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A MODEL STUDYING THE RELATIONSHIP BETWEEN LAKE WATER SALINITY AND WATER LEVEL FLUCTUATIONS

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ABSTRACT. We study a model represented by a system of ordinary differential equations describing fluctuations in the level of a saltwater lake, for example the Caspian Sea, depending on its salinity. The search for an analytical solution using the Laplace transform is discussed, and a numerical method for finding a variety of solutions in the neighborhood of the singular point of the studied model is also shown. The properties of the solutions obtained by simulation for some values of the coefficients of the studied system are discussed. This dependence is also demonstrated by numerically constructed solutions in the neighborhood the singular point of the studied model.

1. Introduction

The Caspian Sea is the largest closed lake on the planet, it is quite well studied, but many mysteries remain in its regime. A peculiarity of the Caspian Sea is the constant fluctuation of the salinity level of its water. The normal concentration of salt in sea water is about 35 ‰, but in the Caspian Sea the average salinity is about 12.8 ‰, while it is very variable and different in different parts of the sea, e.g., in the Kara-Bogaz-Gol bay it is even 300 ‰. One of the most striking phenomena of the Caspian Sea is the periodic variability of its level, which attracts the attention of not only scientists (see [1]). When sea level fell in the 1930s and 1970s, most researchers predicted further decline, and then when sea level began to rise, most predicted near-linear and even accelerating sea level rise. Although the predictions were based on completely different approaches, there was not a single reliable prediction (see [2]).

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The aim of the presented work is not to comprehensively predict the fluctuation of the level of the Caspian Sea, it is a very demanding task and requires close cooperation with experts from the different fields (see [3]). We focus on the fluctuation of the sea water level in relation to the concentration of salt in it, and this relation is studied through a system of differential equations. Salt concentration is an important parameter and plays a fundamental role in forecasting the water level, for example, in the Caspian Sea (see [4]). Our goal is to create a simple model that describes the important relationship between sea level salt concentration and sea level change. Then using a numerical method to find a solution and using it and its visualization to predict the change in the size of the sea coast depending on the concentration of salts and maybe vice versa.

2. Deterministic model and its solution

In this Section, we first create the simplest model of a deterministic dynamical system, then we find a solution to the created system and simulate the solutions of several model problems to confirm the interdependence of the variables in the system.

2.1. Creation of the mathematical model

We choose any point at sea level and place the origin of the coordinate system there. Some point M on the coast then has coordinates $[x_0, y_0]$, see Fig. 1. If the sea coast line changes, the point M will move and get coordinates $[x_1, y_1]$, while the change in coordinate in the x-axis direction is directly proportional to the change in the y-axis direction and vice versa. Let us denote the constant of proportionality by a. It is obvious that $a > 0$.

Consider also that sea water salinity r, expressed in parts per mille, changes periodically over time. If we compare the change in the values of x and y with the change in salinity r, it follows from long-term observations that water salinity is different in different parts of the sea, and while it can increase in one direction and decrease in the other. We denote the constant of proportionality by $b > 0$ and our reasoning can be written as follows

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}r} = \left(a - \dfrac{b}{r}\right) y, \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}r} = \left(a + \dfrac{b}{r}\right) x. \end{cases} \tag{1}$$

System of differential equations (1) expresses the change in the sea coast depending on the change in its salinity. It is obvious that for very high values of water salinity $r \gg 1$ the ratio b/r will be very small and the solutions to the system (1) have similar properties to the solutions to the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y},$$

for which the point $O = [0, 0]$ is an unstable equilibrium point. Depending on the initial values, the solutions behave differently, while in one direction $|x|$ and $|y|$ go to zero. That means the sea is dead.

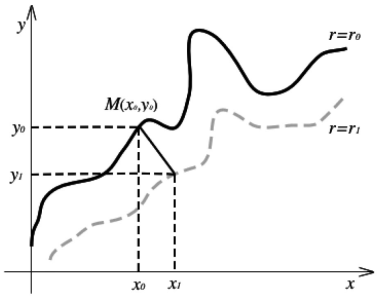

FIGURE 1. The origin of the coordinate system in some point on the sea level. When the salinity of the water changes, the sea coast (solid line) shifts (dashed line).

