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FOURTH-ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS: OSCILLATION VIA NEW CANONICAL TRANSFORMS

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ABSTRACT. This paper presents a new method of transforming fourth-order nonlinear semi-canonical neutral difference equations into canonical type equations. This technique reduces the set of nonoscillatory solutions to only two. By eliminating these two remaining types of nonoscillatory solutions, the authors are able to obtain new criteria for the oscillation of all solutions of the equation under investigation. Examples are provided to demonstrate the significance and novelty of the main results.

1. Introduction

This paper is concerned with the oscillatory behavior of solutions of the fourth-order neutral difference equation

$$D_4 z(n) + q(n) x^{\alpha}(n-\sigma) = 0, \quad n \ge n_0 > 0, \tag{E}$$

where

$$D_0 z(n) = z(n), \quad D_j z(n) = a_j(n)\Delta(D_{j-1}z(n)), \quad j = 1,2,3,$$
$$D_4 z(n) = \Delta(D_3 z(n)) \quad \text{and} \quad z(n) = x(n) + p(n)x(n-\tau).$$

In the sequel, and without further mention, we will assume that:

(H_1) $\{a_j(n)\}$, $j = 1,2,3$, are sequences of positive real numbers;

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(H_2) $\{p(n)\}$ and $\{q(n)\}$ are non-negative real sequences with $0 \le p(n) < 1$;

(H_3) σ and τ are positive integers, and α is the ratio of odd positive integers;

(H_4) equation (E) is in semi-canonical form, that is,

$$\sum_{n=n_0}^{\infty} \frac{1}{a_3(n)} = \sum_{n=n_0}^{\infty} \frac{1}{a_2(n)} = \infty \text{ and } \sum_{n=n_0}^{\infty} \frac{1}{a_1(n)} < \infty.$$

Let $\theta = \max\{\sigma, \tau\}$. By a *solution* of (E), we mean a real sequence $\{x(n)\}$ defined for all $n \ge n_0 - \theta$ that satisfies (E) for all $n \ge n_0$. We consider only solutions that are nontrivial for all large n. A solution of (E) is said to be *oscillatory* if it is neither eventually positive nor eventually negative, and *nonoscillatory* otherwise.

Due to the increase in applications of difference equations to real-world problems, the qualitative theory of such equations has received a great deal of attention in recent years; see, for example, the monographs [1, 3, 9], the papers [2, 4, 5, 7, 8, 12, 13, 15–20], and the references therein. In particular, oscillation theory for fourth-order functional difference equations has attracted a good bit of attention in the last several years; this is clearly evident from the numerous research papers that have appeared in the literature; again, see [2, 4, 5, 7, 8, 12, 13, 15–20] and the papers cited therein.

From a review of the literature, it is clear that most of the results on equations of type (E) are for the case, where

$$\sum_{n=n_0}^{\infty} \frac{1}{a_j(n)} = \infty, \quad \text{for} \quad j = 1, 2, 3.$$

There are a few results available in the literature [6, 16, 18] if (E) satisfies

$$\sum_{n=n_0}^{\infty} \frac{1}{a_3(n)} < \infty, \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_2(n)} = \sum_{n=n_0}^{\infty} \frac{1}{a_1(n)} = \infty, \tag{1}$$

or

$$\sum_{n=n_0}^{\infty} \frac{1}{a_3(n)} = \infty, \quad \sum_{n=n_0}^{\infty} \frac{1}{a_2(n)} < \infty, \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_1(n)} = \infty. \tag{2}$$

Following the terminology introduced in [6] and [11] and employed in [10], we will say that equation (E) is in canonical form if

$$\sum_{n=n_0}^{\infty} \frac{1}{a_j(n)} = \infty, \quad \text{for} \quad j = 1, 2, 3, \tag{C}$$

in noncanonical form if

$$\sum_{n=n_0}^{\infty} \frac{1}{a_j(n)} < \infty, \quad \text{for} \quad j = 1, 2, 3, \tag{NC}$$

in semi-canonical form if either

$$\sum_{n=n_0}^{\infty} \frac{1}{a_1(n)} = \infty, \quad \sum_{n=n_0}^{\infty} \frac{1}{a_2(n)} = \infty, \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_3(n)} < \infty, \tag{SC1}$$

