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BOUNDEDNESS AND REGULARITY OF SOLUTIONS FOR NONLINEAR PARABOLIC PROBLEM WITH TWO SINGULARITIES

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ABSTRACT. We examine the boundedness and regularity of weak solutions to a type of nonlinear parabolic equations with singular natural growth gradient terms. These equations also have a singular right-hand side with $f \in L^m(Q_T)$, $(m \geq 1)$. With suitable test functions, we can use Stampacchia's lemma to prove that the solutions $u(x,t)$ are bounded when $m > \frac{N}{p} + 1$. Furthermore, we can establish a regularity result if $f \in L^1(Q_T)$.

1. Introduction and The Main Result

In the present paper, we study the following nonlinear parabolic problem with double singularities

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\left(e(x,t)|\nabla u|^{p-2}\nabla u\right) + d(x,t)\frac{|\nabla u|^p}{u^\theta} = \frac{f}{u^\gamma} & \text{in } Q_T, \\ u = 0 & \text{on } \Gamma, \\ u(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $Q_T = \Omega \times (0,T)$, $(T > 0)$, Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 3$, Γ is the lateral surface $\partial\Omega \times (0,T)$ and p, θ, γ are real number such that

$$2 \leq p < N, \quad 0 < \theta < 1, \quad 0 \leq \gamma < p - 1. \tag{2}$$

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We also assume that $e(x,t)$ and $d(x,t)$ are measurable functions on Q_T that satisfy the following conditions

$$0 < \alpha \leq e(x,t) \leq \beta \quad \text{and} \quad 0 < \mu \leq d(x,t) \leq \delta, \tag{3}$$

where $\alpha, \beta, \mu, \delta$ are fixed real numbers. The function f belongs to $L^m(Q_T)$, with $m \geq 1$, and it meets the requirement

$$f \geq 0, \quad \text{and} \quad f \not\equiv 0 \quad \text{in } Q_T. \tag{4}$$

It is important to note that (4) means that f is a function that is strictly positive on every subset contained in Q_T.

Singular equations are a significant research topic in mathematics, especially in the field of partial differential equations. They are widely used to describe and analyze natural phenomena such as fluid mechanics, pseudo-plastic flow, chemical reactions, nerve impulses, population dynamics, combustion, morphogenesis, and genetics.

When the singular lower-order term does not appear in (1) (i.e., $b(t,x)=0$) and $\gamma = 0$ in the nonlinear right-hand side, the existence and regularity of solutions to problem (1) were established in [16]. If $\gamma = 0$ and $a(t,x) \equiv 1$, the existence and regularity results for nonlinear parabolic problems of type (1) were proved in [15], and when the operator is degenerate and $\gamma = 0$, problem (1) was studied in [33].

In the case of the p-Laplacian operator with $d(t,x) = B > 0$ and $\gamma = 0$, that is, in the evolutionary setting, equations of the form

$$\partial_t u - \Delta_p u + B\frac{|\nabla u|^p}{u^\theta} = f \quad \text{in } Q_T,$$

have been studied under various assumptions on the summability of the source f for $p = 2$ and $\theta < 1$ (see [22]). When $\theta = 1$, the existence of solutions was investigated in [29, 30] for smooth strictly positive data, while degenerate problems were analyzed in [32] in the one-dimensional case with $p > 2$. In [6], the authors addressed existence and nonexistence of solutions for a general class of singular homogeneous problems (i.e., $f \equiv 0$) in the critical case $\theta = 1$ and $p \geq 2$.

Furthermore, in [11] the authors investigated similar parabolic problems where the lower-order term is of the form $a(x,t)+u^q$ instead of $a(x,t)$, but without singularity in the right-hand side, i.e., f instead of f/u^γ. In [13], a related parabolic problem was considered with $a(x,t)+u^q$ and a nonlinear gradient term of the type $g(x,t,u)$ replacing $d(x,t)|\nabla u|^p/u^\theta$, together with the presence of a singular datum f/u^γ. Other related parabolic models are discussed in [9, 10, 12].

We also recall that in [7] the authors studied the existence and regularity for the problem

$$\begin{cases} \partial_t u - \Delta_p u = \frac{f}{u^\gamma} & \text{in } Q_T = (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_T = (0,T)\times\partial\Omega, \end{cases}$$

with

$$\gamma > 0, \quad p \geq 2, \quad 0 < f \in L^m(Q_T), \quad m \geq 1, \quad \text{and} \quad u_0 \in L^\infty(\Omega).$$

In the elliptic setting, the authors of [5] proved the existence of solutions to elliptic equations with degenerate coercivity and a singular right-hand side of the form

$$-\mathrm{div}\left(\frac{a(x)\nabla u}{(1+|u|)^p}\right) = \frac{f}{|u|^\gamma},$$

where $f \in L^m(\Omega)$ is a nonnegative function and $a\colon \Omega \to \mathbb{R}$ is a measurable coefficient satisfying $0 < \alpha \leq a(x) \leq \beta$. Further contributions concerning degenerate parabolic (resp. elliptic) equations with singular natural growth in the gradient and $\gamma = 0$ were obtained in [17, 24] (resp. [18–20]). For additional results in the elliptic case, we also refer to [27].

The present paper is inspired by the works [15, 23], where parabolic problems with a zero Dirichlet boundary condition and the same type of right-hand side were studied. In these cases, both the gradient term and the source term become singular as u approaches zero. The proof method consists in approximating (1) by a sequence of nonsingular problems (for $f \in L^m(Q_T), m > \frac{N}{p} + 1$). Using Schauder's fixed point theorem and solving the approximate problems, we prove that (1) admits at least one solution

$$u \in L^p\big(0,T; W_0^{1,p}(\Omega)\big) \cap L^\infty(\Omega).$$

In the case where $f \in L^1(Q_T)$, the solution

$$u \in L^\eta\big(0,T; W_0^{1,\eta}(\Omega)\big)$$

holds under the condition $\gamma < \theta$, where $\eta = N(p-\theta+\gamma)/(N-\theta+\gamma)$.

Lemma 1.1 (See [4]). *Let $M_1, \nu, \rho, \vartheta, k_0$ be real positive numbers where $\vartheta > 1$ and $\rho \in [0,1)$. Let $\Lambda\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function such that*

$$\Lambda(h) \leq \frac{M_1 k^{\nu\rho}}{(h-k)^\nu}[\Lambda(k)]^\vartheta, \quad \forall\, h > k \geq k_0.$$

Then, there exists $k^* > 0$ such that $\Lambda(k^*) = 0$.

Lemma 1.2 (See [8]). *Let $v \in L^{\kappa}\big(0,T;W_0^{1,\kappa}(\Omega)\big) \cap L^{\infty}\big(0,T;L^{\varrho}(\Omega)\big)$, $\kappa,\varrho \geq 1$. Then, v belongs to $L^q(Q)$, where $q = \kappa\frac{N+\varrho}{N}$, and there exists a positive constant M_2 depending only on N,κ,ϱ such that*

$$\int\limits_Q |v(x,t)|^q \,\mathrm{d}x\,\mathrm{d}t \leq M_2\|v\|^{\frac{\kappa}{N}}_{L^{\infty}(0,T;L^{\varrho}(\Omega))} \int\limits_Q |\nabla v(x,t)|^{\kappa} \,\mathrm{d}x\,\mathrm{d}t. \tag{5}$$

Now, let us define a weak solution to (1) in the following way.

Definition 1.3. A function $u \in L^1\big(0,T;W_0^{1,1}(\Omega)\big)$ is a weak solution to problem (1), if

$$u > 0 \text{ in } Q_T, \quad \frac{|\nabla u|^p}{u^{\theta}} \in L^1(Q_T), \quad \frac{f}{u^{\gamma}} \in L^1(Q_T)$$

and

$$-\int\limits_{Q_T} u\frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{Q_T} e(x,t)|\nabla u|^{p-2}\nabla u.\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{Q_T} d(x,t)\frac{|\nabla u|^p}{u^{\theta}}\varphi\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int\limits_{Q_T}\frac{f}{u^{\gamma}}\varphi\,\mathrm{d}x\,\mathrm{d}t, \quad \forall\varphi \in L^p\big(0,T;W_0^{1,p}(\Omega)\big) \cap L^{\infty}(Q_T). \tag{6}$$

We will now present our main finding.

