considered and estimated in this paper.

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1. Introduction

Consider the following double sums

$$\sum_{[d,\delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon+it}}, \quad (1.1)$$

in which $J_\varepsilon(d)$ is the generalized Jordan function, $J_\varepsilon(d) = \sum_{h|d} h^\varepsilon \mu(\frac{d}{h})$, μ being Möbius function.

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These sums are naturally connected with zeta approximating sums, hence our interest in their study. Consider indeed two different zeta approximating sums

$$\sum_{n=1}^N \frac{1}{n^{\sigma+it}}, \quad \text{and} \quad \sum_{n=1}^M \frac{1}{n^{\sigma+\varepsilon+it}},$$

where $N \geq 1$, $M \geq 1$ and σ, ε, t are real numbers. By perturbing the first of a factor n^ε and using Möbius inversion formula, we get

$$\sum_{n=1}^N \frac{1}{n^{\sigma+it}} = \sum_{1 \leq d \leq N} \frac{J_\varepsilon(d)}{d^{\sigma+\varepsilon+it}} \sum_{m=1}^{\lfloor \frac{N}{d} \rfloor} \frac{1}{m^{\sigma+\varepsilon+it}}. \quad (1.2)$$

Replacing ε by 2ε in (1.2) yields similarly,

$$\sum_{n=1}^N \frac{1}{n^{\sigma+it}} = \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]^{\sigma+2\varepsilon+it}} \sum_{m=1}^{\lfloor \frac{N}{[d, \delta]} \rfloor} \frac{1}{m^{\sigma+2\varepsilon+it}}. \quad (1.3)$$

For $\sigma = 0$ this gives,

$$\sum_{n=1}^N \frac{1}{n^{it}} = \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]^{2\varepsilon+it}} \sum_{m=1}^{\lfloor \frac{N}{[d, \delta]} \rfloor} \frac{1}{m^{2\varepsilon+it}}.$$

The notation (d, δ) , $[d, \delta]$, respectively, stands for the greatest common divisor and least common multiple of the numbers d and δ . This provides a way to connect a zeta approximating sum with another of larger exponent, thus of a smoother behavior, and thereby, by approximation, to connect the Riemann zeta function $\zeta(\sigma + it)$ at two different values of σ , which is a quite important question. It is expected from the study made in our paper that in a subsequent work, new results for zeta approximating sums and the Riemann zeta function can be derived.

PROPOSITION 1.1.

(i) For any $N \geq 1$, $k \geq 1$ and any real numbers σ, ε, t ,

$$\sum_{n=1}^N \frac{1}{n^{\sigma+it}} = \sum_{[d_1, \dots, d_k] \leq N} \frac{J_\varepsilon(d_1) \dots J_\varepsilon(d_k)}{[d_1, \dots, d_k]^{\sigma+k\varepsilon+it}} \sum_{m=1}^{\lfloor \frac{N}{[d_1, \dots, d_k]} \rfloor} \frac{1}{m^{\sigma+k\varepsilon+it}}.$$

(ii) For any $N \geq 1$, and any real numbers σ, ε, t , we have

$$\left| \sum_{n=1}^N \frac{1}{n^{\sigma+it}} \right| \leq \left(\sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]^{\sigma+2\varepsilon}} \right) \max_{M \in \mathcal{E}(N)} \left| \sum_{m=1}^M \frac{1}{m^{\sigma+2\varepsilon+it}} \right|,$$

where $\mathcal{E}(N)$ denotes the set of integers of the form $\lfloor \frac{N}{d} \rfloor$, d integer, $1 \leq d \leq N$.

(iii) Let $\zeta(\sigma + it)$ be the Riemann zeta-function. Let $\sigma_0 > 0$. Then for any $|t| \geq 1$, $\sigma \geq \sigma_0$,

$$|\zeta(\sigma + it)| \leq \left(\sum_{[d, \delta] \leq |t|} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]^{\sigma+2\varepsilon}} \right) \max_{M \in \mathcal{E}(|t|)} \left| \sum_{m=1}^M \frac{1}{m^{\sigma+2\varepsilon+it}} \right| + C_{\sigma_0} |t|^{-\sigma},$$

where C_{σ_0} depends on σ_0 only.

Proof. Assertions (i) and (ii) are immediate. Now from the approximation formula for the Riemann zeta-function we have

$$\sup_{|t| \geq 1, \sigma \geq \sigma_0} |t|^\sigma \left| \zeta(\sigma + it) - \sum_{k=1}^{|t|} \frac{1}{k^{\sigma+it}} \right| < \infty.$$

See [6, Th. 4.11]. The third inequality then follows from (ii). $\square$

We study the double sums

$$\sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]^{\sigma+2\varepsilon}},$$

as well as the related summatory functions

$$\begin{aligned} \text{(I)} \quad & \sum_{1 \leq d \leq x} \frac{J_\varepsilon(d)}{d^\varepsilon}, \\ \text{(II)} \quad & \sum_{n \leq x} \left(\frac{J_\varepsilon(n)}{n^\varepsilon} \right)^2, \\ \text{(III)} \quad & \sum_{n=1}^N \frac{J_\varepsilon^2(n)}{n^{\theta+2\varepsilon}}, \quad 0 < \theta < 1. \end{aligned}$$

We obtain in the following sections sharp estimates of these sums.

2. Estimates of the double sum $\sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]^{\sigma+2\varepsilon}}$

For some range of values of σ, ε at least, these sums can be estimated directly starting from (1.2) and (1.3).

THEOREM 2.1.

(i) Assume that $0 < \varepsilon < 1/2$. Then,

$$c_1(\varepsilon) N^{2\varepsilon} \leq \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d) J_\varepsilon(\delta)}{[d, \delta]} \leq c_2(\varepsilon) N^{2\varepsilon}.$$

Also,

$$c_3(\varepsilon) N^\varepsilon \leq \sum_{1 \leq d \leq N} \frac{J_\varepsilon(d)}{d} \leq c_4(\varepsilon) N^\varepsilon.$$

If $\varepsilon > 1/2$, then

$$N \leq \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d) J_\varepsilon(\delta)}{[d, \delta]} \leq c_5(\varepsilon) N^{2\varepsilon}$$

(ii) If $0 < \sigma < 1$, then

$$\begin{aligned} \sum_{[d, \delta] \leq N} \frac{1}{[d, \delta]^\sigma} &= \frac{3}{\pi^2} \cdot \frac{1}{1 - \sigma} N^{1-\sigma} (\log N)^2 \\ &\quad + \mathcal{O}(N^{1-\sigma} (\log N)), \end{aligned}$$

and

$$\begin{aligned} c_6(\sigma) e^{-c(\varepsilon) \frac{(\log N)^{1-\varepsilon}}{\log \log N}} N^{1-\sigma} (\log N)^2 &\leq \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d) J_\varepsilon(\delta)}{[d, \delta]^{\sigma+2\varepsilon}} \\ &\leq c_7(\sigma) N^{1-\sigma} (\log N)^2. \end{aligned}$$