or

$$\sum_{n=n_0}^{\infty} \frac{1}{a_1(n)} = \infty, \quad \sum_{n=n_0}^{\infty} \frac{1}{a_2(n)} < \infty, \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_3(n)} = \infty, \tag{SC2}$$

or

$$\sum_{n=n_0}^{\infty} \frac{1}{a_1(n)} < \infty, \quad \sum_{n=n_0}^{\infty} \frac{1}{a_2(n)} = \infty, \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_3(n)} = \infty, \tag{SC3}$$

and is in semi-noncanonical form if either

$$\sum_{n=n_0}^{\infty} \frac{1}{a_1(n)} = \infty, \quad \sum_{n=n_0}^{\infty} \frac{1}{a_2(n)} < \infty, \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_3(n)} < \infty, \tag{SN1}$$

or

$$\sum_{n=n_0}^{\infty} \frac{1}{a_1(n)} < \infty, \quad \sum_{n=n_0}^{\infty} \frac{1}{a_2(n)} = \infty, \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_3(n)} < \infty, \tag{SN2}$$

or

$$\sum_{n=n_0}^{\infty} \frac{1}{a_1(n)} < \infty, \quad \sum_{n=n_0}^{\infty} \frac{1}{a_2(n)} < \infty, \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_3(n)} = \infty. \tag{SN3}$$

In [6], the authors studied the oscillatory properties of (E) if either (1) or (2) holds, that is, the equation is in semi-canonical forms (SC1) or (SC2). They obtained results using a transformation technique that converted equation (E) with (1) or (2) into a canonical type equation. This approach significantly reduced the effort required in the sense that it reduced the possible types of nonoscillatory solutions of (E) to two instead of four. Therefore, to find oscillation criteria for (E), it was only necessary to eliminate the remaining two types of nonoscillatory solutions instead of the original four.

Following this idea, our aim in this paper is to transform equation (E) with (H_4) holding (that is, (E) is of type (SC3)), into a canonical equation and then apply techniques known for canonical equations to obtain oscillation criteria for the semi-canonical equation (E). Examples are provided to illustrate the novelty of the main results.

2. Main Results

In view of (H_4), we introduce the notation

$$A_1(n) = \sum_{s=n}^{\infty} \frac{1}{a_1(s)} \quad \text{and} \quad A_{21}(n) = \sum_{s=n}^{\infty} \frac{A_1(s+1)}{a_2(s)}.$$

We begin with two theorems about transformations.

Theorem 2.1. *Assume that*

$$A_{21}(n_0) = \infty. \tag{3}$$

Then the semi-canonical operator $D_4 z(n)$ can be written in a canonical form as

$$D_4 z(n) = \Delta\left(a_3(n)\Delta\left(\frac{a_2(n)}{A_1(n+1)}\Delta\left(a_1(n)A_1(n)A_1(n+1)\Delta\left(\frac{z(n)}{A_1(n)}\right)\right)\right)\right). \tag{4}$$

Proof. By a direct calculation, we have

$$\begin{aligned}\Delta\left(a_1(n)A_1(n)A_1(n+1)\Delta\left(\frac{z(n)}{A_1(n)}\right)\right) &= \Delta\big(a_1(n)A_1(n)\Delta z(n) + z(n)\big)\\ &= A_1(n+1)\,\Delta\big(a_1(n)\,\Delta z(n)\big)\end{aligned}$$

and hence

$$\Delta\left(a_3(n)\Delta\left(\frac{a_2(n)}{A_1(n+1)}\Delta\left(a_1(n)A_1(n)A_1(n+1)\Delta\left(\frac{z(n)}{A_1(n)}\right)\right)\right)\right) = D_4 z(n).$$

Furthermore,

$$\sum_{n=n_0}^{\infty} \frac{1}{a_1(n)A_1(n)A_1(n+1)} = \sum_{n=n_0}^{\infty} \Delta\left(\frac{1}{A_1(n)}\right) = \lim_{n\to\infty} \frac{1}{A_1(n)} - \frac{1}{A_1(n_0)} = \infty$$

$$\sum_{n=n_0}^{\infty} \frac{A_1(n+1)}{a_2(n)} = \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{1}{a_3(n)} = \infty$$

by (3) and (H_4). Hence, the right-hand side of (4) is in canonical form, and this completes the proof of the theorem. □