Theorem 1.4. *Assume that (2) and (4) hold. Let $f \in L^m(Q_T)$ with $m > \frac{N}{p}+1$. Then, there exists a weak solution $u \in L^p\big(0,T;W_0^{1,p}(\Omega)\big) \cap L^{\infty}(Q_T)$ to problem (1) in the sense of Definition 1.3.*

Remark 1.5. If $\gamma = 0$, the result of Theorem 1.4 coincides with the classical boundedness results for non degenerate parabolic equations [15, Theorem 1].

The next result deals with the case where the data f belongs to $L^1(Q_T)$.

Theorem 1.6. *If hypotheses (3)–(4) hold, let $f \in L^1(Q_T)$ and $\gamma < \theta$. Then, there exists a weak solution $u \in L^{\eta}\big(0,T;W_0^{1,\eta}(\Omega)\big) \cap L^{\eta^*}(Q_T)$ to problem (1) in the sense of Definition 1.3, where*

$$\eta = \frac{N(p-\theta+\gamma)}{N-\theta+\gamma}. \tag{7}$$

Here, η^* is the Sobolev conjugate exponent of η, $(1 < \eta < N)$, i.e., $\eta^* = \frac{N\eta}{N-\eta}$.

2. Proof of Theorems 1.4

We recall that

$$T_j(r) = \max\{-j,\min\{j,r\}\} \quad \text{and} \quad G_j(r) = r - T_j(r) \quad \text{for every} \quad r \in \mathbb{R}.$$

2.1. Approximation problems

For all $n \in \mathbb{N}$, we set $f_n = \frac{f}{1+\frac{1}{n}f} \in L^\infty(Q_T)$. Consider the following approximated problem

$$\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}\big(e(x,t)|\nabla u_n|^{p-2}\nabla u_n\big) + d(x,t)\frac{u_n|\nabla u_n|^p}{\left(|u_n|+\frac{1}{n}\right)^{\theta+1}} = \frac{f_n}{\left(|u_n|+\frac{1}{n}\right)^{\gamma}} & \text{in } Q_T, \\ u_n = 0 & \text{on } \Gamma, \\ u_n(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{8}$$

such that

$$\begin{cases} \|f_n\|_{L^m(Q_T)} \le \|f\|_{L^m(Q_T)}, \\ f_n \to f \text{ strongly in } L^m(Q_T), \quad m \ge 1. \end{cases} \tag{9}$$

The following lemma gives the existence of solutions to the approximate problem (8).

Lemma 2.1. *Let f be in the space $L^m(Q_T)$, where $m \ge 1$. If (3) holds true. Then, the approximating problem (8) has a solution*

$$u_n \in L^p\big(0,T;W_0^{1,p}(\Omega)\big) \cap L^\infty(Q_T)$$

for every fixed $n \in \mathbb{N}$, and it satisfies the weak formulation

$$\int_{Q_T} \left\langle \frac{\partial u_n}{\partial t}, \varphi \right\rangle \mathrm{d}x\,\mathrm{d}t + \int_{Q_T} e(x,t)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_{Q_T} d(x,t)\frac{u_n|\nabla u_n|^p}{\left(|u_n|+\frac{1}{n}\right)^{\theta+1}}\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} \frac{f_n}{\left(|u_n|+\frac{1}{n}\right)^{\gamma}}\varphi\,\mathrm{d}x\,\mathrm{d}t, \tag{10}$$

for every $\varphi \in L^p\big(0,T;W_0^{1,p}(\Omega)\big) \cap L^\infty(Q_T)$.

Proof. Let $n \in \mathbb{N}$ and $v \in L^p(Q_T)$ be fixed. Consider the nonlinear parabolic problem

$$\begin{cases} \frac{\partial w}{\partial t} - \operatorname{div}\big(e(x,t)|\nabla w|^{p-2}\nabla w\big) + d(x,t)\frac{w|\nabla w|^p}{\left(|w|+\frac{1}{n}\right)^{\theta+1}} = \frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\gamma}} & \text{in } Q_T, \\ w = 0 & \text{on } \Gamma, \\ w(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{11}$$

The term on the right-hand side of (11) is in the space $L^\infty(Q_T)$. Therefore, the existence of a weak solution $w \in L^p\big(0,T;W_0^{1,p}(\Omega)\big) \cap L^\infty(Q_T)$ to (11) is demonstrated as in [25] (see also [23]). In addition, there exists a positive constant C_∞ that does not depend on v and w (although it may depend on n), such that

$$\|w\|_{L^\infty(Q_T)} \le C_\infty.$$

The map $P\colon L^p(Q_T) \to L^p(Q_T)$, with $P(v) = w$, defines a weak solution w for problem (11) associated with v. Our aim is to prove the existence of a fixed point for the map P. To start, we must prove that P is continuous and compact on the bounded set of $L^p(Q_T)$. Substituting w into the weak formulation of problem (11) yields

$$\frac{1}{2}\int_{\Omega} w(T)^2 \,\mathrm{d}x + \int_{Q_T} e(x,t)|\nabla w|^p \,\mathrm{d}x\,\mathrm{d}t \\ + \int_{Q_T} d(x,t)\frac{w^2|\nabla w|^p}{\big(|w|+\frac{1}{n}\big)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} \frac{f_n}{\big(|v|+\frac{1}{n}\big)^{\gamma}} w \,\mathrm{d}x\,\mathrm{d}t. \tag{12}$$

By (3), $\frac{f_n}{\big(|v|+\frac{1}{n}\big)^{\gamma}} \leq n^{\gamma+1}$ and dropping the first and third nonnegative term on the left-hand side in (12), we obtain

$$\alpha\int_{Q_T} |\nabla w|^p \,\mathrm{d}x\,\mathrm{d}t \leq n^{\gamma+1}\int_{Q_T} |w| \,\mathrm{d}x\,\mathrm{d}t. \tag{13}$$

Using the Hölder's inequality on the right-hand side of the above estimate, we get

$$\int_{Q_T} |\nabla w|^p \,\mathrm{d}x\,\mathrm{d}t \leq \frac{n^{\gamma+1}}{\alpha}|Q_T|^{\frac{1}{p'}}\left(\int_{Q_T} |w|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}}.$$

Using Poincaré inequality, we have

$$\|w\|_{L^p(Q_T)} \leq C(n,|Q_T|) = R.$$

We take w_k as a test function in the problem solved by w_k

$$\begin{cases} \frac{\partial w_k}{\partial t} - \operatorname{div}\big(e(x,t)|\nabla w_k|^{p-2}\nabla w_k\big) + d(x,t)\frac{w_k|\nabla w_k|^p}{\big(|w_k|+\frac{1}{n}\big)^{\theta+1}} = \frac{f_n}{\big(|v_k|+\frac{1}{n}\big)^{\gamma}} & \text{in } Q_T, \\ w_k = 0 & \text{on } \Gamma, \\ w_k(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{14}$$

where v_k is a bounded sequence in B_R (B_R is a ball of $L^p(Q_T)$ of radius R). Since the ball is invariant for P, we have w_k in B_R and (13) implies that

$$\int_0^T \|\nabla w_k\|^p_{L^p(\Omega)} \,\mathrm{d}t \leq C\int_0^T \left(\int_{\Omega} |w_k|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \mathrm{d}t, \tag{15}$$

so, we obtain that w_k is bounded in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$. Furthermore, choosing $\frac{T_h(w_k)}{h}$ as a test function (14), dropping nonnegative terms, and letting h tend to zero, Fatou's lemma yields

$$\mu \int_{Q_T} \frac{w_k |\nabla w_k|^p}{\left(|w_k| + \frac{1}{n}\right)^{\theta+1}} \, \mathrm{d}x \, \mathrm{d}t \le C. \tag{16}$$

Going back to (11), the sequence

$$\partial_t w_k \text{ is bounded in } L^{p'}\big(0,T; W^{-1,p'}(\Omega)\big) + L^1(Q_T), \quad p' = \frac{p}{p-1},$$

using compactness argument in ([28] Corollary 4), we obtain that

$$w_k \longrightarrow w \quad \text{a.e. in } Q_T. \tag{17}$$

From (17), (15) and by the Dominated Convergence Theorem, we have w_k converges strongly to w in $L^p(\Omega)$ and so, P is compact. Let $w_k = P(v_k)$, where v_k is a sequence of functions such that

$$v_k \longrightarrow v \quad \text{strongly in } \; L^p(Q_T) \text{ and, a.e in } Q_T.$$