Further, if $\sigma = 1$,

$$\sum_{[d, \delta] \leq N} \frac{1}{[d, \delta]} = \frac{1}{\pi^2} (\log N)^3 + \mathcal{O}((\log N)^2),$$

and

$$c_8(\varepsilon) e^{-c(\varepsilon) \frac{(\log N)^{1-\varepsilon}}{\log \log N}} (\log N)^3 \leq \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d) J_\varepsilon(\delta)}{[d, \delta]^{1+2\varepsilon}} \leq c_9(\varepsilon) (\log N)^3.$$

(iii) If $\sigma + 2\varepsilon > 1$, then

$$\begin{aligned} c_{10}(\sigma) e^{-c(\varepsilon) \frac{(\log N)^{1-\varepsilon}}{\log \log N}} N^{1-\sigma} &\leq \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d) J_\varepsilon(\delta)}{[d, \delta]^{\sigma+2\varepsilon}} \\ &\leq c_{11}(\sigma) N^{1-\sigma}. \end{aligned}$$

Here $c_i(\varepsilon)$ (resp. $c_j(\sigma)$) are positive constants depending on ε (resp. σ) only.

Before doing the proof, we notice that

$$\begin{aligned} \sum_{n=1}^N \frac{1}{n^\sigma} &= \sum_{1 \leq d \leq N} \frac{J_\varepsilon(d)}{d^{\sigma+\varepsilon}} \sum_{m=1}^{\frac{N}{d}} \frac{1}{m^{\sigma+\varepsilon}} \\ &= \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]^{\sigma+2\varepsilon}} \sum_{m=1}^{\frac{N}{[d, \delta]}} \frac{1}{m^{\sigma+2\varepsilon}}, \end{aligned} \quad (2.1)$$

which follows from (1.2), (1.3), letting $t = 0$.

Proof. Using (2.1) with $\sigma = 1 - 2\varepsilon$ yields, for $\varepsilon \geq 0$,

$$\begin{aligned} C(\varepsilon)N^{2\varepsilon} &\geq \sum_{n=1}^N \frac{1}{n^{1-2\varepsilon}} = \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]} \sum_{m=1}^{\frac{N}{[d, \delta]}} \frac{1}{m} \\ &\geq \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]}. \end{aligned}$$

(i) Assume that $\varepsilon < 1/2$. Using (2.1) with $\sigma = 0$ gives

$$\begin{aligned} N &= \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]^{2\varepsilon}} \sum_{m=1}^{\frac{N}{[d, \delta]}} \frac{1}{m^{2\varepsilon}} \\ &\leq C(\varepsilon) \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]^{2\varepsilon}} \left(\frac{N}{[d, \delta]} \right)^{1-2\varepsilon} \\ &= C(\varepsilon)N^{1-2\varepsilon} \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]}. \end{aligned}$$

Thus

$$\sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]} \geq C(\varepsilon)N^{2\varepsilon}.$$

So that

$$c_1(\varepsilon)N^{2\varepsilon} \leq \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]} \leq c_2(\varepsilon)N^{2\varepsilon}.$$

Now trivially

$$\sum_{1 \leq d \leq N} \frac{J_\varepsilon(d)}{d} \leq C(\varepsilon)N^\varepsilon.$$

Next, by letting $\sigma = 1 - 2\varepsilon$ in (2.1),

$$\begin{aligned}
C(\varepsilon)N^\varepsilon \sum_{1 \leq d \leq N} \frac{J_\varepsilon(d)}{d} &\geq \sum_{1 \leq d \leq N} \frac{J_\varepsilon(d)}{d^{1-\varepsilon}} \sum_{m=1}^{\frac{N}{d}} \frac{1}{m^{1-\varepsilon}} \\
&= \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]} \sum_{m=1}^{\frac{N}{[d, \delta]}} \frac{1}{m} \geq c(\varepsilon)N^{2\varepsilon}.
\end{aligned}$$

Consequently,

$$c_1(\varepsilon)N^\varepsilon \leq \sum_{1 \leq d \leq N} \frac{J_\varepsilon(d)}{d} \leq c_2(\varepsilon)N^\varepsilon.$$

If $\varepsilon > 1/2$. Similarly,

$$\begin{aligned}
N &= \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]^{2\varepsilon}} \sum_{m=1}^{\frac{N}{[d, \delta]}} \frac{1}{m^{2\varepsilon}} \\
&\leq \left(\sum_{m \geq 1} \frac{1}{m^{2\varepsilon}} \right) \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]^{2\varepsilon}}.
\end{aligned}$$

Whence

$$c(\varepsilon)N^{2\varepsilon} \geq \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]} \geq \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d, \delta]^{2\varepsilon}} \geq N.$$

(ii) Assume that $0 < \sigma \leq 1$, and let $R(u) = \#\{(\delta, d) \in \mathbb{N}^2 : [d, \delta] = u\}$. Clearly, $R(u) \leq d^2(u)$ (in fact $\sum_{v|u} R(v) = d^2(u)$, where d is the divisor function; which by Möbius inversion formula implies $R = \mu * d^2$, whence R is a multiplicative function). Consequently,

$$\begin{aligned}
\sum_{[d, \delta] \leq N} \frac{1}{[d, \delta]^\sigma} &= \sum_{u \leq N} \frac{1}{u^\sigma} \#\{(\delta, d) \in \mathbb{N}^2 : [d, \delta] = u\} \\
&= \sum_{u \leq N} \frac{R(u)}{u^\sigma} = \sum_{u \leq N} \frac{1}{u^\sigma} \sum_{d|u} \mu(d) d^2 \left(\frac{u}{d} \right) \\
&= \sum_{d \leq N} \frac{\mu(d)}{d^\sigma} \sum_{u \leq N/d} \frac{d^2(u)}{u^\sigma}. \tag{2.2}
\end{aligned}$$

For $0 < \sigma < 1$, by using Abel summation, and the formula ([7]),

$$\sum_{n \leq x} d^2(n) = \frac{1}{\pi^2} x (\log x)^3 + Bx (\log x)^2 + Cx \log x + \mathcal{O}(x),$$

where B, C are the constants.