On the other hand, the value $r = 0$ is a singular value of the system, and it is near this point that we are interested in the properties of the solution. Therefore, when investigating the properties of the solution, we will consider the neighborhood of the point $r = 0$, i.e., small values of $r \in (0, 1]$.

2.2. Solution to system of differential equations

System (1) is a linear system, but its coefficients are not constant, so known solution methods are not applicable. To determine the solution to system (1), we use the Laplace transformation method.

The use of the Laplace transform will be more convenient if we first modify the system (1) into a different form through substitution

$$r = \mathrm{e}^{-t}, \qquad t \in (0, +\infty), \tag{2}$$

so, the values of $r = 0$ will be in the interval $(0, 1)$. Then we get

$$\frac{\mathrm{d}x}{\mathrm{d}r} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r} = -\frac{1}{r}\frac{\mathrm{d}x}{\mathrm{d}t}$$

then system (1) takes the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -\left(a\mathrm{e}^{-t} - b\right) y, \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = -\left(a\mathrm{e}^{-t} + b\right) x. \end{cases} \tag{3}$$

We denote the Laplace transforms of the unknown functions $x(t)$ and $y(t)$ by $\mathcal{L}\{x(t)\} = f_1(p)$, and $\mathcal{L}\{y(t)\} = f_2(p)$, the initial conditions by $x(0) = x_0$, $y(0) = y_0$, and by applying the Laplace transformation to system (3) we arrive at the system of difference equations

$$pf_1(p) - bf_2(p) = x_0 - af_2(p+1),$$

$$bf_1(p) + pf_2(p) = y_0 - af_1(p+1),$$

or, in the matrix form

$$\begin{pmatrix} p & -b \\ b & p \end{pmatrix} \begin{pmatrix} f_1(p) \\ f_2(p) \end{pmatrix} = -a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_1(p+1) \\ f_2(p+1) \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

If we denote

$$L(p) = \begin{pmatrix} p & -b \\ b & p \end{pmatrix}^{-1} = \frac{1}{p^2 + b^2} \begin{pmatrix} p & b \\ -b & p \end{pmatrix},$$

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \qquad Z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \qquad A = -a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then we obtain the matrix equation

$$F(p) = L(p)Z_0 + L(p)AF(p+1), \tag{4}$$

as a recurrence relation for the vector sequence F of a complex variable p. We determine F using the method of successive approximations, the initial value of which is as follows:

$$F_0(p) = L(p)\, Z_0,$$

and further successive approximations are given by the relationship:

$$F_n(p) = F_{n-1}(p) + L(p) \left[\prod_{k=1}^{n} A\, L(p+k) \right] Z_0, \ n = 1, 2, \dots$$

Using them, we obtain the solution to system (4) in the form

$$\begin{aligned} F(p) = L(p)\, Z_0 + L(p)AL(p+1)\, Z_0 + L(p)AL(p+1)AL(p+2)\, Z_0 \\ + L(p)AL(p+1)AL(p+2)AL(p+3)\, Z_0 + \cdots \end{aligned} \tag{5}$$

The original to the Laplace transform $F(p)$ thus obtained, i.e. to the function that has infinitely many isolated poles $p_1, p_2, \dots, p_j, \dots$ is determined using the residue theorem (see [5] pp.48-50), according to which

$$Z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \sum_{j=1}^{+\infty} \operatorname*{Res}_{p=p_j} \left(F(p) \mathrm{e}^{pt} \right). \tag{6}$$