Consequently, since $\frac{f_n}{\left(|v_k| + \frac{1}{n}\right)^{\gamma}} \le n^{\gamma} f$ and by the Dominated Convergence Theorem, it follows that

$$\frac{f_n}{\left(|v_k| + \frac{1}{n}\right)^{\gamma}} \longrightarrow \frac{f_n}{\left(|v| + \frac{1}{n}\right)^{\gamma}} \quad \text{strongly in } \; L^p(Q_T).$$

Now, let $P(v) = w$. Arguing as in Lemma 2.8 of [23], we obtain a sub-sequence of $(w_k)_k$ such that

$$\nabla w_k \longrightarrow \nabla w \quad \text{a.e. in} \quad Q_T, \tag{18}$$

and from (16), we have as $k \to +\infty$

$$\frac{w_k |\nabla w_k|^p}{\left(|w_k| + \frac{1}{n}\right)^{\theta+1}} \longrightarrow \frac{w |\nabla w|^p}{\left(|w| + \frac{1}{n}\right)^{\theta+1}}, \quad \text{strongly in } L^1(Q_T).$$

From the fact that $e(x,t)|\nabla w_k|^{p-2}\nabla w_k$ is bounded in $L^{p'}(Q_T)$ and from (18), we conclude that

$$e(x,t)|\nabla w_k|^{p-2}\nabla w_k \rightharpoonup e(x,t)|\nabla w|^{p-2}\nabla w \quad \text{weakly in } \; L^{p'}(Q_T).$$

Hence, by the uniqueness of the limit, one deduces that $w_k = P(v_k)$ converges to $w = P(v)$ in $L^p(Q_T)$. This gives the continuity of P. Next, by Schauder's fixed point theorem, for every fixed n, there exists $u_n \in L^p\big(0,T; W_0^{1,p}(\Omega)\big) \cap L^\infty(Q_T)$ such that $u_n = P(u_n)$.

Choosing $u_n^- = u_n \chi_{\{u_n \le 0\}}$, where $\chi_{\{u_n \le 0\}}$ denotes the characteristic function of $\{(x,t) \in Q_T : u_n(x,t) \le 0\}$ as a test function in (8) and using assumption (3), we have

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_0^T (u_n^-)^2\mathrm{d}t + \alpha\int_{Q_T}|\nabla u_n^-|^p\,\mathrm{d}x\,\mathrm{d}t + \mu\int_{Q_T}\frac{(u_n^-)^2|\nabla u_n|^p}{(|u_n|+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t \le \int_{Q_T}\frac{f_n u_n^-}{\left(|u_n|+\frac{1}{n}\right)^\gamma}\,\mathrm{d}x\,\mathrm{d}t.$$

Using that $\frac{f_n}{\left(|u_n|+\frac{1}{n}\right)^\gamma} > 0$ and dropping nonnegative terms, we get

$$\alpha\|u_n^-\|_{L^p(0,T;W_0^{1,p}(\Omega))} \le 0,$$

so that $u_n \ge 0$ almost everywhere in Q_T. Therefore, u_n solves

$$\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}\left(e(x,t)|\nabla u_n|^{p-2}\nabla u_n\right) + d(x,t)\frac{u_n|\nabla u_n|^p}{\left(u_n+\frac{1}{n}\right)^{\theta+1}} = \frac{f_n}{\left(u_n+\frac{1}{n}\right)^\gamma} & \text{in } Q_T,\\ u_n = 0 & \text{on } \Gamma,\\ u_n(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{19}$$

We shall denote by C various constants depending only on the structure of p, θ, γ, T, $|\Omega|$. □

2.2. A priori estimates

Let $u_n \in L^p\left(0,T;W_0^{1,p}(\Omega)\right)\cap L^\infty(Q_T)$ be a solution to problem (19). In the following Lemma, give the $L^\infty-$estimates for approximate solutions u_n.

Lemma 2.2. *Suppose that* (3)–(4) *hold true and the condition* $p-2 \le \gamma < p-1$ *is satisfied. If* $f \in L^m(Q)$ *with* $m > \frac{N}{p}+1$*, then it follows that* u_n *is bounded in* $L^\infty(Q_T)\cap L^p\left(0,T;W_0^{1,p}(\Omega)\right)$.

P r o o f. We start by examining the scenario where $p=2$ and for any $\tau \in (0,T]$, we can select $\varphi(u_n) = G_j(u_n)\chi_{(0,\tau)}$ as a test function in (19). Then, by assumption (3), we obtain

$$\begin{aligned} \int_{A_{n,j}(\tau)}\left|G_j\left(u_n(t)\right)\right|^2\mathrm{d}x + \alpha\int_0^\tau\int_{A_{n,j}(t)}|\nabla G_j(u_n)|^2\,\mathrm{d}x\,\mathrm{d}t \\ + \mu\int_0^\tau\int_{A_{n,j}(t)}\frac{u_n|\nabla u_n|^2}{\left(u_n+\frac{1}{n}\right)^{\theta+1}}\,G_j(u_n)\,\mathrm{d}x\,\mathrm{d}t \\ \le \int_0^\tau\int_{A_{n,j}(t)}\frac{f_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\,G_j(u_n)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

where

$$A_{n,j}(t) = \{x\in\Omega\colon u_n(x,t) > j\},\quad t\in(0,T).$$

Dropping the third non-negative term, we obtain

$$\int_{A_{n,j}(\tau)} \big|G_j\big(u_n(t)\big)\big|^2 \,\mathrm{d}x + \alpha\int_0^\tau \int_{A_{n,j}(t)} |\nabla G_j(u_n)|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\le \int_0^\tau \int_{A_{n,j}(t)} \frac{f_n}{\big(u_n+\frac{1}{n}\big)^\gamma}\, G_j(u_n)\,\mathrm{d}x\,\mathrm{d}t.$$

Using Hölder's inequality with exponent m in the last equality, and the fact that $u_n + \frac{1}{n} \ge j \ge 1$ on the set $A_{n,j}$, we get

$$\|G_j(u_n)\|^2_{L^\infty(0,T;L^2(A_{n,j}(t)))} + \alpha\int_0^\tau \int_{A_{n,j}(t)} |\nabla G_j(u_n)|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\le C\left(\int_0^T \int_{A_{n,j}(t)} |G_j(u_n)|^{m'} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{m'}}.$$

From this inequality, we can prove the boundedness of the sequence $\{u_n\}_n$ in $L^\infty(Q)$ in the same way as in [15, Lemma 4].

Now, let us consider $p > 2$. We take $\varphi(u_n) = [(1+u_n)^{p-1} - 1]G'_j(u_n)\chi_{(0,\tau)}$ as a test function in (19). Using assumption (3) and the fact that

$$\Phi(u_n) = \int_0^{u_n} \big((1+y)^{p-1} - 1\big)\, G'_j(y)\,\mathrm{d}y \ge \frac{1}{p} G_j(u_n)^p G'_j(u_n),$$

we obtain

$$\int_\Omega \Phi(u_n)\,\mathrm{d}x + \alpha(p-1)\int_0^\tau \int_\Omega |\nabla u_n|^p (1+u_n)^{p-2} G'_j(u_n)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \mu\int_0^\tau \int_\Omega \frac{u_n|\nabla u_n|^p}{\big(u_n+\frac{1}{n}\big)^{\theta+1}}[(1+u_n)^{p-1} - 1]G'_j(u_n)\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \int_0^\tau \int_\Omega f_n(1+u_n)^{p-1-\gamma} G'_j(u_n)\,\mathrm{d}x\,\mathrm{d}t.$$