We get

$$\sum_{u \leq N} \frac{d^2(u)}{u^\sigma} = D_1 N^{1-\sigma} (\log N)^3 + D_2 N^{1-\sigma} (\log N)^2 \\ + D_3 N^{1-\sigma} \log N + \mathcal{O}(N^{1-\sigma}),$$

where

$$D_1 = \frac{1}{(1-\sigma)} \cdot \frac{1}{\pi^2}, \quad D_2 = B + \frac{\sigma B}{1-\sigma} - \frac{3\sigma}{\pi^2(1-\sigma)^2}$$

and

$$D_3 = \frac{C}{1-\sigma} - \frac{2\sigma B}{(1-\sigma)^2} + \frac{6\sigma}{\pi^2(1-\sigma)^3},$$

$$\begin{aligned} \sum_{[d, \delta] \leq N} \frac{1}{[d, \delta]^\sigma} &= \sum_{d \leq N} \frac{\mu(d)}{d^\sigma} \left(D_1 \left(\frac{N}{d} \right)^{1-\sigma} \left(\log \left(\frac{N}{d} \right) \right)^3 \right. \\ &\quad + D_2 \left(\frac{N}{d} \right)^{1-\sigma} \left(\log \left(\frac{N}{d} \right) \right)^2 \\ &\quad \left. + D_3 \left(\frac{N}{d} \right)^{1-\sigma} \left(\log \left(\frac{N}{d} \right) \right) + \mathcal{O} \left(\left(\frac{N}{d} \right)^{1-\sigma} \right) \right) \\ &= D_1 N^{1-\sigma} \sum_{d \leq N} \frac{\mu(d)}{d} \left(\log \left(\frac{N}{d} \right) \right)^3 \\ &\quad + D_2 N^{1-\sigma} \sum_{d \leq N} \frac{\mu(d)}{d} \left(\log \left(\frac{N}{d} \right) \right)^2 \\ &\quad + D_3 N^{1-\sigma} \sum_{d \leq N} \frac{\mu(d)}{d} \left(\log \left(\frac{N}{d} \right) \right) + \mathcal{O} (N^{1-\sigma} (\log N)) \\ &= 3D_1 N^{1-\sigma} (\log N)^2 + \mathcal{O} (N^{1-\sigma} (\log N)) \\ &= \frac{3}{\pi^2} \cdot \frac{1}{1-\sigma} N^{1-\sigma} (\log N)^2 + \mathcal{O} (N^{1-\sigma} (\log N)), \end{aligned}$$

where we have used the estimates for $k = 2, 3$ (see [5])

$$\sum_{n \leq x} \frac{\mu(n)}{n} \left(\log \frac{x}{n} \right)^k = k(\log x)^{k-1} + \sum_{i=1}^{k-2} c_i^{(k)} ((\log x)^i) + \mathcal{O}(1),$$

and

$$\sum_{n \leq x} \frac{\mu(n)}{n} \log \frac{x}{n} = \mathcal{O}(1).$$

For $\sigma = 1$, a similar proof of the case $0 < \sigma < 1$ yields

$$\sum_{[d, \delta] \leq N} \frac{1}{[d, \delta]} = \frac{1}{\pi^2} (\log N)^3 + \mathcal{O}((\log N)^2).$$

The remaining lower bound part is trivial. It is obvious now that for any $\varepsilon > 0$,

$$\sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d) J_\varepsilon(\delta)}{[d, \delta]^{1+2\varepsilon}} \leq (\log N)^3.$$

As to the claimed lower bound, it now follows from Lemma 3.1.

(iii) If $\sigma + 2\varepsilon > 1$, then

$$c(\sigma) N^{1-\sigma} = \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d) J_\varepsilon(\delta)}{[d, \delta]^{\sigma+2\varepsilon}} \sum_{m=1}^{\frac{N}{[d, \delta]}} \frac{1}{m^{\sigma+2\varepsilon}} \leq c(\sigma + 2\varepsilon) \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d) J_\varepsilon(\delta)}{[d, \delta]^{\sigma+2\varepsilon}}.$$

Also we have

$$c(\sigma) N^{1-\sigma} = \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d) J_\varepsilon(\delta)}{[d, \delta]^{\sigma+2\varepsilon}} \sum_{m=1}^{\frac{N}{[d, \delta]}} \frac{1}{m^{\sigma+2\varepsilon}} \geq \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d) J_\varepsilon(\delta)}{[d, \delta]^{\sigma+2\varepsilon}}.$$

Then we obtain

$$c_2(\sigma) N^{1-\sigma} \leq \sum_{[d, \delta] \leq N} \frac{J_\varepsilon(d) J_\varepsilon(\delta)}{[d, \delta]^{\sigma+2\varepsilon}} \leq c_3(\sigma) N^{1-\sigma}. \quad \square$$

3. Estimates of $J_\varepsilon(n)$ and $\sum_{1 \leq d \leq x} \frac{J_\varepsilon(d)}{d^\varepsilon}$

Recall before continuing that $J_\varepsilon(n)$ is a multiplicative function, $J_\varepsilon(1)=1$, $J_\varepsilon(p^\alpha) = p^{\alpha\varepsilon}(1 - p^{-\varepsilon})$ for p prime, $\alpha \geq 1$. Further recall that $J_1 = \phi$, the Euler totient function and $J_0 = \delta$, the inverse Möbius function. We first collect individual estimates.

LEMMA 3.1. *We have*

$$C \frac{v}{\log \log v} \leq \phi(v) \leq v.$$

If $0 < \varepsilon < 1$,

$$v^\varepsilon \exp \left\{ -c(\varepsilon) \frac{(\log v)^{1-\varepsilon}}{\log \log v} \right\} \leq J_\varepsilon(v) \leq v^\varepsilon.$$

Further,

$$\sum_{1 \leq d \leq x} \frac{J_\varepsilon(d)}{d^\varepsilon} = \frac{1}{\zeta(1+\varepsilon)} x + \mathcal{O}(x^{1-\varepsilon}),$$

and as $\varepsilon \rightarrow 0$,

$$\sum_{1 \leq d \leq x} \frac{J_\varepsilon(d)}{d^\varepsilon} = x \sum_{u=1}^{\infty} \frac{\mu(u)}{u^{1+\varepsilon}} + \mathcal{O}(x) \ll \frac{x}{\varepsilon}.$$

P r o o f. The first estimate is well-known and is proved in [4, p. 55] using “record’s breaking numbers”. The second estimate can be proved using the same method. Now,

$$\frac{J_\varepsilon(d)}{d^\varepsilon} = \sum_{u|d} \frac{\mu(u)}{u^\varepsilon} = \prod_{p|d} (1 - p^{-\varepsilon}).$$

Further,

$$\sum_{u \leq x} \frac{\mu(u)}{u^{1+\varepsilon}} = \frac{1}{\zeta(1+\varepsilon)} + \mathcal{O}(x^{-\varepsilon}). \quad (3.1)$$