Theorem 2.2. *Assume that*

$$A_{21}(n_0) < \infty \quad \textit{and} \quad \sum_{n=n_0}^{\infty} \frac{A_{21}(n+1)}{a_3(n)} = \infty. \tag{5}$$

Then the semi-canonical operator $D_4 z(n)$ can be written in the canonical form

$$D_4 z(n) = \Delta\left(\frac{a_3(n)}{A_{21}(n+1)}\Delta\left(\frac{a_2(n)A_{21}(n)A_{21}(n+1)}{A_1(n+1)}\right.\right.$$
$$\left.\left.\Delta\left(\frac{a_1(n)A_1(n)A_1(n+1)}{A_{21}(n)}\Delta\left(\frac{z(n)}{A_1(n)}\right)\right)\right)\right). \tag{6}$$

P r o o f. A direct calculation gives

$$\Delta\left(\frac{a_2(n)A_{21}(n)A_{21}(n+1)}{A_1(n+1)}\Delta\left(\frac{a_1(n)A_1(n)A_1(n+1)}{A_{21}(n)}\Delta\left(\frac{z(n)}{A_1(n)}\right)\right)\right)$$
$$= \Delta\left(\left(\frac{1}{A_1(n+1)}\Big[A_{21}(n)A_1(n+1)a_2(n)\Delta\big(a_1(n)\Delta z(n)\big)\right.\right.$$
$$- A_{21}(n)a_2(n)\Delta z(n) + A_1(n+1)A_1(n)a_1(n)\Delta z(n)$$
$$\left.\left. + A_{21}(n)a_2(n)\Delta z(n) + A_1(n+1)z(n)\Big]\right)\right)$$
$$= \Delta\Big(A_{21}(n)a_2(n)\Delta\big(a_1(n)\Delta z(n)\big) + A_1(n)a_1(n)\Delta z(n) + z(n)\Big)$$
$$= A_{21}(n+1)\Delta\Big(a_2(n)\Delta\big(a_1(n)\Delta z(n)\big)\Big),$$

and so

$$\Delta\left(\frac{a_3(n)}{A_{21}(n+1)}\Delta\left(\frac{a_2(n)A_{21}(n)A_{21}(n+1)}{A_1(n+1)}\right.\right.$$
$$\left.\left.\Delta\left(\frac{a_1(n)A_1(n)A_1(n+1)}{A_{21}(n)}\Delta\left(\frac{z(n)}{A_1(n)}\right)\right)\right)\right)$$
$$= \Delta\Big(a_3(n)\Delta\Big(a_2(n)\Delta\big(a_1(n)\Delta z(n)\big)\Big)\Big).$$

In addition,

$$\sum_{n=n_0}^{\infty} \frac{A_1(n+1)}{a_2(n)A_{21}(n)A_{21}(n+1)} = \sum_{n=n_0}^{\infty} \Delta\Big(\frac{1}{A_{21}(n)}\Big) = \lim_{n\to\infty} \frac{1}{A_{21}(n)} - \frac{1}{A_{21}(n_0)} = \infty$$

and

$$\sum_{n=n_0}^{\infty} \frac{A_{21}(n)}{a_1(n)\, A_1(n)A_1(n\,+\,1)} = \sum_{n=n_0}^{\infty} A_{21}(n)\Delta\Big(\frac{1}{A_1(n)}\Big)$$

$$= \frac{A_{21}(n)}{A_1(n)}\Big|_{n_0}^{\infty} + \sum_{n=n_0}^{\infty} \frac{1}{a_2(n)} > \sum_{n=n_0}^{\infty} \frac{1}{a_2(n)} = \infty$$

by (H_4). Also, from (5), we see that

$$\sum_{n=n_0}^{\infty} \frac{A_{21}(n+1)}{a_3(n)} = \infty.$$

Hence, the right hand side of (6) is in canonical form, which ends the proof. □

Based on Theorems 2.1 and 2.2, we can write the equation (E) in an equivalent canonical form as