Dropping the third non-negative term and using Hölder's inequality on the right--hand side of the previous inequality (19), and the fact that

$$1 + u_n \le 2\big(j + G_j(u_n)\big) \quad \text{as} \quad j \ge 1,$$

one has

$$\int_{A_{n,j}(\tau)} G_j\big(u_n(\tau)\big)^p \,\mathrm{d}x + \int_0^\tau \int_{A_{n,j}(t)} \frac{|\nabla u_n|^p}{(1+u_n)^{2-p}} \,\mathrm{d}x\,\mathrm{d}t \le C \left(\int_0^T \int_{A_{n,j}(t)} \big(j + G_j(u_n)\big)^{(p-1-\gamma)m'} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{m'}}.$$

Hence,

$$\|G_j(u_n)\|^p_{L^\infty(0,T;L^p(A_{n,j}(t)))} + \int_0^T \int_{A_{n,j}(t)} \frac{|\nabla u_n|^p}{(1+u_n)^{2-p}} \,\mathrm{d}x\,\mathrm{d}t \le C \left(\int_0^T \int_{A_{n,j}(t)} \big(j + G_j(u_n)\big)^{(p-1-\gamma)m'} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{m'}}. \tag{20}$$

Since $p > 2$, we write, by (20)

$$\int_0^T \int_\Omega \nabla G_j(u_n)^p \,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \int_{A_{n,j}(t)} |\nabla u_n|^p (1+u_n)^{p-2} \,\mathrm{d}x\,\mathrm{d}t \le C \left(\int_0^T \int_{A_{n,j}(t)} \big(j + G_j(u_n)\big)^{(p-1-\gamma)m'} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{m'}}.$$

From Lemma 1.2 (here $v = G_j(u_n)$, $\kappa = \varrho = p$), the previous inequality and (20) yield

$$\begin{aligned}\int_0^T \int_{A_{n,j}(t)} G_j(u_n)^{\frac{(N+p)p}{N}} \,\mathrm{d}x\,\mathrm{d}t &\le \Big(\|G_j(u_n)\|^p_{L^\infty(0,T;L^p(A_{n,j}(t)))} \Big)^{\frac{p}{N}} \int_0^T \int_{A_{n,j}(t)} |\nabla G_j(u_n)|^p \,\mathrm{d}x\,\mathrm{d}t \\ &\le C \left(\int_0^T \int_{A_{n,j}(t)} (j + G_j(u_n))^{m'(p-1-\gamma)} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{(N+p)}{Nm'}}.\end{aligned}$$

Thanks to Hölder's inequality with the exponents $\frac{(N+p)p}{Nm'(p-1-\gamma)}$, $\left(\frac{(N+p)p}{Nm'(p-1-\gamma)}\right)'$ (since $m > \frac{N}{p}+1$ and $p > p-1-\gamma$), we have for all $j \geq 1$

$$\int_0^T \int_{A_{n,j}(t)} G_j(u_n)^{\frac{(N+p)p}{N}}\,\mathrm{d}x\,\mathrm{d}t \leq C\left(\int_0^T \int_{A_{n,j}(t)} \big(j+G_j(u_n)\big)^{\frac{(N+p)p}{N}}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p-1-\gamma}{p}} \times \left(\int_0^T |A_{n,j}(t)|\,\mathrm{d}t\right)^{\frac{N+p}{Nm'}-\frac{p-1-\gamma}{p}}.$$

Therefore, by the previous inequality, we can write for all $j \geq 1$

$$\mathbf{G}_{n,j} \leq C\mathbf{G}_{n,j}^{\frac{p-1-\gamma}{p}} \Lambda_n(j)^{\frac{(N+p)}{Nm'}-\frac{p-1-\gamma}{p}} + Cj^{\frac{(N+p)p}{N}\left(\frac{p-1-\gamma}{p}\right)} \Lambda_n(j)^{\frac{(N+p)}{Nm'}}, \tag{21}$$

where

$$\Lambda_n(j) = \int_0^T |A_{n,j}(t)|\,\mathrm{d}t, \qquad \mathbf{G}_{n,j} = \int_0^T \int_{A_{n,j}(t)} G_j(u_n)^{\frac{(N+p)p}{N}}\,\mathrm{d}x\,\mathrm{d}t.$$

By virtue of $\gamma > 0$, we have $\frac{p}{p-1-\gamma} > 1$. Then, using Young's inequality for all $\varepsilon > 0$,

$$\mathbf{G}_{n,j}^{\frac{p-1-\gamma}{p}} \Lambda_n(j)^{\frac{(N+p)}{Nm'}-\frac{p-1-\gamma}{p}} \leq \varepsilon\mathbf{G}_{n,j} + C(\varepsilon)\Lambda_n(j)^{\frac{(N+p)p}{Nm'(1+\gamma)}-\frac{p-1-\gamma}{1+\gamma}}. \tag{22}$$

Taking $\varepsilon = \frac{1}{2C}$ in (22) and applying (21) to (22), we get

$$\mathbf{G}_{n,j} \leq C\Lambda_n(j)^{\frac{(N+p)p}{Nm'(1+\gamma)}-\frac{p-1-\gamma}{1+\gamma}} + Cj^{\frac{(N+p)p}{N}\left(\frac{p-1-\gamma}{p}\right)} \Lambda_n(j)^{\frac{(N+p)}{Nm'}}.$$

The assumptions $\gamma < p-1$ and $m > \frac{N}{p}+1$ imply that

$$\frac{(N+p)p}{Nm'(1+\gamma)} - \frac{p-1-\gamma}{1+\gamma} > \frac{(N+p)}{Nm'} > 1.$$

Since $|\Lambda_n(j)| \leq T|\Omega|$ and $j \geq 1$, then

$$\mathbf{G}_{n,j} \leq Cj^{\frac{(N+p)p}{N}\left(\frac{p-1-\gamma}{p}\right)} \Lambda_n(j)^{\frac{(N+p)}{Nm'}}. \tag{23}$$

Recalling the property of $G_j(s)$, if $h > j$, we have $G_j(u_n) > h-j$ on $A_{n,h}(t)$, and $A_{n,h}(t) \subset A_{n,j}(t)$. By virtue of $\frac{p-1-\gamma}{p} < 1$, (23) can be written as

$$\Lambda_n(j) \leq \frac{Cj^{\frac{(N+p)p}{N}\left(\frac{p-1-\gamma}{p}\right)} \Lambda_n(j)^{\frac{N+p}{Nm'}}}{(h-j)^{\frac{(N+p)p}{N}}}, \quad \forall\, h > j \geq 1.$$

Applying Lemma 1.1 to

$$\rho = \frac{p-1-\gamma}{p}, \quad \nu = \frac{(N+p)p}{N}, \quad \text{and} \quad \vartheta = \frac{N+p}{Nm'},$$

we have that there exists a positive constant k^* such that

$$\Lambda_n(k^*) = 0.$$

By the fact that $|\Lambda_n(j)| \le T|\Omega|$ (see the proof of Lemma A.1 of [4]), there exists a positive constant d_0 independent of n such that $k^* \le d_0$, so that

$$\Lambda_n(d_0) = 0. \tag{24}$$

Therefore, from (24), it follows the boundedness of the sequence u_n in $L^\infty(Q_T)$.

Now, taking u_n as a test function for problem (19) and using (3), we obtain

$$\frac{1}{2}\int_\Omega u_n(T)^2\,\mathrm{d}x + \alpha\int_{Q_T}|\nabla u_n|^p\,\mathrm{d}x\,\mathrm{d}t + \mu\int_{Q_T}\frac{u_n^2|\nabla u_n|^p}{\big(u_n+\frac{1}{n}\big)^{\theta+1}} \le \int_{Q_T} f_n u_n^{1-\gamma}\,\mathrm{d}x\,\mathrm{d}t. \tag{25}$$

Dropping the non-negative term and using Hölder's inequality on the right-hand side of inequality (25), together with the boundedness of the sequence u_n in $L^\infty(Q_T)$, we obtain

$$\int_{Q_T}|\nabla u_n|^p\,\mathrm{d}x\,\mathrm{d}t \;\le\; \|f\|_{L^m(Q_T)}|Q|^{\frac{1}{m'}}\|u_n^{1-\gamma}\|_{L^\infty(Q_T)} \;\le\; C\|f\|_{L^m(Q_T)}. \tag{26}$$

Consequently, the sequence $\{u_n\}_n$ is bounded in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$. □

In the next proposition, we will prove that the solutions of the approximated problems are uniformly bounded away from zero in every subset $(0,T)\times\omega$ of $(0,T)\times\Omega$ with $\omega \subset\subset \Omega$.