Hence by [4, (2.3), p. 36],

$$\sum_{1 \leq d \leq x} \frac{J_\varepsilon(d)}{d^\varepsilon} = x \sum_{u=1}^{\infty} \frac{\mu(u)}{u^{1+\varepsilon}} + \mathcal{O}(x^{1-\varepsilon}) = \frac{1}{\zeta(1+\varepsilon)} x + \mathcal{O}(x^{1-\varepsilon}).$$

Letting $x \rightarrow \infty$ in (3.1),

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{1+\varepsilon}} \ll \frac{1}{\varepsilon} + 1. \quad (3.2)$$

Then for $\varepsilon \rightarrow 0$,

$$\sum_{1 \leq d \leq x} \frac{J_\varepsilon(d)}{d^\varepsilon} = x \sum_{u=1}^{\infty} \frac{\mu(u)}{u^{1+\varepsilon}} + \mathcal{O}(x) \ll \frac{x}{\varepsilon} + \mathcal{O}(x) \ll \frac{x}{\varepsilon}. \quad \square$$

NOTE. When $\varepsilon = 0$, the series in the right-term is formally equal to $\sum_{u=1}^{\infty} \frac{\mu(u)}{u} = 0$ and we recall [1, p. 209] that

$$\sum_{u \leq x} \frac{\mu(u)}{u} = \mathcal{O}\left(\left(\sqrt{\log x}\right) e^{-c\sqrt{\log x}}\right).$$

Obviously, $\frac{J_\varepsilon(d)}{d^\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$. The summatory function $\sum_{1 \leq d \leq x} \frac{J_\varepsilon(d)}{d^\varepsilon}$ naturally tightly depends on the smallness of ε .

4. The summatory function $\sum_{n \leq x} \left(\frac{J_\varepsilon(n)}{n^\varepsilon} \right)^2$.

THEOREM 4.1. For $\varepsilon \geq 1/2$,

$$\begin{aligned} \sum_{n \leq x} \left(\frac{J_\varepsilon(n)}{n^\varepsilon} \right)^2 &= \frac{x}{\zeta(1+\varepsilon)^2} \sum_{t \leq x} \frac{|\mu(t)| \phi^{-1}(t)}{J_{1+\varepsilon}^2(t)} \\ &\quad + \mathcal{O}_\varepsilon \left(\frac{x^{1-\varepsilon}}{\varepsilon \zeta(1+\varepsilon)} \right) \\ &\quad + \mathcal{O}(\Delta_\varepsilon(x)), \end{aligned}$$

where $\phi^{-1}(t)$ denotes the inverse of $\phi(t)$ and $\Delta_\varepsilon(x)$ is defined by (4.1).

THEOREM 4.2.

$$x \exp \left\{ -c(\varepsilon) \frac{(\log x)^{1-\varepsilon}}{\log \log x} \right\} \ll \sum_{1 \leq d \leq x} \left(\frac{J_\varepsilon(d)}{d^\varepsilon} \right)^2 \ll x.$$

Proof of Theorem 4.1. As $\frac{J_\varepsilon(n)}{n^\varepsilon} = \sum_{u|n} \frac{\mu(u)}{u^\varepsilon}$, we have

$$\left(\frac{J_\varepsilon(n)}{n^\varepsilon} \right)^2 = \sum_{\substack{u|n \\ v|n}} \frac{\mu(u)\mu(v)}{(uv)^\varepsilon}.$$

Thus

$$\begin{aligned} \sum_{n \leq x} \left(\frac{J_\varepsilon(n)}{n^\varepsilon} \right)^2 &= \sum_{n \leq x} \sum_{\substack{u|n \\ v|n}} \frac{\mu(u)\mu(v)}{(uv)^\varepsilon} = \sum_{\substack{u \leq x \\ v \leq x}} \frac{\mu(u)\mu(v)}{(uv)^\varepsilon} \left(\sum_{\substack{n \leq x \\ [u,v]|n}} 1 \right) \\ &= \sum_{\substack{u \leq x \\ v \leq x}} \frac{\mu(u)\mu(v)}{(uv)^\varepsilon} \left(\frac{x}{[u,v]} + \mathcal{O}(1) \right) \\ &= x \sum_{\substack{u \leq x \\ v \leq x}} \frac{\mu(u)\mu(v)}{(uv)^\varepsilon [u,v]} + \mathcal{O} \left(\left(\sum_{u \leq x} \frac{|\mu(u)|}{u^\varepsilon} \right)^2 \right) \\ &= x \sum_{\substack{u \leq x \\ v \leq x}} \frac{\mu(u)\mu(v)}{(uv)^\varepsilon [u,v]} + \mathcal{O}(\Delta_\varepsilon(x)), \end{aligned}$$

where

$$\Delta_\varepsilon(x) = \begin{cases} x^{2-2\varepsilon} & \text{for } \frac{1}{2} \leq \varepsilon < 1, \\ \log x & \text{for } \varepsilon = 1, \\ 1 & \text{for } \varepsilon > 1. \end{cases} \quad (4.1)$$

Now

$$\begin{aligned} \sum_{\substack{u \leq x \\ v \leq x}} \frac{\mu(u)\mu(v)}{(uv)^\varepsilon [u, v]} &= \sum_{d \leq x} d \sum_{\substack{u \leq x, v \leq x \\ (u, v) = d}} \frac{\mu(u)\mu(v)}{(uv)^{1+\varepsilon}} \\ &= \sum_{d \leq x} \frac{1}{d^{1+2\varepsilon}} \sum_{\substack{u' \leq x/d, v' \leq x/d \\ (u', v') = 1}} \frac{\mu(u'd)\mu(v'd)}{(u'v')^{1+\varepsilon}} \\ &= \sum_{d \leq x} \frac{1}{d^{1+2\varepsilon}} \sum_{u' \leq x/d, v' \leq x/d} \frac{\mu(u'd)\mu(v'd)}{(uv)^{1+\varepsilon}} \left(\sum_{\substack{w|u' \\ w|v'}} \mu(w) \right) \\ &= \sum_{d \leq x} \frac{1}{d^{1+2\varepsilon}} \sum_{w \leq x/d} \mu(w) \left(\sum_{\substack{u' \leq x/d \\ w|u'}} \frac{\mu(u'd)}{(u')^{1+\varepsilon}} \right)^2 \\ &= \sum_{d \leq x} \frac{1}{d^{1+2\varepsilon}} \sum_{w \leq x/d} \mu(w) \left(d^{1+\varepsilon} \sum_{\substack{h \leq x \\ dw|h}} \frac{\mu(h)}{h^{1+\varepsilon}} \right)^2 \\ &= \sum_{d \leq x} d \sum_{w \leq x/d} \mu(w) \left(\sum_{\substack{h \leq x \\ dw|h}} \frac{\mu(h)}{h^{1+\varepsilon}} \right)^2. \end{aligned} \quad (4.2)$$