$$L_4y(n) + q(n)x^{\alpha}(n-\sigma) = 0, \tag{E_c}$$

where

$$L_0y(n) = y(n), \qquad L_jy(n) = b_j(n)\Delta\big(L_{j-1}y(n)\big),$$
$$j = 1, 2, 3, \qquad L_4y(n) = \Delta\big(L_3y(n)\big),$$

and

$$b_1(n) = \begin{cases} a_1(n)A_1(n)A_1(n+1), & \text{if } (3) \text{ holds}, \\ \frac{a_1(n)A_1(n)A_1(n+1)}{A_{21}(n)}, & \text{if } (5) \text{ holds}, \end{cases}$$

$$b_2(n) = \begin{cases} \frac{a_2(n)}{A_1(n+1)}, & \text{if } (3) \text{ holds}, \\ \frac{a_2(n)A_{21}(n)A_{21}(n+1)}{A_1(n+1)}, & \text{if } (5) \text{ holds}, \end{cases}$$

$$b_3(n) = \begin{cases} a_3(n), & \text{if } (3) \text{ holds}, \\ \frac{a_3(n)}{A_{21}(n+1)}, & \text{if } (5) \text{ holds}, \end{cases}$$

and

$$y(n) = \frac{z(n)}{A_1(n)}.$$

From the previous theorems, we obtain the following corollary.

Corollary 2.3. *Let* (3) *or* (5) *hold. If the semi-canonical equation* (E) *has a solution* $\{x(n)\}$*, then the canonical equation* (E_c) *also has the same solution.*

Corollary 2.4. *Let* (3) *or* (5) *hold. The semi-canonical equation* (E) *has an eventually positive solution if and only if the canonical equation* (E_c) *has an eventually positive solution.*

Our next lemma presents the classification of the nonoscillatory solution of the equation (E_c).

Lemma 2.5. *Let* $\{x(n)\}$ *be an eventually positive nonoscillatory solution of* (E_c)*. Then the corresponding sequence* $\{y(n)\}$ *is also eventually positive and either*

(C_1) $L_1y(n) > 0,\ L_2y(n) < 0,\ L_3y(n) > 0,\ L_4y(n) \le 0$

or

(C_2) $L_1y(n) > 0,\ L_2y(n) > 0,\ L_3y(n) > 0,\ L_4y(n) \le 0,$

for sufficiently large n*.*

P r o o f. This lemma can be found as Lemma 3 in [12] and so the details of the proof are omitted. □

Before finding a relationship between $\{x(n)\}$ and the corresponding sequence $\{y(n)\} = \left\{\frac{z(n)}{A_1(n)}\right\}$, let us make the additional covering assumption that

(H_5) $$E(n) = 1 - \frac{p(n)A_1(n-\tau)}{A_1(n)} > 0 \text{ for all } n \ge n_0.$$

Lemma 2.6. *Let* $\{x(n)\}$ *be an eventually positive solution of* (E_c)*. Then*

$$x(n) \ge A_1(n)E(n)y(n-\tau) \tag{7}$$

for all $n \ge n_1 \ge n_0$*.*

P r o o f. From the definition of $y(n)$, we have

$$A_1(n)y(n) = z(n) = x(n) + p(n)x(n-\tau),$$

or

$$x(n) \ge A_1(n)y(n) - p(n)A_1(n-\tau)y(n-\tau)$$
$$\ge A_1(n)E(n)y(n-\tau)$$

since in both of the cases (C_1) and (C_2), the sequence $\{y(n)\}$ is increasing. This completes the proof. □

Theorem 2.7. *Let* (3) *or* (5) *hold. If*

$$\sum_{n=n_1}^{\infty} q(n)A_1^{\alpha}(n-\sigma)E^{\alpha}(n-\sigma)=\infty, \tag{8}$$

then equation (E) *is oscillatory.*

Proof. Let $\{x(n)\}$ be an eventually positive solution of (E). Then by Corollary 2.3, we see that $\{x(n)\}$ is also an eventually positive solution of (E_c). Then, by Lemma 2.5, the corresponding sequence $\{y(n)\}$ satisfies either (C_1) or (C_2) for all $n \ge n_1 \ge n_0$.