Proposition 2.3. *Let $p \ge 2$. For every subset $\omega \Subset \Omega$. Then, there exists a positive constant $C_{\omega,T}$, independent of n, such that $u_n \ge C_{\omega,T}$ in $(0,T)\times\omega$.*

Proof. Let us define, for $s \ge 0$, the functions

$$H(s) = \int_0^s \frac{d\tau}{\tau^\theta}, \quad \Theta(s) \,=\, e^{-\delta\frac{H(s)}{\beta}}, \quad 0<\theta<1.$$

Next, we consider the decreasing function ϕ

$$\phi(s) = \int_s^1 \Theta(t)\,\mathrm{d}t.$$

Taking

$$\varphi = \Theta(u_n)v, \quad \text{with} \quad 0 \le v \in L^p\big(0,T;W_0^{1,p}(\Omega)\big) \cap L^\infty(Q_T),$$

as a test function in (19) and using that

$$\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}} \le \frac{|\nabla u_n|^p}{u_n^\theta}, \qquad \frac{f_n}{\big(u_n+\frac{1}{n}\big)^\gamma} \le f_n u_n^{-\gamma},$$

(3) and the fact that

$$\nabla\Theta(u_n)v = \Theta(u_n)\left[-\frac{\delta}{\beta u_n^\theta} v\nabla u_n + \nabla v\right],$$

we obtain,

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \Theta(u_n)v \right\rangle \mathrm{d}t + \beta \int_{Q_T} |\nabla u_n|^{p-2}\nabla u_n \nabla v \Theta_n(u_n)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \ge \delta \int_{Q_T} \frac{|\nabla u_n|^p}{u_n^\theta}\Theta(u_n)v - \delta \int_{Q_T} \frac{|\nabla u_n|^p}{u_n^\theta}\Theta(u_n)v\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} f_n u_n^{-\gamma}\Theta(u_n)v\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{Q_T} f_n u_n^{-\gamma}\Theta(u_n)v\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

By the definition of the function ϕ, we obtain

$$\begin{aligned}
&-\int_0^T \left\langle \frac{\partial\phi(u_n)}{\partial t}, v \right\rangle \mathrm{d}t - \beta \int_{Q_T} |\nabla u_n|^{p-2}\nabla\phi(u_n)\nabla v\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \ge \int_{Q_T} \big(\Theta(u_n)-1\big) f_n u_n^{-\gamma} v\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Using the fact that $f_n \le f$, we find

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial\psi(u_n)}{\partial t}, v \right\rangle \mathrm{d}t + \beta \int_{Q_T} |\nabla u_n|^{p-2}\nabla\phi(u_n)\nabla v\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le \int_{Q_T} \big(1-\Theta_n(u_n)\big) f u_n^{-\gamma} v\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We define, for every $z \ge 0$,

$$\widetilde{a}(t,x,z,\nabla z) = \frac{\beta}{\phi'(\phi^{-1}(z))^{p-2}}|\nabla z|^{p-2}\nabla z,$$

and

$$b(s) = \Big(1-\Theta_n\big(\phi^{-1}(s)\big)\Big)\big(\phi^{-1}(s)\big)^{-\gamma} f,$$

for every $\lim_{s\longrightarrow+\infty}\phi(s) = \phi_\infty < 0$.

Thus, see [2] for instance, we deduce that $z = \phi(u_n)$ is a *sub-solution* of

$$\partial_t z - \operatorname{div}\big(\widetilde{a}(t,x,z,\nabla z)\big) = b(z) \quad \text{in} \quad Q_T.$$

Since b is nonnegative term and f satisfies (4), then, by Lemma 3.12 of [21] applied to the previous equation, we obtain the existence of $c_{\omega,T} > 0$ such that

$$z = \phi(u_n) \le c_{\omega,T}, \quad \forall (t,x) \in (0,T) \times \omega, \quad \forall n > 1.$$

Therefore, it follows

$$u_n \ge \phi^{-1}(c_{\omega,T}) = C_{\omega,T} > 0 \quad \text{in} \quad (0,T) \times \omega. \tag{27}$$

Therefore,

$$\frac{u_n}{(u_n + 1/n)^{\theta+1}} \le \max_{u_n \in [C_{\omega,T}, j]} \frac{1}{u_n^\theta} = C_{j,T}(\omega), \quad \forall n >> 1, \tag{28}$$

for every

$$(t,x) \in (0,T) \times \omega \quad \text{such that} \quad u_n(t,x) \le j.$$

This completes the proof of Proposition 2.3. $\square$

The following lemma gives a control of the lower-order term.

Lemma 2.4. *Let $p \ge 2$ and let u_n be the solutions to problems* (19)*. Then we have*

$$\mu \int_{Q_T} \frac{u_n |\nabla u_\varepsilon|^p}{(u_n + \frac{1}{n})^{\theta+1}} \,\mathrm{d}x \,\mathrm{d}t \le \frac{C}{(C_{w,T})^\gamma} \|f\|_{L^m(Q_T)}, \tag{29}$$

where $C_{w,T}$ is defined as in (27)*.*

P r o o f. Choosing $\frac{T_h(u_n)}{h}$ as a test function in (19), dropping the nonnegative terms, we obtain

$$\mu \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \frac{T_h(u_n)}{h} \,\mathrm{d}x \,\mathrm{d}t \le \int_{Q_T} \frac{f_n}{\big(u_n + \frac{1}{n}\big)^\gamma} \left| \frac{T_h(u_n)}{h} \right| \mathrm{d}x \,\mathrm{d}t.$$

Since

$$\frac{1}{\big(u_n + \frac{1}{n}\big)^\gamma} \le \frac{1}{u_n^\gamma} \le \frac{1}{C_{\omega,T}^\gamma},$$

and the fact that $\left| \frac{T_h(u_n)}{h} \right| \le 1$, we have

$$\mu \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \frac{T_h(u_n)}{h} \,\mathrm{d}x \,\mathrm{d}t \le \frac{1}{(C_{w,T})^\gamma} \int_{Q_T} f_n \,\mathrm{d}x \,\mathrm{d}t.$$

Applying Hölder's inequality for $m \geq 1$, we obtain

$$\mu \int_{Q_T} \frac{u_n |\nabla u_n|^p}{(u_n + 1/n)^{\theta+1}} \frac{T_h(u_n)}{h} \,\mathrm{d}x\,\mathrm{d}t \leq \frac{C}{(C_{w,T})^\gamma} \|f\|_{L^m(Q_T)}.$$

Let h tend to 0, and according to Fatou's lemma, we deduce (29). □

2.3. Passage to the limit

From (19), we deduce that $\left\{\frac{\partial u_n}{\partial t}\right\}_n$ is bounded in

$$L^{p'}\big(0,T; W^{-1,p'}(\Omega)\big) + L^m(Q_T).$$

Using Aubin-Simon compactness arguments (see again [28, Corollary 4]), we have

$$u_n \to u \quad \text{strongly in } L^p(Q_T), \text{ and a.e. in } Q_T. \tag{30}$$

Now, we prove that

$$T_j(u_n) \longrightarrow T_j(u) \quad \text{in} \quad L^p\big(0,T; W_0^{1,p}(\Omega)\big). \tag{31}$$

We introduce a time-regularization of the function $T_j(u)$ by

$$\big(T_j(u)\big)_\nu(t,x) := \nu \int_{-\infty}^{t} e^{\nu(s-t)} T_j\big(u(s,x)\big)\,\mathrm{d}s,$$

where $T_j\big(u(s,x)\big)$ is the zero extension of u for $s < 0$ (see [31]). We recall that $\big(T_j(u_n)\big)_\nu$ converges to $T_j(u_n)$ strongly in $L^p\big(0,T; W_0^{1,p}(\Omega)\big)$ as ν tends to infinity. Set

$$\psi_\lambda(r) = re^{\lambda r^2}, \quad \text{with} \quad \lambda > \frac{(2^{p-1}\mu C_{j,T}(\omega))^2}{4(2^{2-p}\alpha)^2},$$

where $C_{j,T}(\omega)$ is as in (28). We will denote by $\tau(\epsilon,n,\nu)$ any quantity such that

$$\lim_{\nu \longrightarrow +\infty} \lim_{n \longrightarrow +\infty} \lim_{\epsilon \longrightarrow +\infty} \tau(\epsilon,n,\nu) = 0.$$