We shall first estimate for a given integer θ the sum

$$\sum_{\substack{h \leq x \\ \theta|h}} \frac{\mu(h)}{h^{1+\varepsilon}},$$

Let $\mathcal{P}_x$ denotes the collection of prime numbers less than x and $\mathcal{P} = \mathcal{P}_\infty$. If θ is not squarefree, then $\mu(h) = 0$ and this sum reduces to 0. Otherwise, let $\mathcal{Q}_x$ denote the set of integers θ such that

$$\theta = p_1 \dots p_r, \quad \text{where} \quad p_j \in \mathcal{P}_x, \quad \theta \leq x.$$

Only those integers h of type $h = p_1 \dots p_r \cdot q_1 \dots q_s$, with $q_i \in \mathcal{P}_x \setminus \{p_i\}$, contribute to the sum. We thereby get the sum

$$\frac{\mu(\theta)}{\theta^{1+\varepsilon}} \cdot \sum_{\substack{k \leq x \\ (k, \theta)=1}} \frac{\mu(k)}{k^{1+\varepsilon}},$$

if $\theta \in \mathcal{Q}_x$, and this sum is 0 otherwise. Thus

$$\sum_{\substack{h \leq x \\ \theta | h}} \frac{\mu(h)}{h^{1+\varepsilon}} = \begin{cases} \frac{\mu(\theta)}{\theta^{1+\varepsilon}} \cdot \sum_{\substack{k \leq x \\ (k, \theta)=1}} \frac{\mu(k)}{k^{1+\varepsilon}}, & \theta \in \mathcal{Q}_x, \\ 0, & \theta \notin \mathcal{Q}_x. \end{cases} \quad (4.3)$$

Let r be a squarefree positive integer. By formula p. 196 in [3], for $s > 1$,

$$\sum_{n=1}^{\infty} \frac{\mu(nr)}{n^s} = \mu(r) \sum_{\substack{n=1 \\ (n, r)=1}}^{\infty} \frac{\mu(n)}{n^s} = \frac{\mu(r)r^s}{J_s(r)\zeta(s)}. \quad (4.4)$$

Thus

$$\begin{aligned} \sum_{\substack{h \leq x \\ \theta | h}} \frac{\mu(h)}{h^{1+\varepsilon}} &= \frac{\mu(\theta)}{\theta^{1+\varepsilon}} \left(\sum_{\substack{k=1 \\ (k, \theta)=1}}^{\infty} \frac{\mu(k)}{k^{1+\varepsilon}} - \sum_{\substack{k > x \\ (k, \theta)=1}} \frac{\mu(k)}{k^{1+\varepsilon}} \right) \\ &= \frac{\mu(\theta)}{\theta^{1+\varepsilon}} \cdot \left(\frac{\theta^{1+\varepsilon}}{J_{1+\varepsilon}(\theta)\zeta(1+\varepsilon)} + \mathcal{O}_{\varepsilon}(x^{-\varepsilon}) \right) \\ &= \frac{\mu(\theta)}{J_{1+\varepsilon}(\theta)\zeta(1+\varepsilon)} \\ &\quad + \mathcal{O}_{\varepsilon}(\theta^{-(1+\varepsilon)}x^{-\varepsilon}). \end{aligned} \quad (4.5)$$

In (4.2), $\theta = dw$ and thus must belong to $\mathcal{Q}_x$, in particular we must have $(d, w) = 1$. Thus

$$d|h, \quad w|h, \quad d, w \in \mathcal{Q}_x \quad \text{and} \quad (d, w) = 1.$$

Whence

$$\sum_{\substack{h \leq x \\ dw|h}} \frac{\mu(h)}{h^{1+\varepsilon}} = \frac{\mu(d)\mu(w)}{J_{1+\varepsilon}(d)J_{1+\varepsilon}(w)\zeta(1+\varepsilon)} + \mathcal{O}_{\varepsilon}\left(\frac{1}{(dw)^{1+\varepsilon}x^{\varepsilon}}\right).$$

By reporting

$$\begin{aligned}
 \sum_{\substack{u \leq x \\ v \leq x}} \frac{\mu(u)\mu(v)}{(uv)^\varepsilon [u, v]} &= \sum_{d \leq x} d \sum_{\substack{w \leq x/d \\ (d, w)=1}} \mu(w) \left(\sum_{\substack{h \leq x \\ dw|h}} \frac{\mu(h)}{h^{1+\varepsilon}} \right)^2 \\
 &= \sum_{d \leq x} d \sum_{\substack{w \leq x/d \\ (d, w)=1}} \mu(w) \left(\frac{\mu(d)\mu(w)}{J_{1+\varepsilon}(d)J_{1+\varepsilon}(w)\zeta(1+\varepsilon)} + \mathcal{O}_\varepsilon \left(\frac{1}{(dw)^{1+\varepsilon}x^\varepsilon} \right) \right)^2 \\
 &= \frac{1}{\zeta(1+\varepsilon)^2} \sum_{d \leq x} \frac{d\mu(d)}{J_{1+\varepsilon}^2(d)} \sum_{\substack{w \leq x/d \\ (d, w)=1}} \frac{|\mu(w)|}{J_{1+\varepsilon}^2(w)} \\
 &\quad + \mathcal{O}_\varepsilon \left(\frac{1}{\zeta(1+\varepsilon)x^\varepsilon} \sum_{d \leq x} \frac{1}{d^\varepsilon J_{1+\varepsilon}(d)} \sum_{\substack{w \leq x/d \\ (d, w)=1}} \frac{1}{w^{1+\varepsilon} J_{1+\varepsilon}(w)} \right) \\
 &\quad + \mathcal{O}_\varepsilon \left(\frac{1}{x^{2\varepsilon}} \sum_{d \leq x} \frac{1}{d^{1+2\varepsilon}} \sum_{\substack{w \leq x/d \\ (d, w)=1}} \frac{1}{w^{2(1+\varepsilon)}} \right) \\
 &= \frac{1}{\zeta(1+\varepsilon)^2} \sum_{d \leq x} \frac{d\mu(d)}{J_{1+\varepsilon}^2(d)} \sum_{\substack{w \leq x/d \\ (d, w)=1}} \frac{|\mu(w)|}{J_{1+\varepsilon}^2(w)} \\
 &\quad + \mathcal{O}_\varepsilon \left(\frac{1}{\varepsilon \zeta(1+\varepsilon)x^\varepsilon} \right) + \mathcal{O}_\varepsilon \left(\frac{1}{\varepsilon x^{2\varepsilon}} \right) \\
 &= \frac{1}{\zeta(1+\varepsilon)^2} \sum_{w \leq x} \frac{|\mu(w)|}{J_{1+\varepsilon}^2(w)} \sum_{\substack{d \leq x/w \\ (d, w)=1}} \frac{d\mu(d)}{J_{1+\varepsilon}^2(d)} \\
 &\quad + \mathcal{O}_\varepsilon \left(\frac{1}{\varepsilon \zeta(1+\varepsilon)x^\varepsilon} \right). \tag{4.6}
 \end{aligned}$$

