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UNIFORM DISTRIBUTION OF αn MODULO ONE FOR A FAMILY OF INTEGER SEQUENCES

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ABSTRACT. We show that the sequence $(\alpha n)_{n\in\mathcal{B}}$ is uniformly distributed modulo 1, for every irrational α, provided $\mathcal{B}$ belongs to a certain family of integer sequences, which includes the prime, almost prime, squarefree, practical, densely divisible and lexicographical numbers. We also give an estimate for the discrepancy if α has finite irrationality measure.

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1. Introduction

We say that a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers is uniformly distributed modulo 1 (u.d. mod1) if

$$\lim_{x\to\infty} \frac{|\{n \le x : \{a_n\} \in [a,b]\}|}{x} = b - a$$

for all $0 \le a < b \le 1$, where $\{u\}$ denotes the fractional part of u. Weyl's criterion asserts that this is equivalent to

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} e(la_n) = 0,$$

for every fixed non-zero integer l, where $e(y) := e^{2\pi i y}$.

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Let α be any irrational real number. Weyl's criterion shows at once that the sequence $(\alpha n)_{n\in\mathbb{N}}$ is u.d. mod1. Vinogradov [11, Ch. XI] proved that the same holds for the sequence $(\alpha p)_{p\in\mathbb{P}}$, where p runs through the prime numbers.

We consider the following family of integer sequences. For each natural number n, let $I(n)$ be an arbitrary interval of real numbers, possibly empty, or the union of a bounded number (uniformly in n) of such intervals. Let $\mathcal{B} = \mathcal{B}_I$ be the set of positive integers containing $n = 1$ and all those $n \geq 2$ with prime factorization $n = p_1 \cdots p_k$, $p_1 \leq p_2 \leq \ldots \leq p_k$, which satisfy

$$p_j \in I\left(\prod_{1\leq i<j} p_i\right) \qquad (1 \leq j \leq k).$$

This setting is more general than in [12], where I was of the form $I(n) = [1, \theta(n)]$ for some function $\theta(n)$. If $I(1) = [1, \infty)$ and $I(n) = \emptyset$ for $n > 1$, then $\mathcal{B} \setminus \{1\}$ is the set of primes. If $I(1) = I(p) = [1, \infty)$ for primes p and $I(n) = \emptyset$ for composite n, then $\mathcal{B}$ is the set of integers with at most two prime factors, counted with multiplicity. Similarly, one can obtain more general almost-primes. Squarefree numbers can be generated with $I(n) = (P(n), \infty)$, where $P(n)$ denotes the largest prime factor of $n > 1$ and $P(1) = 1$. Integers whose divisors grow by factors of at most t, which are called *t-densely divisible* [7] or *t-dense* [14], arise from $I(n) = [1, nt]$, while the *practical numbers* result from $I(n) = [1, \sigma(n) + 1]$, where $\sigma(n)$ is the sum of the positive divisors of n. This family also includes the *lexicographical numbers* [9], where $I(n) = \{P(n)\} \cup (n, \infty)$. Define

$$\mathcal{B}(x) := \mathcal{B} \cap [1, x], \quad B(x) := |\mathcal{B}(x)|,$$

$$B_d(x) := |\{n \in \mathcal{B}(x) : d|n\}|.$$

Theorem 1.1. *Let α be any irrational real number. Assume there exist constants $A \geq 0$ and $\delta > 0$ such that*

$$B(x) \gg x(\log x)^{-A}, \qquad B_d(x) \ll d^{-\delta} B(x), \qquad (x, d \geq 2). \tag{1}$$

Then the sequence $(\alpha n)_{n\in\mathcal{B}}$ is u.d. mod1.

When $\alpha > 1$, the integer sequence $(\lfloor \alpha n \rfloor)_{n\in\mathbb{N}}$ contains a proportion of $\frac{1}{\alpha}$ of the natural numbers. Corollary 1.2, which follows easily from Theorem 1.1 (see Sec. 3), shows that it contains the same proportion of members of $\mathcal{B}$.