Let

$$0 \leq \phi \in C_c^\infty(Q_T).$$

By the same technique as in [1, 22], we have that

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \psi_\lambda(T_j(u_n) - \big(T_j(u_\epsilon)\big)_\nu)\phi \right\rangle \mathrm{d}t \geq \tau(\epsilon,n,\nu). \tag{32}$$

Choosing $\varphi_\lambda = \psi_\lambda\big(T_j(u_n) - (T_j(u_\epsilon))_\nu\big)\phi$ as a test function in (8), and using (3) and (32), we obtain

$$\begin{aligned}
&\alpha\int_{Q_T} |\nabla u_n|^{p-2}\nabla u_n \nabla\Big(T_j(u_n) - \big(T_j(u_\epsilon)\big)_\nu\Big)\\
&\quad\times \psi'_\lambda\Big(T_j(u_n) - \big(T_j(u_\epsilon)\big)_\nu\Big)\phi \,\mathrm{d}x\,\mathrm{d}t + \mu\int_{Q_T}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\varphi_\lambda \,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_{Q_T}\frac{f_n}{\big(u_n+\frac{1}{n}\big)^\gamma}\varphi_\lambda\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \alpha\int_{Q_T}|\nabla u_n|^{p-2}\nabla u_n\nabla\phi\psi_\lambda\Big(T_j(u_n) - \big(T_j(u_\epsilon)\big)_\nu\Big)\,\mathrm{d}x\,\mathrm{d}t - \tau(\epsilon,n,\nu). \qquad (33)
\end{aligned}$$

By (30), the sequence $\big(T_j(u_\epsilon)\big)_\nu$ converges to $\big(T_j(u)\big)_\nu$ almost everywhere in Q_T, and the fact that

$$\left|\frac{f_n}{\big(u_n+\frac{1}{n}\big)^\gamma}\varphi_\lambda\right| \le \frac{C}{(C_{w,T})^\gamma}\|f\|_{L^m}\psi_\lambda(2j) \in L^1(Q_T), \quad m\ge 1,$$

by the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{\nu\longrightarrow+\infty}\lim_{n\longrightarrow+\infty}\lim_{\epsilon\longrightarrow+\infty}\int_{Q_T}\frac{f_n}{\big(u_n+\frac{1}{n}\big)^\gamma}\varphi_\lambda = 0.$$

By putting $Q_T = \{u_n \le j\}\cup\{u_n > j\}$ and using the approach employed in [1], we have

$$\lim_{\nu\longrightarrow+\infty}\lim_{n\longrightarrow+\infty}\lim_{\epsilon\longrightarrow+\infty}\int_{Q_T}|\nabla u_n|^{p-2}\nabla u_n\nabla\phi\psi_\lambda\Big(T_j(u_n) - \big(T_j(u_\epsilon)\big)_\nu\Big) = 0.$$

Therefore, we can write that

$$\begin{aligned}
\int_{Q_T}\frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi_\lambda - \alpha\int_{Q_T}|\nabla u_n|^{p-2}\nabla u_n\nabla\phi\psi_\lambda\Big(T_j(u_n) - \big(T_j(u_\epsilon)\big)_\nu\Big)&\\
= \tau(\epsilon,n,\nu).& \qquad (34)
\end{aligned}$$

Additionally, using the definition of $C_{j,T}(\omega)$ and the fact that $\psi_\lambda\big(j - (T_j(u))_\nu\big)$ is non-negative, we may state that

$$\begin{aligned}
\lim_{\epsilon\longrightarrow+\infty}\int_{Q_T}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\varphi_\lambda\,\mathrm{d}x\,\mathrm{d}t &\ge \int_{\{C_{\omega,T}\le u_n\le j\}}\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\varphi_\lambda\,\mathrm{d}x\,\mathrm{d}t\\
&\ge -C_{j,T}(\omega)\int_{Q_T}|\nabla T_j(u_n)|^p|\varphi_\lambda|\,\mathrm{d}x\,\mathrm{d}t. \qquad (35)
\end{aligned}$$

Now, from this convergence

$$\nabla\big(T_j(u_\epsilon)\big)_\nu \rightharpoonup \nabla\big(T_j(u)\big)_\nu \quad \text{weakly in} \quad \big(L^p(Q_T)\big)^N \text{ as } \epsilon \longrightarrow +\infty,$$

and by using (33)–(35), we obtain that

$$\alpha\int_{Q_T} |\nabla u_n|^{p-2}\nabla u_n \nabla\Big(T_j(u_n)-\big(T_j(u)\big)_\nu\Big)\psi'_\lambda\Big(T_j(u_n)-\big(T_j(u)\big)_\nu\Big)\phi \,\mathrm{d}x\,\mathrm{d}t \\ - \mu C_{j,T}(\omega)\int_{Q_T} |\nabla T_j(u_n)|^p|\varphi_\lambda| \,\mathrm{d}x\,\mathrm{d}t \le \tau(\nu,n), \quad (36)$$

with $\tau(\nu,n) = 0$ as $\nu, n \longrightarrow +\infty$. Note that

$$\alpha\int_{Q_T} |\nabla u_n|^{p-2}\nabla u_n \nabla\Big(T_j(u_n)-\big(T_j(u)\big)_\nu\Big) \\ \times \psi'_\lambda\Big(T_j(u_n)-\big(T_j(u)\big)_\nu\Big)\phi\chi_{\{u_n\ge j\}} \,\mathrm{d}x\,\mathrm{d}t \\ = -\alpha\int_{Q_T} |\nabla u_n|^{p-2}\nabla u_n \nabla\big(T_j(u)\big)_\nu \psi'_\lambda\Big(j-\big(T_j(u)\big)_\nu\Big)\phi\chi_{\{u_n\ge j\}} \,\mathrm{d}x\,\mathrm{d}t \\ = \tau(\nu,n).$$

So, adding

$$-\alpha\int_{Q_T} |\nabla\big(T_j(u)\big)_\nu|^{p-2}\nabla\big(T_j(u)\big)_\nu \nabla\Big(T_j(u_n)-\big(T_j(u)\big)_\nu\Big) \\ \times \psi'_\lambda\Big(T_j(u_n)-\big(T_j(u)\big)_\nu\Big)\phi \,\mathrm{d}x\,\mathrm{d}t = \tau(\nu,n),$$

to both sides of (36) and since

$$\int_{Q_T} |\nabla T_j(u_n)|^p|\varphi_\lambda| \,\mathrm{d}x\,\mathrm{d}t \\ \le 2^{p-1}\int_{Q_T} \Big|\nabla\Big(T_j(u_n)-\big(T_j(u)\big)_\nu\Big)\Big|^p |\varphi_\lambda| \,\mathrm{d}x\,\mathrm{d}t + 2^{p-1}\int_{Q_T} |\nabla\big(T_j(u)\big)_\nu|^p|\varphi_\lambda| \,\mathrm{d}x\,\mathrm{d}t \\ = 2^{p-1}\int_{Q_T} \Big|\nabla\Big(T_j(u_n)-\big(T_j(u)\big)_\nu\Big)\Big|^p |\varphi_\lambda| \,\mathrm{d}x\,\mathrm{d}t + \tau(\nu,n),$$

and from this inequality

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta) \ge 2^{2-p}|\xi-\eta|^p, \quad \forall p \ge 2,$$

we find

$$2^{2-p}\alpha\int\limits_{Q_T}\left|\nabla\Big(T_j(u_n)-\big(T_j(u)\big)_\nu\Big)\right|^p \psi'_\lambda\Big(T_j(u_n)-\big(T_j(u)\big)_\nu\Big)\phi\,\mathrm{d}x\,\mathrm{d}t$$
$$-\,2^{p-1}\mu C_{j,T}(\omega)\int\limits_{Q_T}\left|\nabla\Big(T_j(u_n)-\big(T_j(u)\big)_\nu\Big)\right|^p \left|\psi_\lambda\Big(T_j(u_n)-\big(T_j(u)\big)_\nu\Big)\right|\phi\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \tau(\nu,n).$$