For the main term we have

$$\begin{aligned}
 \frac{1}{\zeta(1+\varepsilon)^2} \sum_{w \leq x} \frac{|\mu(w)|}{J_{1+\varepsilon}^2(w)} \sum_{\substack{d \leq x/w \\ (d, w)=1}} \frac{d\mu(d)}{J_{1+\varepsilon}^2(d)} &= \frac{1}{\zeta(1+\varepsilon)^2} \sum_{t \leq x} \frac{|\mu(w)|}{J_{1+\varepsilon}^2(t)} \sum_{\substack{w \leq x, d \leq x/w \\ (d, w)=1, dw=t}} d\mu(d) \\
 &= \frac{1}{\zeta(1+\varepsilon)^2} \sum_{t \leq x} \frac{|\mu(t)|}{J_{1+\varepsilon}^2(t)} \sum_{\substack{d \leq x, \\ d|t}} d\mu(d),
 \end{aligned}$$

where we have used $|\mu(d)| = \mu(d)^2$, $\mu(w)|\mu(d)| = \mu(w)^3|\mu(d)| = |\mu(wd)|\mu(w)$ for $(w, d) = 1$; and $J_\varepsilon(n)$ is multi-function. Further the inner sum

$$\sum_{\substack{d \leq x \\ d|t}} d\mu(d)$$

simplifies as follows

$$\sum_{d|t} d\mu(d) = \phi^{-1}(t),$$

since $t \leq x$. Thus

$$\frac{1}{\zeta(1+\varepsilon)^2} \sum_{w \leq x} \frac{|\mu(w)|}{J_{1+\varepsilon}^2(w)} \sum_{\substack{d \leq x/w \\ (d,w)=1}} \frac{d\mu(d)}{J_{1+\varepsilon}^2(d)} = \frac{1}{\zeta(1+\varepsilon)^2} \sum_{t \leq x} \frac{|\mu(t)|\phi^{-1}(t)}{J_{1+\varepsilon}^2(t)}.$$

So that

$$\sum_{\substack{u \leq x \\ v \leq x}} \frac{\mu(u)\mu(v)}{(uv)^\varepsilon[u, v]} = \frac{1}{\zeta(1+\varepsilon)^2} \sum_{t \leq x} \frac{|\mu(t)|\phi^{-1}(t)}{J_{1+\varepsilon}^2(t)} + \mathcal{O}_\varepsilon\left(\frac{1}{\varepsilon \zeta(1+\varepsilon)x^\varepsilon}\right).$$

Therefore

$$\begin{aligned} \sum_{n \leq x} \left(\frac{J_\varepsilon(n)}{n^\varepsilon} \right)^2 &= x \sum_{\substack{u \leq x \\ v \leq x}} \frac{\mu(u)\mu(v)}{(uv)^\varepsilon[u, v]} + \mathcal{O}(\Delta_\varepsilon(x)) \\ &= \frac{x}{\zeta(1+\varepsilon)} \sum_{t \leq x} \frac{|\mu(t)|\phi^{-1}(t)}{J_{1+\varepsilon}^2(t)} + \mathcal{O}_\varepsilon\left(\frac{x^{1-\varepsilon}}{\varepsilon \zeta(1+\varepsilon)}\right) + \mathcal{O}(\Delta_\varepsilon(x)). \end{aligned}$$

This achieves the proof. $\square$

Proof of Theorem 4.2. Theorem 4.2 immediately follows from Lemma 3.1 and Theorem 4.1. $\square$

5. The summatory function $\sum_{n=1}^N \frac{J_\varepsilon^2(n)}{n^{\theta+2\varepsilon}}$, $0 < \theta < 1$.

THEOREM 5.1. *Let θ, ε be positive reals such that $0 < \theta + \varepsilon < 1$. Then,*

$$\sum_{n=1}^N \frac{J_\varepsilon^2(n)}{n^{\theta+2\varepsilon}} \leq C_{-\theta} \left(N^{1-\theta} \left(\exp\{-H(\varepsilon)\} + \frac{1}{\varepsilon N^\varepsilon} \right) + N^{1-\theta-\varepsilon} (\log N)^2 \right),$$

where $C_{-\theta}$ is defined in (5.3) and

$$H(\varepsilon) = \sum_p \frac{(2-p^{-\varepsilon})}{p^{1+\varepsilon}}.$$

$$\sum_{n=1}^N \frac{J_\varepsilon^2(n)}{n^{\theta+2\varepsilon}} \leq C_{1-\theta} \left(N^{1-\theta} \left(\exp\{-H(\varepsilon)\} + \frac{1}{\varepsilon N^\varepsilon} \right) + N^{1-\theta-\varepsilon} (\log N)^2 \right). \quad (5.1)$$

PROPOSITION 5.2. *We have*

$$H(\varepsilon) = \frac{3}{\log 2} \cdot \frac{1}{\varepsilon} - \log \left(\frac{1}{\varepsilon} \right) + \mathcal{O}(1), \quad \text{as } \varepsilon \rightarrow 0.$$

Proposition applies notably when ε is small, depending on N , the typical range of values being given by the interval $[1/\log N, 1/\log \log N]$.

Proof of Theorem 5.1. Use the simple notation $P = P(p) = \frac{p^\varepsilon}{2-p^\varepsilon}$, and note that $1 - P^{-1} = (1 - p^{-\varepsilon})^2$.

Define an auxiliary (completely multiplicative) arithmetical function h as follows,

$$h(p^\alpha) = P^\alpha, \quad h(n) = \prod_{p^\alpha || n} P^\alpha. \quad (5.2)$$

By arguing as in [4, p. 36],

$$\frac{J_\varepsilon^2(n)}{n^{2\varepsilon}} = \prod_{p|n} (1 - p^{-\varepsilon})^2 = \prod_{p|n} (1 - P^{-1}) = \sum_{u|n} \frac{\mu(u)}{h(u)}.$$

Besides,

$$\begin{aligned} \sum_{n=1}^N \frac{J_\varepsilon^2(n)}{n^{\theta+2\varepsilon}} &= \sum_{n=1}^N \frac{1}{n^\theta} \sum_{d|n} \frac{\mu(d)}{h(d)} = \sum_{d \leq N} \frac{\mu(d)}{h(d)} \sum_{\substack{n=1 \\ d|n}}^N \frac{1}{n^\theta} \\ &= \sum_{d \leq N} \frac{\mu(d)}{h(d) d^\theta} \sum_{m=1}^{N/d} \frac{1}{m^\theta}. \end{aligned}$$