Now, using (7) in (E_c), we obtain

$$L_4 y(n) + q(n)A_1^{\alpha}(n-\sigma)E^{\alpha}(n-\sigma)y^{\alpha}(n-\tau-\sigma) \le 0 \tag{9}$$

for all $n \ge n_1$. In both cases $\{y(n)\}$ is increasing, so there exists a constant $M > 0$ and an integer $n_2 \ge n_1$ such that $y(n-\tau-\sigma) \ge M$ for $n \ge n_2$. Using this in (9), we obtain

$$-L_4 y(n) \ge M^{\alpha} q(n)A_1^{\alpha}(n-\sigma)E^{\alpha}(n-\sigma).$$

Summing the last inequality from n_2 to n gives

$$M^{\alpha}\sum_{s=n_2}^{n} q(s)A_1^{\alpha}(s-\sigma)E^{\alpha}(s-\sigma) \le L_3 y(n_2) - L_3 y(n+1) \le L_3 y(n_2)$$

since in both cases $L_3 y(n) > 0$. This contradicts (8) and completes the proof of the theorem. □

Remark 1. The above theorem is independent of α and σ, and so it is applicable to linear, sub-linear, or super-linear equations as well as to ordinary, delay, or advanced-type equations.

Before we state and prove our main oscillation results, let us introduce the notation

$$Q_1(n) = \left(\frac{1}{b_2(n)}\sum_{s=n}^{\infty}\frac{1}{b_3(s)}\sum_{t=s}^{\infty} q(t)\, A_1^{\alpha}(t-\sigma)E^{\alpha}(t-\sigma)\right)\left(\sum_{s=n_*}^{n-\tau-\sigma-1}\frac{1}{b_1(s)}\right)^{\alpha}$$

and

$$Q_2(n) = q(n)A_1^{\alpha}(n-\sigma)E^{\alpha}(n-\sigma)\left(\sum_{s=n_*}^{n-\sigma-\tau-1}\frac{1}{b_1(s)}\sum_{s_1=n_*}^{s-1}\frac{1}{b_2(s_1)}\sum_{s_2=n_*}^{s_1-1}\frac{1}{b_3(s_2)}\right)^{\alpha},$$

where $n_* \ge n_0$ is a sufficiently large integer.

Theorem 2.8. *Let* (3) *or* (5) *hold. If both of the first-order delay difference equations*

$$\Delta\eta(n) + Q_1(n)\eta^{\alpha}(n-\tau-\sigma) = 0 \tag{10}$$

and

$$\Delta\zeta(n) + Q_2(n)\zeta^{\alpha}(n-\tau-\sigma) = 0 \tag{11}$$

are oscillatory, then (E) *is oscillatory.*

P r o o f. Let $\{x(n)\}$ be an eventually positive solution of (E), say $x(n) > 0$ for $n \geq n_1 \geq n_0$. Then by Corollary 2.3, we see that $\{x(n)\}$ is also a positive solution of (E_c), and by Lemma 2.5, the sequence $\{y(n)\}$ is positive and satisfies either (C_1) or (C_2). Using (7) in (E_c), we obtain

$$L_4y(n) + q(n)A_1^{\alpha}(n-\sigma)E^{\alpha}(n-\sigma)y^{\alpha}(n-\tau-\sigma) \leq 0, \tag{12}$$

for all $n \geq n_2 \geq n_1 + \tau + \sigma$.

First, assume that $y(n)$ satisfies (C_1). Note that $L_1y(n)$ is positive and decreasing, we have

$$y(n) \geq \sum_{s=n_2}^{n-1} \frac{1}{b_1(s)} L_1y(s) \geq L_1y(n) \sum_{s=n_2}^{n-1} \frac{1}{b_1(s)}. \tag{13}$$

Summing (12) from n to ∞, we obtain

$$L_3y(n) \geq \sum_{s=n}^{\infty} q(s)A_1^{\alpha}(s-\sigma)E^{\alpha}(s-\sigma)y^{\alpha}(s-\tau-\sigma). \tag{14}$$

Since $y(n-\tau-\sigma)$ is increasing, the last inequality implies

$$L_2y(n) \geq \frac{y^{\alpha}(n-\tau-\sigma)}{b_3(n)} \sum_{s=n}^{\infty} q(s)A_1^{\alpha}(s-\sigma)E^{\alpha}(s-\sigma).$$