Since $\lambda > \frac{(2^{p-1}\mu C_{j,T}(\omega))^2}{4(2^{2-p}\alpha)^2}$, it holds that

$$2^{2-p}\alpha\psi'_\lambda(r) - 2^{p-1}\mu C_{j,T}(\omega)|\psi_\lambda(r)| \geq \frac{2^{2-p}\alpha}{2}$$

for every $r \in \mathbb{R}$, and by setting $\nu \to +\infty$, we obtain (31). From this result, we deduce that (up to subsequences)

$$\nabla u_n \longrightarrow \nabla u \ \text{ a.e. in } \ Q_T. \tag{37}$$

Corollary 2.5. *Let $p \geq 2$ and $0 < \theta < 1$. Then, we have*

$$\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}} \longrightarrow \frac{|\nabla u|^p}{u^\theta} \quad \textit{strongly in} \quad L^1(Q_T). \tag{38}$$

Proof. We will start by determining the local equi-integrability of $\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}$ on Q_T before proceeding. To this end, we introduce the function $\psi_\eta\colon\mathbb{R}^+ \to \mathbb{R}$ by

$$\psi_\eta(\sigma) = (\eta-1)\int\limits_0^\sigma \frac{dt}{(1+t)^\eta} = \left(1 - \frac{1}{(1+\sigma)^{\eta-1}}\right) \geq 0, \quad \sigma \in \mathbb{R}^+, \quad \eta > 1.$$

This function is globally Lipschitzian and $\psi_\eta(0) = 0$, $0 \leq \psi_\eta(\sigma) \leq 1$. Let $j > 0$. We also define $\varphi_j\colon\mathbb{R}^+ \to \mathbb{R}$ by

$$\varphi_j(\sigma) = \psi_\eta(\sigma - j) \quad \text{if} \qquad \sigma \geq j$$

and

$$\varphi_j(\sigma) = 0 \quad \text{if} \quad 0 < \sigma < j.$$

Choosing $\varphi_j(u_n)$ as a test function in (19), we obtain

$$\int_\Omega \mathrm{d}x \int_0^{u_n(T,x)} \varphi_j(\sigma)\mathrm{d}\sigma + \int_0^T\int_\Omega e(t,x)|\nabla u_n|^p \varphi_j'(u_n)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\int_\Omega \mathrm{d}(t,x)\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\varphi_j(u_n)\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \int_0^T\int_\Omega \varphi_j(u_n)\frac{f_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\,\mathrm{d}x\,\mathrm{d}t.$$

As $|\varphi_j| \leq 1$ and (3), we have

$$\int_{B_R} \mathrm{d}x \int_0^{u_n(T,x)} \varphi_j(\sigma)\mathrm{d}\sigma + \alpha\int_0^T\int_\Omega |\nabla u_n|^p \varphi_j'(u_n)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\mu\int_0^T\int_\Omega \frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\varphi_j(u_n)\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq j^{-\gamma}\int_0^T\int_{\Omega\cap\{u_n\geq j\}} |f_n|\,\mathrm{d}x\,\mathrm{d}t.$$

Dropping the non-negative terms and using (9), we deduce from the above inequality that

$$\int_0^T\int_\Omega \frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\varphi_j(u_n)\,\mathrm{d}x\,\mathrm{d}t \to 0 \ \text{ as } \ j\to+\infty$$

uniformly with respect to n. Using the properties of the function φ_j, we get as $j\to+\infty$

$$\int_{\Omega\cap\{u_n\geq 2j\}} \frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \frac{1}{\varphi_j(2j)}\int_0^T\int_\Omega \frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\varphi_j(u_n)\,\mathrm{d}x\,\mathrm{d}t \to 0. \quad (39)$$

Consequently, by (28), we have

$$\int_0^T \int_\Omega \frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \int_{\Omega\cap\{u_n\ge 2j\}} \frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}\,\mathrm{d}x\,\mathrm{d}t + C_{j,T}(\omega)\int_0^T \int_{\Omega\cap\{u_n\le 2j\}} |\nabla T_{2j}(u_n)|^p\,\mathrm{d}x\,\mathrm{d}t. \tag{40}$$

From (39) and (26), we obtain the equi-integrability of $\frac{u_n|\nabla u_n|^p}{(u_n+1/n)^{\theta+1}}$ on Q_T. By this, together with (30), (37) and Vitali's Theorem, we obtain (38) and $\frac{|\nabla u|^p}{u^\theta} \in L^1(Q_T)$. □

Lemma 2.6. *Let $u_n \in L^p\big(0,T,W_0^{1,p}(\Omega)\big)$ be a sequence of weak solutions to* (19). *Then,*

$$\lim_{n\to+\infty}\int_{Q_T} \frac{f_n}{\big(u_n+\frac{1}{n}\big)^\gamma}\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} \frac{f}{u^\gamma}\varphi\,\mathrm{d}x\,\mathrm{d}t, \quad \forall\,\varphi\in C_0^\infty(Q_T). \tag{41}$$

Proof. We fix $\varphi \in C_0^\infty(Q_T)$. Since $\operatorname{supp}(\varphi)$ is a compact subset of Q_T, thanks to (27), we have that there exists a constant $C_{\operatorname{supp}(\varphi)} > 0$ such that $u_n \ge C_{\operatorname{supp}(\varphi)}$ in $\operatorname{supp}(\varphi)$. Then,

$$\left|\frac{f_n}{\big(u_n+\frac{1}{n}\big)^\gamma}\varphi\right| \le \frac{f}{C^\gamma_{\operatorname{supp}(\varphi)}}\|\varphi\|_{L^\infty(Q_T)}, \qquad \forall\,(x,t)\in\operatorname{supp}(\varphi).$$

As n goes to $+\infty$ and since (30), we can see that it converges almost everywhere to $\frac{f}{u^\gamma}\varphi$. So, according to Lebesgue's Theorem, we conclude that (41) is true. □

The proof of Theorem 1.4 is completed by obtaining that u is a weak solution for (1) and passing to the limit for $n \longrightarrow +\infty$ in the weak formulation of (10) through the convergences (30), (37), (38) and (41).

3. Proof of Theorems 1.6

Lemma 3.1. *For all $n\in\mathbb{N}$, we have the following estimate*

$$\int_{Q_T} d(x,t)\frac{u_n|\nabla u_n|^p}{\big(u_n+\frac{1}{n}\big)^{\theta+1-\gamma}}\,\mathrm{d}x\,\mathrm{d}t \le \|f\|_{L^1(Q_T)}. \tag{42}$$

P r o o f. Let us consider $\varphi(u_n) = (u_n + \frac{1}{n})^\gamma \frac{T_h(u_n)}{h}$, $h > 0$ fixed (this is possible since $u_n \in L^\infty(Q_T)$), as a test function in the weak formulation of problems (19). Using (3) and the fact that

$$\nabla\varphi(u_n) = \frac{1}{h}\left(u_n + \frac{1}{n}\right)^{\gamma-1}\left[\gamma T_h(u_n) + \left(u_n + \frac{1}{n}\right)\right]\nabla u_n \chi_{]-h,h[}(u_n),$$

we obtain

$$\int_\Omega \Phi(u_n)\,\mathrm{d}x + \frac{\alpha}{h}\int_{Q_T}\left(u_n + \frac{1}{n}\right)^{\gamma-1}\left[\gamma T_h(u_n) + \left(u_n + \frac{1}{n}\right)\right]|\nabla u_n|^p \chi_{]-h,h[}(u_n)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{Q_T} d(x,t)\frac{u_n|\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1-\gamma}}\frac{T_h(u_n)}{h}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \int_{Q_T} f_n \frac{T_h(u_n)}{h}\,\mathrm{d}x\,\mathrm{d}t, \quad (43)$$

where

$$\Phi(s) = \int_0^s \left(\zeta + \frac{1}{n}\right)^\gamma \frac{T_h(\zeta)}{h} d\zeta \geq 0, \quad \forall \zeta \geq 0. \quad (44)$$