Corollary 1.2. *Suppose that the assumptions of Theorem 1.1 are satisfied and $\alpha > 1$. Then*

$$|\{n \in \mathbb{N} : \lfloor \alpha n \rfloor \in \mathcal{B}(x)\}| \sim \frac{1}{\alpha} B(x) \quad (x \to \infty). \tag{2}$$

If $I(n) = [1, \theta(n)]$, the following assumption implies (1), by Lemma 4.1: There exist constants $C, J, K \geq 1$ and $0 \leq \eta < 1$, such that for all $m, n \in \mathbb{N}$,

$$\theta(n) \leq \theta(mn) \leq C m^J \theta(n), \quad \max(2, n) \leq \theta(n) \leq Kn \exp((\log n)^\eta). \tag{3}$$

Corollary 1.3. *Let α be irrational. Assume $I(n) = [1, \theta(n)]$ and (3) holds. Then the sequence $(\alpha n)_{n \in \mathcal{B}}$ is u.d.* mod1*. If $\alpha > 1$, then (2) holds.*

Table 1. Examples of $\mathcal{B}$ with $(\alpha n)_{n \in \mathcal{B}}$ u.d. mod1. It's easy to verify (1) for the first two examples, and for the third with [9, Thm. 2]. For the others it follows from Corollary 1.3.

Sequence $\mathcal{B}$	$I(n)$	OEIS [6]
almost prime	see above	A037143
squarefree	$(P(n), \infty)$	A005117
lexicographical	$\{P(n)\} \cup (n, \infty)$	A361232
t-dense ($t \geq 2$)	$[1, nt]$	A174973 ($t = 2$)
practical	$[1, \sigma(n) + 1]$	A005153
practical & φ-practical	$[1, n + 1]$	A359420
inverse prime numbers	$[1, \max(2, n)]$	A047836

Corollary 1.4. *Let α be irrational. If $\mathcal{B}$ is one of the sequences in Table 1, then the sequence $(\alpha n)_{n \in \mathcal{B}}$ is u.d.* mod1*. If $\alpha > 1$, then (2) holds.*

The proof of Theorem 1.1 is divided into two cases. If α is very close to a rational number a/q with a small denominator q (relative to x), we say α belongs to the *major arcs*. If not, we say α belongs to the *minor arcs*.

Theorem 1.5 (Major Arcs)**.** *Let $g(x) = \log x \log\log x$. There exists a constant $c > 0$ such that the following holds. Let $U > 0$ be fixed. For $q \leq (\log x)^U$, $\alpha = a/q + \beta$, where $(a, q) = 1$ and $|\beta| \leq \exp\big(c\sqrt[3]{g(x)}\big)/x$, we have*

$$\sum_{n \in \mathcal{B}(x)} e(\alpha n) = \sum_{n \in \mathcal{B}(x)} \frac{\mu(q/(n,q))}{\varphi(q/(n,q))} e(\beta n) + O\Big(x \exp\big(-2c\sqrt[3]{g(x)}\big)\Big), \tag{4}$$

where $(n, q) = \gcd(n, q)$, μ is the Möbius function and φ is Euler's totient function.

If, in addition,

$$B(x) \gg x \exp\left(-c\sqrt[3]{g(x)}\right) \quad \textit{and} \quad B_d(x) \ll d^{-\delta} B(x)$$

for $x, d \geq 2$ *and some constants* $0 < \varepsilon < \delta \leq 1$*, then*

$$\sum_{n \in \mathcal{B}(x)} e(\alpha n) \ll \frac{B(x)}{q^{\delta-\varepsilon}}.$$