Again summing the last inequality from n to ∞, we are led to

$$-\Delta\big(L_1y(n)\big) \geq \frac{y^{\alpha}(n-\tau-\sigma)}{b_2(n)} \sum_{s=n}^{\infty} \frac{1}{b_3(s)} \sum_{t=s}^{\infty} q(t)A_1^{\alpha}(t-\sigma)E^{\alpha}(t-\sigma). \tag{15}$$

Using (13) in (15), we have

$$-\Delta\big(L_1y(n)\big) \geq Q_1(n)\big(L_1y^{\alpha}(n-\tau-\sigma)\big)^{\alpha}.$$

So the sequence

$$\{\eta(n)\} = \{L_1y(n)\}$$

is a positive solution of the delay difference inequality

$$\Delta\eta(n) + Q_1(n)\eta^{\alpha}(n-\tau-\sigma) \leq 0.$$

But by Lemma 3 of [7], the associated delay difference equation (10) also has a positive solution, which is a contradiction.

Next, assume that $y(n)$ satisfies (C_2). Since $L_3y(n)$ is positive and decreasing, we see that

$$L_2y(n) \geq \sum_{s=n_2}^{n-1} \frac{1}{b_3(s)} L_3y(s) \geq L_3y(n) \sum_{s=n_2}^{n-1} \frac{1}{b_3(s)}.$$

Summing the above inequality, we have

$$\Delta y(n) \geq L_3y(n) \frac{1}{b_1(n)} \sum_{s=n_2}^{n-1} \frac{1}{b_2(s)} \sum_{t=n_2}^{s-1} \frac{1}{b_3(t)}.$$

Summing once more, we see that $\zeta(n) = L_3y(n)$ satisfies

$$y(n) \geq \zeta(n) \sum_{s=n_2}^{n-1} \frac{1}{b_1(s)} \sum_{t=n_2}^{s-1} \frac{1}{b_2(s)} \sum_{u=n_2}^{t-1} \frac{1}{b_3(u)}.$$

Substituting this into (12), we see that $\zeta(n)$ is a positive solution of the difference inequality

$$\Delta\zeta(n) + Q_2(n)\zeta^{\alpha}(n - \tau - \sigma) \leq 0.$$

Again, by Lemma 3 of [7], the corresponding difference equation (11) also has a positive solution. This contradiction completes the proof. $\square$

Next, we present explicit oscillation criteria for equation (E) if α is linear, sub-linear, or super-linear.

Corollary 2.9. *Let* (3) *or* (5) *hold. If*

$$\liminf_{n\to\infty} \sum_{s=n-\tau-\sigma}^{n-1} Q(s) > \left(\frac{\tau+\sigma}{\tau+\sigma+1}\right)^{\tau+\sigma+1} \qquad \textit{for} \quad \alpha = 1, \tag{16}$$

or

$$\sum_{n=n_*}^{\infty} Q(n) = \infty, \qquad \textit{for} \quad 0 < \alpha < 1, \tag{17}$$

where

$$Q(n) = \min\{Q_1(n), Q_2(n)\},$$

then equation (E) *is oscillatory.*

P r o o f. It is clear that (see Theorem 7.6.1 of [9] and Theorem 1 of [14], respectively) conditions (16) and (17) ensure the oscillation of equations (10) and (11) in the cases $\alpha = 1$ and $0 < \alpha < 1$, respectively. The conclusion then follows from Theorem 2.8. $\square$

Corollary 2.10. *Assume that* (3) *or* (5) *hold. If* $\alpha > 1$ *and there exists a constant*

$$\lambda > \frac{1}{\tau+\sigma}\ln\alpha$$

such that

$$\liminf_{n\to\infty}\left[Q(n)\exp(-e^{\lambda n})\right] > 0, \tag{18}$$

where $Q(n)$ *is given in Corollary 2.9, then equation* (E) *is oscillatory.*

The conclusion follows from Theorem 2 of [14] and Theorem 2.8.

Remark 2. It is important to point out that any set of conditions that implies the oscillation of all solutions of the canonical equation (E_c) guarantees the same property for the semi-canonical equation (E). Therefore, it is possible to find many oscillation results for equation (E). Hence, this has proved to be a highly successful approach and a significant result in oscillatory theory for fourth-order neutral difference equations.