The poles of the function F are poles of the first order, and at the same time they are singularity points of the matrices $L(p+k)$, $k \in \mathbb{N}_0$, i.e., such points at which $\det L(p+k) = 0$. Therefore, we can determine them by the simple calculation

$$\det L(p+k) = \begin{vmatrix} p+k & b \\ -b & p+k \end{vmatrix} = (p+k)^2 + b^2 = 0,$$

from where we obtain $p = -k \pm i\,b,\ k = 0, 1, 2, \ldots$

First, we determine the residues for $k = 0$ according to [5, p. 52]. We obtain

$$\begin{aligned}
\operatorname*{Res}_{p=ib} F(p) &= \lim_{p\to ib} (p - ib)F(p) \\
&= \lim_{p\to ib} (p - ib)L(p)\,[E + AL(p+1) + AL(p+1)AL(p+2) + \cdots]\ Z_0 \\
&= \lim_{p\to ib} \frac{p-ib}{p^2+b^2} \begin{pmatrix} p & b \\ -b & p \end{pmatrix} \\
&\qquad\qquad \times \lim_{p\to ib} [E + AL(p+1) + AL(p+1)AL(p+2) + \cdots]\ Z_0 \\
&= \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix} [E + AL(ib+1) + AL(ib+1)AL(ib+2) + \cdots]\ Z_0,
\end{aligned}$$

and analogously

$$\begin{aligned}
\operatorname*{Res}_{\bar p=-ib} F(p) &= \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix} [E + AL(-ib+1) + AL(-ib+1)AL(-ib+2) + \cdots]\ Z_0 \\
&= \overline{\operatorname*{Res}_{p=ib} F(p)}.
\end{aligned}$$

Similarly, it is possible to determine the residues if $k \neq 0$. To simplify calculations, it is convenient to use the relationship between the residues of the Laplace transform $F(p)$ at two singular points whose difference is equal to 1, i.e., at the points $p = -k + ib$ and $p + 1 = -(k-1) + ib$, $k = 0, 1, 2, \ldots$ Using relation (4) after modifications we obtain

$$\begin{aligned}
\operatorname*{Res}_{p=-k+ib} F(p) &= \operatorname*{Res}_{p=-k+ib} [L(p)Z_0 + L(p)AF(p+1)] \\
&= \underbrace{\operatorname*{Res}_{p=-k+ib} [L(p)Z_0]}_{=0} + \operatorname*{Res}_{p=-k+ib} [L(p)AF(p+1)] \\
&= L(-k+ib)A \operatorname*{Res}_{p=-k+ib} F(p+1) \\
&= L(-k+ib)A \operatorname*{Res}_{p=-(k-1)+ib} F(p).
\end{aligned}$$

To calculate the residues $F(p)$ at the conjugate points $p = -k + ib$, $\bar p = -k - ib$, $k = 1, 2, \ldots$, an analogous relationship can be used as for $k = 0$:

$$\operatorname*{Res}_{\bar p=-k-ib} F(p) = \overline{\operatorname*{Res}_{p=-k+ib} F(p)}.$$

It is obvious that the analytical expression of the solution in this way is possible, but it requires a lot of operations. Using the notation

$$\operatorname*{Res}_{p=p_k} F(p) = U_k + iV_k,$$

the original $Z(t)$ to the Laplace transform $F(p)$, in the sense of (6), can be written in the form

$$Z(t) = \sum_{k=0}^{+\infty} \operatorname*{Res}_{p=p_k} \big(F(p)\mathrm{e}^{pt}\big) = \sum_{k=0}^{+\infty} (U_k + iV_k)\,\mathrm{e}^{p_k t}.$$

If we return to the original variable r from relation (2), given

$$\mathrm{e}^{p_k t} = \mathrm{e}^{(-k \pm i\, b)t} = r^k \mathrm{e}^{\pm i\, b(-\ln r)} = r^k\big(\cos(b\ln r) \pm i\sin(-b\ln r)\big),$$

the original $Z(r)$ to the Laplace transform $F(p)$ can be obtained in the form

$$Z(r) = \left[2\cos(b\ln r)\sum_k U_k r^k + 2\sin(b\ln r)\sum_k V_k r^k\right] Z_0, \quad r \in (0,1]. \tag{7}$$

To express the solution to system (1) in the form (7) it is necessary to determine the initial condition Z_0 for $r = 1$, i.e. the values of $x_0 = x(1)$ and $y_0 = y(1)$, which we determine numerically.