Let $-1 < \alpha < 0$. Then for $M \geq 1$,

$$\begin{aligned} \sum_{k=1}^M k^\alpha &= \frac{M^{\alpha+1}}{\alpha+1} + \mathcal{O}((M^\alpha)) + \left(\frac{1}{2} - \frac{1}{\alpha+1} \right) \\ &\quad + \alpha(\alpha-1) \sum_{k=1}^{\infty} \int_0^1 \frac{t-t^2}{2} (k+t)^{\alpha-2} dt - \sum_{k=M}^{\infty} \mathcal{O}(k^{\alpha-2}) \\ &= \frac{M^{\alpha+1}}{\alpha+1} + C_\alpha + \mathcal{O}(M^\alpha), \end{aligned}$$

where

$$C_\alpha = \frac{1}{2} - \frac{1}{\alpha + 1} + \alpha(\alpha - 1) \sum_{k=1}^{\infty} \int_0^1 \frac{t - t^2}{2} (k + t)^{\alpha-2} dt. \quad (5.3)$$

Thus

$$\sum_{m=1}^{N/d} \frac{1}{m^\theta} = \frac{1}{1-\theta} \left(\frac{N}{d} \right)^{1-\theta} + C_{-\theta} + \mathcal{O} \left(\left(\frac{N}{d} \right)^{-\theta} \right).$$

Whence,

$$\begin{aligned} \sum_{n=1}^N \frac{J_\varepsilon^2(n)}{n^{\theta+2\varepsilon}} &= \frac{N^{1-\theta}}{1-\theta} \sum_{d \leq N} \frac{\mu(d)}{h(d)d} + C_{-\theta} \sum_{d \leq N} \frac{\mu(d)}{h(d)d^\theta} \\ &\quad + \mathcal{O} \left(N^{-\theta} \sum_{d \leq N} \frac{|\mu(d)|}{h(d)} \right). \end{aligned} \quad (5.4)$$

We first estimate the sum $\sum_{d \leq N} \frac{\mu(d)}{h(d)d}$. Now

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{h(d)d} = \prod_p \left(1 - \frac{1}{h(p)p} \right) = \prod_p \left(1 - \frac{(2-p^{-\varepsilon})}{p^{1+\varepsilon}} \right)$$

Thus $0 < \sum_{n=1}^{\infty} \frac{\mu(d)}{h(d)d} < \infty$ and

$$\begin{aligned} \sum_{d=1}^{\infty} \frac{\mu(d)}{h(d)d} &= \exp \left\{ \sum_p \left(1 - \frac{(2-p^{-\varepsilon})}{p^{1+\varepsilon}} \right) \right\} \leq \exp \left\{ - \sum_p \frac{(2-p^{-\varepsilon})}{p^{1+\varepsilon}} \right\} \\ &= \exp\{-H(\varepsilon)\}. \end{aligned}$$

We have

$$h(n) = \prod_{p^\alpha \parallel n} P^\alpha = \prod_{p^\alpha \parallel n} \left(\frac{p^\varepsilon}{2-p^{-\varepsilon}} \right)^\alpha \geq \frac{n^\varepsilon}{2^{\Omega(n)}},$$

where $\Omega(n)$ is the sum of prime divisor function. Thus

$$\sum_{d > N} \frac{1}{h(d)d} \leq \sum_{d > N} \frac{2^{\Omega(d)}}{d^{1+\varepsilon}}.$$

We use the fact that

$$\sum_{n \leq x} 2^{\Omega(n)} = C_0 x (\log x)^2 + \mathcal{O}(x \log x),$$

where C_0 is numerical. Applying Abel summation gives,

$$\sum_{m=N+1}^{N+M} \frac{2^{\Omega(d)}}{d^{1+\varepsilon}} \leq C \left\{ \sum_{\mu=N+1}^{N+M} \frac{(\log \mu)^2}{\mu^{1+\varepsilon}} + \frac{(\log(M+N))^2}{(M+N)^\varepsilon} \right\}.$$

Hence, by letting M tend to infinity,

$$\sum_{d>N} \frac{1}{h(d)d} \leq C \sum_{\mu>N} \frac{(\log \mu)^2}{\mu^{1+\varepsilon}} \leq \frac{C}{\varepsilon N^\varepsilon}.$$

We deduce that

$$\left| \sum_{d \leq N} \frac{\mu(d)}{h(d)d} \right| \leq \exp\{-H(\varepsilon)\} + \frac{C}{\varepsilon N^\varepsilon}.$$

Consider now the second sum $\sum_{d \leq N} \frac{\mu(d)}{h(d)d^\theta}$ in the right-term of (5.4), we have by using Abel summation again,

$$\sum_{d \leq N} \frac{1}{h(d)d^\theta} \leq \sum_{d \leq N} \frac{2^{\Omega(d)}}{d^{\theta+\varepsilon}} \leq C_{\theta,\varepsilon} N^{1-\theta-\varepsilon} (\log N)^2.$$

Finally, the third sum $\sum_{d \leq N} \frac{\mu(d)}{h(d)}$ is special case ($\theta = 0$) of the previous and we have,

$$N^{-\theta} \sum_{d \leq N} \frac{|\mu(d)|}{h(d)} \leq N^{-\theta} \sum_{d \leq N} \frac{1}{h(d)} \leq C_{\theta,\varepsilon} N^{1-\theta-\varepsilon} (\log N)^2.$$

By combining these estimates, we get

$$\sum_{n=1}^N \frac{J_\varepsilon^2(n)}{n^{\theta+2\varepsilon}} \leq C_{-\theta} \left(N^{1-\theta} \left(\exp\{-H(\varepsilon)\} + \frac{1}{\varepsilon N^\varepsilon} \right) + N^{1-\theta-\varepsilon} (\log N)^2 \right). \quad \square$$

Proof of Proposition 5.2. Set

$$A(x) = \sum_{n \leq x} a(n), \quad f(x) = \frac{1}{x^{1+\varepsilon}},$$

and $a(n) = 1$ for n is prime, otherwise $a(n) = 0$. Then we have

$$A(x) = \frac{x}{\log x} + \mathcal{O}\left(\frac{x}{\log^2 x}\right).$$

Applying Abel's identity,

$$\begin{aligned} \sum_{p \leq x} \frac{1}{p^{1+\varepsilon}} &= \sum_{n \leq x} \frac{a(n)}{n^{1+\varepsilon}} = A(x)f(x) - \int_2^x A(t)f'(t) dt \\ &= \frac{1}{x^\varepsilon \log x} + \mathcal{O}\left(\frac{1}{x^\varepsilon \log^2 x}\right) + (1+\varepsilon) \int_2^x \frac{1}{t^{1+\varepsilon} \log t} \\ &\quad + \mathcal{O}\left(\frac{1}{t^{1+\varepsilon} \log^2 t}\right) dt. \end{aligned} \tag{5.5}$$