3. Examples

In this section, we present four examples to illustrate the importance of our main results.

Example. Consider the fourth-order neutral delay difference equation

$$\Delta^2\left(\frac{1}{n+1}\Delta\Big(n(n+1)\Delta\big(x(n)+\frac{1}{2}x(n-2)\big)\Big)\right)+\frac{q_0}{n(n+1)}x(n-1)=0, \tag{19}$$

$n \geq 2$, where $q_0 > 0$ is a constant.

Here,

$$a_3(n)=1, \qquad a_2(n)=\tfrac{1}{n+1}, \qquad a_1(n)=n(n+1), \qquad p(n)=\tfrac{1}{2},$$

$$\tau=2, \qquad q(n)=\tfrac{q_0}{n(n+1)}, \qquad \sigma=1, \qquad \text{and} \qquad \alpha=1.$$

This is a semi-canonical equation of type (SC3). Clearly, (H_1)–(H_4) hold. Simple calculations show that

$$A_1(n)=\frac{1}{n} \quad \text{and} \quad A_{21}(2)=\infty,$$

so condition (3) holds. Since

$$b_1=b_2=b_3=1,$$

the transformed equation is

$$\Delta^4 y(n)+\frac{q_0}{n(n+1)}x(n-1)=0, \quad n\geq 2,$$

which, clearly, is in canonical form.

We also see that

$$E(n) \approx \frac{1}{2}, \quad Q_1(n) \approx \frac{q_0(n-3)}{2(n-1)},$$

and

$$Q_2(n) \approx \frac{q_0(n-3)(n-2)}{12n(n+1)}.$$

Therefore,

$$Q(n) \approx \frac{q_0(n-3)(n-2)}{12n(n+1)}.$$

Condition (16) becomes

$$\liminf_{n\to\infty} \sum_{s=n-3}^{n-1} \frac{q_0(s-3)(s-2)}{12s(s+1)} = \frac{q_0}{4} > \left(\frac{3}{4}\right)^4,$$

that is, condition (16) holds if $q_0 > \frac{81}{64}$. Hence, by Corollary 2.9, equation (19) is oscillatory if $q_0 > \frac{81}{64}$.

Example. Consider the fourth-order neutral difference equation

$$\Delta^2\left(n\Delta\left(n(n+1)\Delta\left(x(n)+\frac{1}{2}x(n-2)\right)\right)\right) + \frac{q_0(n-1)}{(n+1)}x^{1/3}(n-1) = 0, \quad (20)$$

$n \geq 2$, where $q_0 > 0$ is a constant, which we will see is of type (SC3). Here,

$$a_3(n) = 1, \qquad a_2(n) = n, \qquad a_1(n) = n(n+1), \qquad p(n) = \tfrac{1}{2},$$

$$\tau = 2, \qquad q(n) = \tfrac{q_0(n-1)}{(n+1)}, \qquad \sigma = 1, \qquad \text{and} \qquad \alpha = \tfrac{1}{3}.$$

We see that (H_1)–(H_4) hold,

$$A_1(n) = \frac{1}{n}, \quad A_{21}(n) = \frac{1}{n},$$

and

$$\frac{A_{21}(n+1)}{a_3(n)} = \frac{1}{n+1}$$

so condition (5) holds. We have

$$b_1 = n, \quad b_2 = 1, \quad \text{and} \quad b_3 = n+1,$$

so the transformed equation is

$$\Delta\Big((n+1)\Delta^2\big(n\Delta y(n)\big)\Big) + \frac{q_0(n-1)}{(n+1)}x^{1/3}(n-1) = 0, \quad n \geq 2,$$

which is in canonical form.

We also see that $E(n) \approx \frac{1}{2}$ and condition (8) becomes

$$\sum_{n=2}^{\infty} \frac{q_0(n-1)^{2/3}}{2^{1/3}(n+1)} = \infty,$$

so it holds. Hence, by Theorem 2.7, equation (20) is oscillatory for all $q_0 > 0$.