For this purpose, if we take $u = x + y,\ v = x - y$, system (1) is transformed into a system of new variables

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}r} = \quad au + \dfrac{b}{r}v \\[2ex] \dfrac{\mathrm{d}v}{\mathrm{d}r} = -av - \dfrac{b}{r}u. \end{cases} \tag{8}$$

We seek a solution to system (8) in the form

$$u(t) = K(t)v(t). \tag{9}$$

After substitution (9) into system (8), we get a nonlinear differential equation with an unknown function K,

$$\frac{\mathrm{d}K(t)}{\mathrm{d}t} = -2a\mathrm{e}^{-t}K(t) - b\left(1 + K^2(t)\right). \tag{10}$$

The solution to this equation can be found using the Runge-Kutta method with the initial condition $v(1) = 1$. Then

$$u(1) = K(1)v(1) = K(1)$$

and for the values $x(1), y(1)$ we have

$$x(1) = \frac{1 + K(1)}{2}, \qquad y(1) = \frac{K(1) - 1}{2}. \tag{11}$$

Finally, if we use the initial conditions (11) thus obtained in relation (7), we obtain a solution to system (1) in the right neighborhood of the singular point $r = 0$. This solution describes the behaviour of the sea water surface shape in regions of very low salt concentration and is useful for studying the differences between coastal and open waters. The considered model can be also used to predict threats to nature and human life. It can be observed that if the salt concentration increases, the sea water level decreases. As the salt concentration decreases, the changes are reversed. The results of the analysis of the solution can be used as auxiliary data for ecological monitoring in the area of some seas, especially the Caspian Sea.

2.3. Simulations of solutions to model problems

We will simulate the solutions to system (1) for different values of parameters a and b, but with the same initial conditions $x_0 = 1, y_0 = 1$. We will show how the values of the parameters can affect the character of the solution, with a significant change depending on the proportionality of the coefficients a, b of the system.

Example. First, we perform simulations for the parameter values $a = 0.1$, $b = 0.9$, i.e., the condition $b > a$ is satisfied. The solution in this case is shown in Fig. 2, with the blue part of the graph corresponding to the given parameter values. The starting point is point $x_0 = 1, y_0 = 1$, as mentioned. It can be observed that if the salt concentration r increases, then at its very small values the sea water level first increases, but after reaching a certain value of the salt concentration the sea water level starts to decrease. It is obvious that in the opposite case, when the salt concentration decreases, the sea water level increases.

Example. The colored part of the graph in Fig. 2 represents the solution to system (1) for the values of parameters $a = 0.2, b = 0.1$. These parameter values are in the opposite ratio to those in the previous case, i.e., the condition $b < a$ is satisfied, and this small change has affected the nature of the solution. The starting point is again the point $x_0 = 1, y_0 = 1$. For very small values of salt concentration r, the values of x and y increase slightly, but before they start to decrease in the previous case, and this decrease is slower. Since the parameter b denotes the coefficient of proportionality of water salinity and position in the sea, such a result is understandable.

Example. In the third case, the ratio between the parameters a and b is increased even more significantly when we consider the values $a = 2, b = 24$. The solution graph is shown in Fig. 3. With such a value of the parameter b, the starting point $x_0 = 1, y_0 = 1$ corresponds to a high salt concentration in seawater, which is outside the interval $(0, 1]$ we are monitoring. The graph shows a cyclical change in the values of x and y for all values of r.