From (43) and (44), and by dropping the non-negative term of the left-hand side of (43), we get

$$0 \leq \int_{Q_T} d(x,t)\frac{u_n|\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1-\gamma}}\frac{T_h(u_n)}{h}\,\mathrm{d}x\,\mathrm{d}t \leq \int_{Q_T} f_n \frac{T_h(u_n)}{h}\,\mathrm{d}x\,\mathrm{d}t.$$

Since $f_n \leq f$ and $|\frac{T_h(u_n)}{h}| \leq 1$, we have

$$\int_{Q_T} d(x,t)\frac{u_n|\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1-\gamma}}\frac{T_h(u_n)}{h}\,\mathrm{d}x\,\mathrm{d}t \leq \|f\|_{L^1(Q_T)}.$$

By Fatou's Lemma, as $h \to 0$, and using the strict positivity of u_n, we obtain (42). □

Lemma 3.2. *Let f belong to $L^1(Q_T)$ and $\gamma < \theta$. Then, the sequence $\{u_n\}_n$ is bounded in $L^{\eta^*}(Q_T) \cap L^\eta\big(0,T;W_0^{1,\eta}(\Omega)\big)$, where η is defined as in (7). Moreover, the sequence $\{T_j(u_n)\}_n$ is bounded in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ for every $j > 0$.*

P r o o f. The estimate (42) and

$$\frac{1}{2^{\theta+1-\gamma}u_n^{\theta-\gamma}} \leq \frac{u_n}{(1+u_n)^{\theta+1-\gamma}} \leq \frac{u_n}{(\frac{1}{n}+u_n)^{\theta+1-\gamma}} \quad \text{in} \quad A_{n,1}(t),$$

give

$$\frac{\mu}{2^{\theta+1-\gamma}} \int\limits_{A_{n,1}(t)} \frac{|\nabla u_n|^p}{u_n^{\theta-\gamma}} \,\mathrm{d}x\,\mathrm{d}t \le \int\limits_{A_{n,1}(t)} d(x,t) \frac{u_n |\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1-\gamma}} \,\mathrm{d}x\,\mathrm{d}t$$

$$\le \|f\|_{L^1(Q)}, \tag{45}$$

where

$$A_{n,1}(t) = \{x \in \Omega \colon u_n(x,t) > 1\}, \quad t \in (0,T).$$

Let $\eta < p$ and using the Hölder inequality, we have

$$\int\limits_{Q_T} |\nabla G_1(u_n)|^\eta \,\mathrm{d}x\,\mathrm{d}t = \int\limits_{A_{n,1}(t)} \frac{|\nabla G_1(u_n)|^\eta}{u_n^{\frac{(\theta-\gamma)\eta}{p}}} u_n^{\frac{(\theta-\gamma)\eta}{p}} \,\mathrm{d}x\,\mathrm{d}t$$

$$\le \Bigg(\int\limits_{A_{n,1}(t)} \frac{|\nabla G_1(u_n)|^p}{u_n^{\theta-\gamma}} \,\mathrm{d}x\,\mathrm{d}t \Bigg)^{\frac{\eta}{p}}$$

$$\times \Bigg(\int\limits_{A_{n,1}(t)} u_n^{\frac{(\theta-\gamma)\eta}{p-\eta}} \,\mathrm{d}x\,\mathrm{d}t \Bigg)^{\frac{p-\eta}{p}}. \tag{46}$$

By (45)–(46) and the fact that $u_n \le G_1(u_n) + 1$, we get

$$\int\limits_{Q_T} |\nabla G_1(u_n)|^\eta \,\mathrm{d}x\,\mathrm{d}t \le C \, \|f\|_{L^1(Q_T)}^{\frac{\eta}{p}} \Bigg(\int\limits_{A_{n,1}(t)} u_n^{\frac{(\theta-\gamma)\eta}{p-\eta}} \,\mathrm{d}x\,\mathrm{d}t \Bigg)^{\frac{p-\eta}{p}}$$

$$\le C \Bigg(\int\limits_{A_{n,1}(t)} \big(G_1(u_n)\big)^{\frac{(\theta-\gamma)\eta}{p-\eta}} \,\mathrm{d}x\,\mathrm{d}t \Bigg)^{\frac{p-\eta}{p}} + C. \tag{47}$$

Inequality (7) implies that $\eta^* = \frac{(\theta-\gamma)\eta}{p-\eta} > 1$ (since $\gamma < \theta$). By Sobolev's inequality and (47), we get

$$\Bigg(\int\limits_{Q_T} \big(G_1(u_n)\big)^{\eta^*} \,\mathrm{d}x\,\mathrm{d}t \Bigg)^{\frac{N-\eta}{N}} \le C \int\limits_{Q_T} |\nabla G_1(u_n)|^\eta \,\mathrm{d}x\,\mathrm{d}t$$

$$\le C \Bigg(\int\limits_{Q_T} \big(G_1(u_n)\big)^{\eta^*} \,\mathrm{d}x\,\mathrm{d}t \Bigg)^{\frac{p-\eta}{p}} + C. \tag{48}$$

Since $p < N$, we have $\frac{N-\eta}{N} > \frac{p-\eta}{p}$, then, from (48), we deduce that the sequence $\{u_n\}_n$ is bounded in $L^{\eta^*}(Q_T)$. From (47), it follows the boundedness of $G_1(u_n)$ in $L^\eta\big(0,T;W_0^{1,\eta}(\Omega)\big)$.

Moreover, using $T_j(u_n)$ as a test function in (8), by (3), we have

$$\frac{1}{2}\int_\Omega \big(T_j(u_n)\big)^2\,\mathrm{d}x + \alpha\int_{\{u_n\le j\}} |T_j(u_n)|^p\,\mathrm{d}x\,\mathrm{d}t + \mu\int_{Q_T} \frac{u_n|\nabla u_n|^p}{\big(u_n+\frac{1}{n}\big)^{\theta+1}}T_j(u_n)\,\mathrm{d}x\,\mathrm{d}t \le \int_{Q_T}\frac{f_n}{(u_n+\frac{1}{n})^\gamma}T_j(u_n)\,\mathrm{d}x\,\mathrm{d}t. \tag{49}$$

Dropping the first and third non-negative term, we obtain

$$\alpha\int_{\{u_n\le j\}} |T_j(u_n)|^p\,\mathrm{d}x\,\mathrm{d}t \le \int_{Q_T}\frac{f_n}{(u_n+\frac{1}{n})^\gamma}T_j(u_n)\,\mathrm{d}x\,\mathrm{d}t. \tag{50}$$

For the right-hand side of (50), for every $j > 0$, one has

$$\int_{Q_T}\frac{f_n}{\big(u_n+\frac{1}{n}\big)^\gamma}T_j(u_n)\,\mathrm{d}x\,\mathrm{d}t = \int_{\{u_n\le j\}}\frac{f_n u_n}{(u_n+\frac{1}{n})^\gamma}\,\mathrm{d}x\,\mathrm{d}t + j\int_{\{u_n>j\}}\frac{f_n}{(u_n+\frac{1}{n})^\gamma}\,\mathrm{d}x\,\mathrm{d}t \le 2j^{1-\gamma}\|f\|_{L^1(Q_T)}. \tag{51}$$

Taking $j = 1$ in (51) and using (50), we have that $T_1(u_n)$ is bounded in the space $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$, then it is bounded in $L^\eta\big(0,T;W_0^{1,\eta}(\Omega)\big)$ (since $\eta < p$). Since

$$u_n = G_1(u_n) + T_1(u_n),$$

we deduce that u_n is bounded in

$$L^\eta\big(0,T;W_0^{1,\eta}(\Omega)\big).$$

Moreover, (50) and (51) imply that

$$\|T_j(u_n)\|_{L^p(0,T;W_0^{1,p}(\Omega))} \le Cj^{\theta+1-\gamma}, \quad \forall n \ge 1. \tag{52}$$

(52) implies that $T_j(u_n)$ is bounded in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ with respect to n for every $j > 0$. This completes the proof of Lemma 3.2.

To complete the proof of Theorem 1.6, letting j tend to $+\infty$ in (52) and applying Fatou's lemma, we deduce that u_n is bounded in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$. Therefore, we can use all the previous convergences (30), (37), (38) and (41) in order to pass to the limit in the weak formulation of (10) for obtaining u as a weak solution for (1). □

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