EXAMPLE. Consider the fourth-order difference equation

$$\Delta\left(\frac{1}{(n+1)}\Delta\Big(n\Delta\big(n(n+1)\Delta z(n)\big)\Big)\right) + q_0(n+1)^2x^3(n-1) = 0, \tag{21}$$

$n \geq 2$, where

$$z(n) = x(n) + (1/2)x(n-2) \quad \text{and} \quad q_0 > 0.$$

It is not difficult to see that

$$a_3(n) = \tfrac{1}{n+1}, \qquad a_2(n) = n, \qquad a_1(n) = n(n+1), \qquad p(n) = \tfrac{1}{2},$$
$$\tau = 2, \qquad q(n) = q_0(n+1)^2, \qquad \sigma = 1, \qquad \text{and} \qquad \alpha = 3.$$

Now, (H_1)–(H_4) hold,

$$A_1(n) = \frac{1}{n}, \quad A_{21}(n) = \frac{1}{n},$$

and

$$\sum_{n=2}^{\infty} \frac{A_{21}(n+1)}{a_3(n)} = \infty,$$

so condition (5) holds. We have

$$b_1 = n, \quad b_2 = 1 = b_3,$$

and the transformed equation is

$$\Delta^3\big(n\Delta y(n)\big) + q_0(n+1)^2x^3(n-1) = 0, \quad n \geq 2,$$

which is in canonical form. Moreover,

$$E(n) = 1 - \frac{1}{2}\left(\frac{n}{n-2}\right) \approx \frac{1}{2}$$

and condition (8) holds since

$$\sum_{n=2}^{\infty} \frac{q_0(n+1)^2}{8(n-1)^3} = \infty.$$

Therefore, by Theorem 2.7, equation (21) is oscillatory for any $q_0 > 0$.

Example. Consider the fourth-order equation

$$\Delta^2 \left(\frac{1}{(n+1)} \Delta\big(n(n+1)\Delta z(n)\big) \right) + \frac{q_0}{n(n+1)} x^{3/5}(n-1) = 0, \tag{22}$$

$n \geq 2$, where

$$z(n) = x(n) + \frac{1}{2} x(n-2) \quad \text{and} \quad q_0 > 0.$$

We have

$$\begin{array}{cccc} a_3(n) = 1, & a_2(n) = \frac{1}{n+1}, & a_1(n) = n(n+1), & p(n) = \frac{1}{2}, \\ \tau = 2, & (n) = \frac{q_0}{n(n+1)}, & \sigma = 1, & \alpha = \frac{3}{5}, \end{array}$$

and (H_1)–(H_4) hold. Also,

$$A_1(n) = \frac{1}{n} \quad \text{and} \quad A_{21}(2) = \infty,$$

that is, condition (3) holds. Since

$$b_1 = b_2 = b_3 = 1$$

, the transformed equation becomes

$$\Delta^4 y(n) + \frac{q_0}{n(n+1)} x^{3/5}(n-1) = 0, \; n \geq 2,$$

which is in canonical form. Furthermore, we have

$$E(n) \approx \frac{1}{2}, \quad Q_1(n) \approx \frac{25}{24} \frac{q_0}{2^{3/5}} \left(\frac{n-3}{n} \right)^{3/5},$$

and

$$Q_2(n) \approx \frac{q_0}{(12)^{3/5}} \, \frac{1}{n^{4/5}}.$$

Hence,

$$eQ(n) \approx \frac{q_0}{(12)^{3/5}} \, \frac{1}{n^{4/5}}$$

and (17) become

$$\sum_{n=2}^{\infty} \frac{q_0}{(12)^{3/5}} \, \frac{1}{n^{4/5}} = \infty.$$

Thus, by Corollary 2.9, equation (22) is oscillatory for all $q_0 > 0$.

4. Conclusion

In this paper, we first transformed equation (E) into an equation in canonical form. This reduced the number of types of possible nonoscillatory solutions from four to two. This was also very useful in finding a relationship between a solution $\{x(n)\}$ of (E) and the corresponding sequence $\{y(n)\}$; this was essential to obtain criteria for the oscillation of all solutions of equation (E).

It should be noted that we can then use all the known oscillation results for canonical equations to obtain similar results for the semi-canonical equation (E). Thus, this is a significant contribution to the theory of oscillation for fourth-order neutral-type difference equations.

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