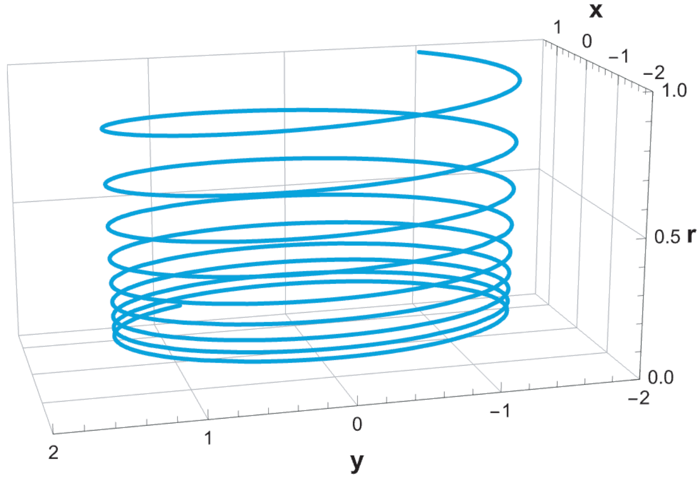

FIGURE 2. Simulations of solutions to system (1) for different values of parameters a, b. The blue part of the graph is for the values $a = 0.1, b = 0.9$ and the colored part for the values $a = 0.2, b = 0.1$. The starting point is $x_0 = 1, y_0 = 1$. In both cases, the values of x, y begin to decrease after reaching a certain value of water salinity, but the difference is in the intensity of the decrease in the water level in the sea.

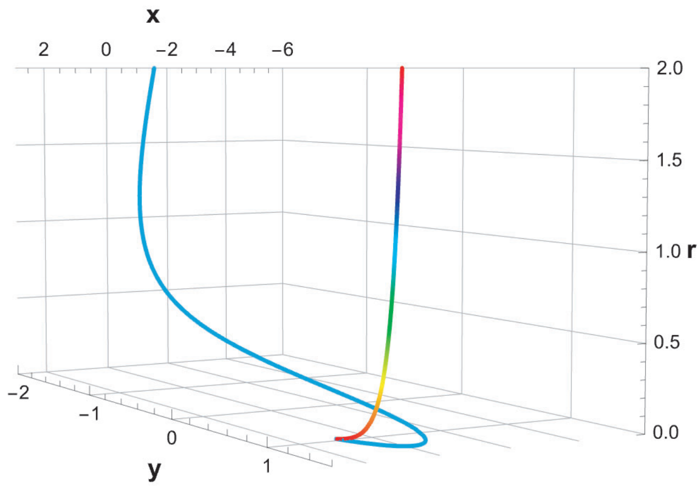

FIGURE 3. Simulations of solutions to system (1) for the parameter values $a = 2, b = 24$. A cyclical change in the values of x and y is visible. The starting point $x_0 = 1, y_0 = 1$ corresponds to a high level concentration.

3. Conclusions and open problems

Analysis of solutions of a system of differential equations near singular points is always a very delicate and difficult matter. The presented method is one of the effective approaches to obtaining an exact solution near a singular point and can be applied to various problems in this area. The considered model, although simple, can also be used to predict threats to nature and human life. It can be observed that if the salt concentration increases, the coast recedes and the sea level drops. When the salt concentration decreases, the changes are the opposite. The results of the analysis of solutions can be used as auxiliary data for monitoring the ecology of the Caspian Sea region.

This simple model makes it possible to describe, to some extent, the phenomena that have been observed in the Caspian Sea for centuries. When constructing the model, many conditions were neglected, and the parameters a, b were chosen without relation to reality. In order to make the model more accurate and more effectively used for predictive purposes, a modification of the model is necessary.

When describing natural processes in the Caspian Sea, we observe stochastic effects that have a significant impact on the amount of seawater and its salinity in different parts of the sea. For this reason, we can add random coefficients to system (1) and study the system of the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}r} = \left(a(\xi) - \dfrac{b(\xi)}{r}\right) y, \\[2mm] \dfrac{\mathrm{d}y}{\mathrm{d}r} = \left(a(\xi) + \dfrac{b(\xi)}{r}\right) x, \end{cases} \tag{12}$$

where $a(\xi), b(\xi)$ are random coefficients dependent on a Markov process ξ. Then coefficient $a(\xi)$ describes a random placement of a point in the sea and $b(\xi)$ describes distance of a point from seaboard, r is the salt concentration. its properties by numerical simulations and the so-called moment equations. The properties of the solutions can then be described using numerical methods, simulations, or by constructing moment equations to determine the first and second order moments. The derivation of moment equations and their application to selected types of systems of differential and difference equations can be found in the monograph [10]. Moment equations were used, for example, in studying the conditions of stability of the foreign exchange market under conditions of uncertainty (see [9]) and also in studying the system of discrete stochastic equations modeling the stabilization of corporate income (see [8]).

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