Theorem 1.6 (Minor Arcs)**.** *Let* $\kappa < 1/\sqrt{6}$ *be a constant. Let* $L = \log x$ *and*

$$1 \leq R \leq \exp(\kappa\sqrt{\log x \log\log x}).$$

Then, for

$$h, a, q \in \mathbb{N}, \quad h \leq R, \quad R^{12}L^{26} < q \leq \frac{x}{R^{11}L^{26}}$$

and

$$|\alpha - a/q| \leq 1/q^2, \quad \textit{where} \quad (a, q) = 1,$$

we have

$$\sum_{n \in \mathcal{B}(x)} e(\alpha h n) \ll \frac{x}{R}.$$

Theorem 1.6 leads to an estimate for the discrepancy of the sequence $(\{\alpha n\})_{n \in \mathcal{B}}$, assuming the denominators of the continued fraction convergents a_j/q_j of α do not grow too quickly. The irrationality measure of an irrational number α can be defined as $\mu(\alpha) = 1 + \limsup_{j \geq 1} \frac{\log q_{j+1}}{\log q_j}$. Every algebraic irrational number α satisfies $\mu(\alpha) = 2$, as well as $\mu(e) = 2$, while the constants $\pi, \pi^2, \log 2, \log 3$ and $\zeta(3)$ are all known to have finite irrationality measure [15]. For $0 \leq y \leq 1$, define

$$B(x, y, \alpha) := |\{n \in \mathcal{B}(x) : \{\alpha n\} \leq y\}|.$$

The following estimate for the discrepancy follows from Theorem 1.6 and the Erdős-Turán inequality (see Lemma 9.1).

Theorem 1.7. *Let* $\kappa < \frac{1}{\sqrt{6}}$ *and assume* α *is irrational with finite irrationality measure. Then, for all* $x \geq x_0(\alpha)$*,*

$$\sup_{0 \leq y \leq 1} |B(x, y, \alpha) - yB(x)| \ll \frac{x}{\exp(\kappa\sqrt{\log x \log\log x})}.$$

In particular, the sequence $(\alpha n)_{n \in \mathcal{B}}$ *is u.d.* mod 1 *provided* $B(x)$ *satisfies* $B(x) \gg x \exp(-\kappa\sqrt{\log x \log\log x})$ *for some* $\kappa < \frac{1}{\sqrt{6}}$*.*

2. Derivation of Theorem 1.1 from Theorems 1.5 and 1.6

By Weyl's criterion, we need to show that

$$T := \frac{1}{B(x)} \sum_{n \in \mathcal{B}(x)} e(l\alpha n) \to 0 \quad (x \to \infty)$$

for every fixed non-zero integer l. It suffices to consider $l = 1$, since the following is equally valid if α is replaced by $l\alpha$. Let $(a_j/q_j)_{j \geq 1}$ be the sequence of continued fraction convergents of α. Then $q_1 < q_2 < \ldots$ and

$$|\alpha - a_j/q_j| \leq \frac{1}{q_j q_{j+1}} \quad (j \geq 1).$$

Let x be sufficiently large and let $R = (\log x)^{A+1}$ and $h = 1$ in Theorem 1.6. Define

$$Q = R^{12} L^{26} = L^{12A+38}, \quad \text{where} \quad L = \log x.$$

If there is at least one j such that $Q < q_j \leq x/Q$, Theorem 1.6 (with $q = q_j$) shows that we have

$$T \ll x(\log x)^{-A-1}/B(x) \ll 1/\log x.$$

If there does not exist a j such that $Q < q_j \leq x/Q$, let i be such that $q_i \leq Q$ and $q_{i+1} > x/Q$. We have

$$|\alpha - a_i/q_i| \leq 1/(q_i q_{i+1}) \leq 1/q_{i+1} < Q/x.$$

Theorem 1.5, with $U = 12A + 38$ and $q = q_i$, shows that $T \ll q_i^{-\delta/2}$. As $x \to \infty$, $i \to \infty$ and $q_i \to \infty$. Thus $T \to 0$ as $x \to \infty$.