We have

$$\begin{aligned}
\int_2^x \frac{1}{t^{1+\varepsilon} \log t} dt &= \frac{1}{\log 2} \int_2^x \frac{1}{t^{1+\varepsilon}} dt + \int_2^x \frac{1}{t^{1+\varepsilon}} \left(\frac{1}{\log t} - \frac{1}{\log 2} \right) dt \\
&= \frac{1}{\varepsilon \log 2} (2^{-\varepsilon} - x^{-\varepsilon}) - \int_2^x \frac{1}{t^{1+\varepsilon}} \int_{\log 2}^{\log t} \frac{1}{u^2} du dt \\
&= \frac{1}{\varepsilon \log 2} (2^{-\varepsilon} - x^{-\varepsilon}) - \int_{\log 2}^{\log x} \frac{1}{u^2} du \int_{e^u}^x \frac{1}{t^{1+\varepsilon}} dt \\
&= \frac{1}{\varepsilon \log 2} (2^{-\varepsilon} - x^{-\varepsilon}) + \frac{1}{\varepsilon} \int_{\log 2}^{\log x} \frac{1}{u^2} (x^{-\varepsilon} - e^{-u\varepsilon}) du \\
&= \frac{1}{\varepsilon \log 2} (2^{-\varepsilon} - x^{-\varepsilon}) + \frac{x^{-\varepsilon}}{\varepsilon} \left(\frac{1}{\log 2} - \frac{1}{\log x} \right) + \frac{1}{\varepsilon} \int_{\log 2}^{\log x} \frac{e^{-u\varepsilon}}{u^2} du,
\end{aligned} \tag{5.6}$$

and for $\varepsilon \geq \frac{1}{\log x}$,

$$\begin{aligned}
\int_{\log 2}^{\log x} \frac{e^{-u\varepsilon}}{u^2} du &= \int_{\log 2}^{\frac{1}{\varepsilon}} \frac{e^{-u\varepsilon}}{u^2} du + \int_{\frac{1}{\varepsilon}}^{\log x} \frac{e^{-u\varepsilon}}{u^2} du \\
&= \int_{\log 2}^{\frac{1}{\varepsilon}} \frac{1}{u^2} (1 - u\varepsilon + \mathcal{O}(u^2\varepsilon^2)) du + \mathcal{O} \left(\varepsilon^2 \int_{\frac{1}{\varepsilon}}^{\log x} e^{-u\varepsilon} du \right) \\
&= \frac{1}{\log 2} + (\log \log 2)\varepsilon - (\varepsilon - (\log \varepsilon)\varepsilon) + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon x^{-\varepsilon}), \tag{5.7}
\end{aligned}$$

for $\varepsilon \leq \frac{1}{\log x}$,

$$\begin{aligned}
\int_{\log 2}^{\log x} \frac{e^{-u\varepsilon}}{u^2} du &= \int_{\log 2}^{\log x} \frac{1}{u^2} (1 - u\varepsilon + \mathcal{O}(u^2\varepsilon^2)) du \\
&= \frac{1}{\log 2} + (\log \log 2)\varepsilon - \left(\frac{1}{\log x} + (\log \log x)\varepsilon \right) + \mathcal{O}(\varepsilon) \\
&= \frac{1}{\log 2} + (\log \log 2)\varepsilon - \frac{1}{\log x} \\
&\quad + \mathcal{O}(\log \log x)\varepsilon + \mathcal{O}(\varepsilon) \\
&= \frac{1}{\log 2} + (\log \log 2)\varepsilon - \frac{1}{\log x} + \mathcal{O} \left(\frac{1}{\log x} \log \log x \right) + \mathcal{O}(\varepsilon) \\
&= \frac{1}{\log 2} + (\log \log 2)\varepsilon + \mathcal{O} \left(\frac{1}{\log x} \log \log x \right) + \mathcal{O}(\varepsilon), \tag{5.8}
\end{aligned}$$

then by (5.5)–(5.8),

$$\begin{aligned}
 \sum_{p \leq x} \frac{1}{p^{1+\varepsilon}} &= \frac{1}{x^\varepsilon \log x} + \mathcal{O}\left(\frac{1}{x^\varepsilon \log^2 x}\right) \\
 &+ (1 + \varepsilon) \left(\frac{1}{\varepsilon \log 2} (2^{-\varepsilon} - x^{-\varepsilon}) + \frac{x^{-\varepsilon}}{\varepsilon} \left(\frac{1}{\log 2} - \frac{1}{\log x} \right) \right) \\
 &+ \frac{1}{\varepsilon \log 2} + (\log \log 2) - (1 - \log \varepsilon) \\
 &+ \mathcal{O}\left(\frac{1}{\varepsilon \log x} \log \log x\right) + \mathcal{O}(1) + \mathcal{O}(x^{-\varepsilon}) \\
 &+ \mathcal{O}\left(\int_2^x \frac{1}{t^{1+\varepsilon} \log^2 t} dt\right). \tag{5.9}
 \end{aligned}$$

Let $x \rightarrow \infty$ in (5.9), thus

$$\sum_{p=1}^{\infty} \frac{1}{p^{1+\varepsilon}} = (1 + \varepsilon) \left(\frac{2^{-\varepsilon}}{\varepsilon \log 2} + \frac{1}{\varepsilon \log 2} - (1 - \log \varepsilon) \right) + \mathcal{O}(1). \tag{5.10}$$

For $\varepsilon \rightarrow 0$, (5.10) yields

$$\sum_{p=1}^{\infty} \frac{1}{p^{1+\varepsilon}} = \frac{2}{\varepsilon \log 2} - \log\left(\frac{1}{\varepsilon}\right) + \mathcal{O}(1). \tag{5.11}$$

By (5.11),

$$\begin{aligned}
 H(\varepsilon) &= \sum_{p=1}^{\infty} \frac{2 - p^{-\varepsilon}}{p^{1+\varepsilon}} = 2 \left(\frac{2}{\varepsilon \log 2} - \log\left(\frac{1}{\varepsilon}\right) + \mathcal{O}(1) \right) \\
 &- \left(\frac{1}{\varepsilon \log 2} - \log\left(\frac{1}{\varepsilon}\right) + \mathcal{O}(1) \right) \\
 &= \frac{3}{\log 2} \cdot \frac{1}{\varepsilon} - \log\left(\frac{1}{\varepsilon}\right) + \mathcal{O}(1). \quad \square
 \end{aligned}$$

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