3. Derivation of Corollary 1.2 from Theorem 1.1

Let $\lambda = \frac{1}{\alpha} < 1$. Since $(\lambda m)_{m \in \mathcal{B}}$ is u.d. mod1, we have

$$\begin{aligned}
\frac{1}{\alpha} B(x) = \lambda B(x) &\sim |\{m \in \mathcal{B}(x) : 1 - \lambda < \{\lambda m\} < 1\}| \\
&= |\{m \in \mathcal{B}(x) : \exists n \in \mathbb{N}, \lambda m < n < \lambda m + \lambda\}| \\
&= |\{m \in \mathcal{B}(x) : \exists n \in \mathbb{N}, m = \lfloor \alpha n \rfloor\}| \\
&= |\{n \in \mathbb{N} : \lfloor \alpha n \rfloor \in \mathcal{B}(x)\}|.
\end{aligned}$$

4. Derivation of Corollary 1.3 from Theorem 1.1

Lemma 4.1. *Assume θ satisfies* (3) *and $I(n)=[1,\theta(n)]$. For $x,d\ge 2$, we have*

$$B(x)=\frac{c_\theta x}{\log x}\left(1+O\left(\frac{1}{(\log x)^{1-\eta}}\right)\right),\qquad B_d(x)\ll B(x)\frac{\log 2d}{d},$$

where $c_\theta>0$.

P r o o f. The first claim follows from [14, Thm. 4]. Lemma 8 of [13] shows that $\{m : md\in\mathcal{B}_\theta\}\subset\mathcal{B}_{\theta_d}$ where $\theta_d(n)\le\theta(dn)$ for all $n\ge 1$. Since $\theta(dn)\le Cd^J\theta(n)$ for some constants C,J, by (3),

$$\{m\le x/d : md\in\mathcal{B}_\theta\}\subset\{m\le x/d : m\in\mathcal{B}_{Cd^J\theta}\}.$$

The result now follows from [8, Prop. 1]. □

5. Lemmas for the major arcs

Let $P(n)$ denote the largest prime factor of $n\ge 2$ and put $P(1)=1$. The following estimate follows from [1, Eqs. (1.3), (1.4), (1.5)].

Lemma 5.1 (de Bruijn). *For $y\ge(\log x)^2$ and $u=\frac{\log x}{\log y}\ge 1$, we have*

$$\Psi(x,y) := \sum_{\substack{n\le x\\ P(n)\le y}} 1\ll x\exp(-u\log u).$$

A key ingredient for the major arcs is the Siegel-Walfisz theorem (see [5, Thm. 6.9 and Cor. 11.21]).

Lemma 5.2 (Siegel-Walfisz). *There exists a constant $c_1>0$ such that the following holds. Let $A>0$. For $q\le(\log x)^A$ and $(a,q)=1$,*

$$\pi(x,a,q) := \sum_{\substack{p\le x\\ p\equiv a \bmod q}} 1 = \frac{\pi(x)}{\varphi(q)}+O_A(xe^{-c_1\sqrt{\log x}}),$$

where $\pi(x):=\pi(x,0,1)$ and φ is Euler's totient function.

Lemma 5.3 (Ramanujan's Sum). *Let $a,q,d\in\mathbb{N}$ with $(a,q)=1$ and $d|q$. Then*

$$\sum_{\substack{n=1\\ (n,q)=d}}^{q} e(na/q)=\mu(q/d),$$

where μ is the Möbius function.

P r o o f. Writing $n' = n/d$ and $q' = q/d$ we have $(a, q') = 1$ and

$$\sum_{\substack{n=1\\(n,q)=d}}^{q} e(na/q) \;=\; \sum_{\substack{n'=1\\(n',q')=1}}^{q'} e(n'a/q') = \mu(q') = \mu(q/d).$$

The last sum is called Ramanujan's sum and is evaluated in [5, Thm. 4.1]. $\square$

6. Proof of Theorem 1.5

If $n \in \mathcal{B}$, $n > 1$, we write $n = mp$ where $p = P(n)$ and $m \in \mathcal{B}$. We have

$$f(\alpha) \;:=\sum_{n\in\mathcal{B}(x)} e(\alpha n) \;=\; e(\alpha) + \sum_{m\in\mathcal{B}(x)} \;\sum_{p\in J(x,m)} e(\alpha pm)$$

where $J(x, m)$ is the (possibly empty) union of a bounded number of intervals

$$J(x, m) := \{y \in \mathbb{R} : P(m) \le y \le x/m\} \cap I(m).$$

Define $g(x) := \log x \log\log x$. If E_0 denotes the contribution to $f(\alpha)$ from primes $p \le \exp\big(g(x)^{2/3}\big)$, then

$$|E_0| \le \Psi\Big(x, \exp\big(g(x)^{2/3}\big)\Big) \ll x \exp\big(-g(x)^{1/3}/4\big),$$

by Lemma 5.1. Thus we may replace $J(x, m)$ by

$$\tilde{J}(x, m) := J(x, m) \cap \Big(\exp\big(g(x)^{2/3}\big), \infty\Big).$$

Note that if $mp \equiv b \bmod q$ where $(b, q) = d$ and the prime p satisfies $p > q$, then $(m, q) = d$. In this case we write $q' = q/d$, $b' = b/d$ and $m' = m/d$. For r with $(r, q) = 1$, let $\bar{r}$ be such that $\bar{r}r \equiv 1 \bmod q$.

We have

$$\begin{aligned} f(a/q+\beta) &= E_0 + \sum_{d\mid q} \sum_{\substack{b=1\\(b,q)=d}}^{q} \sum_{\substack{m\in\mathcal{B}(x)\\(m,q)=d}} \sum_{\substack{p\in\tilde{J}(x,m)\\mp\equiv b \bmod q}} e\Big(\big(\tfrac{a}{q}+\beta\big)pm\Big) \\ &= E_0 + \sum_{d\mid q} \sum_{\substack{b=1\\(b,q)=d}}^{q} e\big(\tfrac{a}{q}\,b\big) \sum_{\substack{m\in\mathcal{B}(x)\\(m,q)=d}} \sum_{\substack{p\in\tilde{J}(x,m)\\p\equiv b'\overline{m'} \bmod q'}} e(\beta pm). \end{aligned}$$

Since $p \ge \exp\big(g(x)^{2/3}\big)$, we have $q \le (\log x)^U \ll (\log p)^{3U/2}$. By the Siegel-Walfisz theorem, in the form of Lemma 5.2, the sum over p is

$$\int_{\rho\in\tilde{J}(x,m)} e(\beta\rho m) d\big(\pi(\rho, b'\overline{m'}, q')\big) = \int_{\rho\in\tilde{J}(x,m)} e(\beta\rho m) d\big(\pi(\rho)/\varphi(q') + E_1\big),$$

where $E_1 \ll \rho e^{-c_1\sqrt{\log \rho}}$. Let $0 < c < c_1/3$. Integration by parts applied to the error term shows that

$$\sum_{\substack{p\in \tilde J(x,m)\\ p\equiv b'\overline{m'} \bmod q'}} e(\beta pm) \;=\; \frac{1}{\varphi(q')} \sum_{p\in \tilde J(x,m)} e(\beta pm) + E_2,$$

where

$$E_2 \ll \frac{x}{m} e^{-c_1\sqrt{\log x/m}} + |\beta| x \int_{\tilde J(x,m)} e^{-c_1\sqrt{\log \rho}} d\rho =: E_3 + E_4,$$

say. Since $m \le x\exp(-g(x)^{2/3})$ if $\tilde J(x,m)$ is non-empty, E_3 contributes

$$\ll q \sum_{m\le x} \frac{x}{m} \exp\bigl(-c_1 \sqrt[3]{g(x)}\bigr) \ll x \exp\bigl(-2c \sqrt[3]{g(x)}\bigr) =: E_5.$$

The contribution from E_4 is

$$\ll q|\beta|x \sum_{m\le x} \int_{\tilde J(x,m)} e^{-c_1\sqrt{\log \rho}} d\rho \;\le\; q|\beta| x \sum_{m\le x} \frac{x}{m} \exp\bigl(-c_1 \sqrt[3]{g(x)}\bigr) \;\ll\; E_5,$$

since $|\beta| x \le \exp\bigl(c\sqrt[3]{g(x)}\bigr)$ and $c < c_1/3$. Thus

$$f(a/q+\beta) = \sum_{d|q} \sum_{\substack{b=1\\(b,q)=d}}^{q} e(\tfrac{a}{q}b) \sum_{\substack{m\in\mathcal{B}(x)\\(m,q)=d}} \frac{1}{\varphi(q/d)} \sum_{p\in \tilde J(x,m)} e(\beta pm) + O(E_5).$$

The two inner sums are now independent of b. Thus we can evaluate the sum over b with Lemma 5.3. We combine the two inner sums, writing $n = mp$, and put back the contribution from $P(n) \le \exp\bigl(g(x)^{2/3}\bigr)$, with an error $\ll E_5$. Thus

$$f(a/q+\beta) = \sum_{d|q} \frac{\mu(q/d)}{\varphi(q/d)} \sum_{\substack{n\in\mathcal{B}(x)\\(n,q)=d}} e(\beta n) + O(E_5),$$

which is (4). Since $|\mu(q/d)| \le 1$ and $|e(\beta n)| \le 1$, the modulus of the main term is

$$\le \sum_{d|q} \frac{1}{\varphi(q/d)} \sum_{\substack{n\in\mathcal{B}(x)\\(n,q)=d}} 1 \le \frac{1}{\varphi(q)} \sum_{d|q} dB_d(x).$$

With the assumption $B_d(x) \ll d^{-\delta} B(x)$, where $0<\varepsilon<\delta\le 1$, this is

$$\ll \frac{B(x)}{\varphi(q)} \sum_{d|q} d^{1-\delta} \ll \frac{B(x)}{\varphi(q)} q^{1-\delta} \sum_{d|q} 1 \ll \frac{B(x)}{q^{\delta-\varepsilon}}.$$

7. Lemmas for the minor arcs

Lemma 7.1. *Assume $|a_n|, |b_n| \le 1$ for $n \ge 1$ and $h, a, q \in \mathbb{N}$. For $|\alpha - a/q| \le 1/q^2$ with $(a, q) = 1$, we have*

$$\sum_{\substack{n>N \\ mn\le x}} \sum_{m>M} a_n b_m e(\alpha hnm) \ll \left(\frac{hx}{M} + \frac{x}{N} + \frac{hx}{q} + q\right)^{\frac{1}{2}} (hx)^{\frac{1}{2}} (\log 2hx)^2.$$

Proof. If $h = 1$, this is [3, Lemma 13.8]. The general case follows from this case with $n' = hn$, $x' = hx$, $N' = hN$, $a'_{n'} = a_{n'/h}$ if $h|n'$, and $a'_{n'} = 0$ otherwise. □

Lemma 7.2. *Let $h, m, a, q \in \mathbb{N}$. For $|\alpha - a/q| \le 1/q^2$ with $(a, q) = 1$, we have*

$$\sum_{p\le x/m} e(\alpha hmp) \ll (\log 2hx)^7 \left(hxq^{-\frac{1}{2}} + (h^2mx)^{\frac{4}{5}} + (hxq)^{\frac{1}{2}}\right).$$

Proof. This follows from [10, Thm. 1], with the substitutions

$$H = hm, \quad N = x/m, \quad D \le H = hm.$$

The factor $\log p$ in that result can be removed with partial summation. □

8. Proof of Theorem 1.6

If $n \in \mathcal{B}$, $n > 1$, we write $n = mp$ where $p = P(n)$ and $m \in \mathcal{B}$. We have

$$\sum_{n\in\mathcal{B}(x)} e(\alpha hn) = e(\alpha h) + \sum_{m\in\mathcal{B}(x)} \sum_{\substack{P(m)\le p\le x/m \\ p\in I(m)}} e(\alpha hmp).$$

We write $L := \log x$. Lemma 5.1 shows that when

$$R = \exp(\kappa\sqrt{\log x \log\log x}), \quad \kappa < 1/\sqrt{6},$$

then

$$\Psi(x, R^3L^6) \ll x/R.$$

This upper bound must also hold for any smaller value of R. Thus the contribution from $p \le R^3L^6$ is $\ll x/R$.

By Lemma 7.2, the contribution from $m \le R^4L^6$ is

$$\le \sum_{m \le R^4 L^6} \Bigg| \sum_{\substack{P(m) \le p \le x/m \\ p \in I(m)}} e(\alpha h m p) \Bigg|$$

$$\ll \; R^4 L^6 L^7 \left(hxq^{-\frac{1}{2}} + x^{\frac{4}{5}+\varepsilon} + (hxq)^{\frac{1}{2}} \right) \; \ll \; \frac{x}{R},$$

provided

$$h \le R \quad \text{and} \quad R^{12}L^{26} < q \le \frac{x}{R^{11}L^{26}}.$$

Let $M = R^4L^6$, $N = R^3L^6$, and a_m (resp. b_n) be the characteristic functions of $\mathcal{B}$ (resp. of the primes). It remains to estimate

$$S := \sum_{m > M} \sum_{\substack{n > N,\, n \in \tilde{I}(m) \\ mn \le x}} a_m b_n e(\alpha h m n),$$

where $\tilde{I}(m) = I(m) \cap [P(m), \infty)$. Applying a strategy called *cosmetic surgery* in [2, Sec. 3.2], we can remove the condition $n \in \tilde{I}(m)$ at the expense of a factor $\ll \log x$. This leads to

$$S \ll (\log x) \Bigg| \sum_{\substack{m > M \\ mn \le x}} \sum_{n > N} \tilde{a}_m \tilde{b}_n e(\alpha h m n) \Bigg| + O(1),$$

for suitable $\tilde{a}_n$, $\tilde{b}_n$ with $|\tilde{a}_m|, |\tilde{b}_n| \le 1$. An application of Lemma 7.1 with $R^4L^6 < q \le \frac{x}{R^3L^6}$ and $h \le R$ yields $S \ll x/R$.

9. Proof of Theorem 1.7

For the following upper bound see [4, Thm. 2.5 of Ch. 2].

Lemma 9.1 (Erdős-Turán)**.** *With the notation of Theorem 1.7 and $m \in \mathbb{N}$ we have*

$$\sup_{0 \le y \le 1} |B(x, y, \alpha) - yB(x)| \le \frac{6B(x)}{m+1} + \frac{4}{\pi} \sum_{h=1}^{m} \left(\frac{1}{h} - \frac{1}{m+1} \right) \Bigg| \sum_{n \in \mathcal{B}(x)} e(\alpha h n) \Bigg|.$$

Proof of Theorem 1.7. Let $\kappa < 1/\sqrt{6}$ and pick $\eta > 0$ such that $\kappa + \eta < 1/\sqrt{6}$. Let $R = \exp\big((\kappa+\eta)\sqrt{\log x \log\log x}\big)$. Since α has finite irrationality measure μ, the continued fraction convergents a_j/q_j of α satisfy $q_{j+1} \ll_\alpha q_j^{\mu}$.

It follows that for all $x \geq x_0(\alpha)$, there is a q_j satisfying the condition

$$Q < q_j \leq \frac{x}{Q} \quad \text{where} \quad Q := R^{12}(\log x)^{26}.$$

Theorem 1.6 shows that for $h \leq R$ and $|\alpha - a_j/q_j| \leq 1/q_j^2$,

$$\sum_{n \in \mathcal{B}(x)} e(\alpha h n) \ll x/R.$$

The result now follows from Lemma 9.1 with $m = \lfloor R \rfloor$. □

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